Michael E. Taylor

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Nonlinear Equations

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Nonlinear Equations

Second Edition

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To my wife and daughter, Jane Hawkins and Diane Taylor

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Preface

Partial differential equations is a many-faceted subject. Created to describe the mechanical behavior of objects such as vibrating strings and blowing winds, it has developed into a body of material that interacts with many branches of mathematics, such as differential geometry, complex analysis, and harmonic analysis, as well as a ubiquitous factor in the description and elucidation of problems in mathematical physics.

This work is intended to provide a course of study of some of the major aspects of PDE. It is addressed to readers with a background in the basic introductory graduate mathematics courses in American universities: elementary real and complex analysis, differential geometry, and measure theory.

Chapter 1 provides background material on the theory of ordinary differential equations (ODE). This includes both very basic material–on topics such as the existence and uniqueness of solutions to ODE and explicit solutions to equations with constant coefficients and relations to linear algebra–and more sophisticated results–on flows generated by vector fields, connections with differential geometry, the calculus of differential forms, stationary action principles in mechanics, and their relation to Hamiltonian systems. We discuss equations of relativistic motion as well as equations of classical Newtonian mechanics. There are also applications to topological results, such as degree theory, the Brouwer fixed-point theorem, and the Jordan-Brouwer separation theorem. In this chapter we also treat scalar first-order PDE, via Hamilton–Jacobi theory.

Chapters 2–6 constitute a survey of basic linear PDE. Chapter 2 begins with the derivation of some equations of continuum mechanics in a fashion similar to the derivation of ODE in mechanics in Chap. 1, via variational principles. We obtain equations for vibrating strings and membranes; these equations are not necessarily linear, and hence they will also provide sources of problems later, when nonlinear PDE is taken up. Further material in Chap. 2 centers around the Laplace operator, which on Euclidean space $\mathbb{R}^n$ is

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \tag{1}$$

and the linear wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0. \tag{2}$$

We also consider the Laplace operator on a general Riemannian manifold and the wave equation on a general Lorentz manifold. We discuss basic consequences of Green's formula, including energy conservation and finite propagation speed for solutions to linear wave equations. We also discuss Maxwell's equations for electromagnetic fields and their relation with special relativity. Before we can establish general results on the solvability of these equations, it is necessary to develop some analytical techniques. This is done in the next couple of chapters.

Chapter 3 is devoted to Fourier analysis and the theory of distributions. These topics are crucial for the study of linear PDE. We give a number of basic applications to the study of linear PDE with constant coefficients. Among these applications are results on harmonic and holomorphic functions in the plane, including a short treatment of elementary complex function theory. We derive explicit formulas for solutions to Laplace and wave equations on Euclidean space, and also the heat equation,

$$\frac{\partial u}{\partial t} - \Delta u = 0. \tag{3}$$

We also produce solutions on certain subsets, such as rectangular regions, using the method of images. We include material on the discrete Fourier transform, germane to the discrete approximation of PDE, and on the fast evaluation of this transform, the FFT. Chapter 3 is the first chapter to make extensive use of functional analysis. Basic results on this topic are compiled in Appendix A, Outline of Functional Analysis.

Sobolev spaces have proven to be a very effective tool in the existence theory of PDE, and in the study of regularity of solutions. In Chap. 4 we introduce Sobolev spaces and study some of their basic properties. We restrict attention to L^2-Sobolev spaces, such as $H^k(\mathbb{R}^n)$, which consists of L^2 functions whose derivatives of order $\leq k$ (defined in a distributional sense, in Chap. 3) belong to $L^2(\mathbb{R}^n)$, when k is a positive integer. We also replace k by a general real number s. The L^p-Sobolev spaces, which are very useful for nonlinear PDE, are treated later, in Chap. 13.

Chapter 5 is devoted to the study of the existence and regularity of solutions to linear elliptic PDE, on bounded regions. We begin with the Dirichlet problem for the Laplace operator,

$$\Delta u = f \ \text{ on } \Omega, \qquad u = g \ \text{ on } \partial\Omega, \tag{4}$$

and then treat the Neumann problem and various other boundary problems, including some that apply to electromagnetic fields. We also study general boundary problems for linear elliptic operators, giving a condition that guarantees regularity and solvability (perhaps given a finite number of linear conditions on the data). Also in Chap. 5 are some applications to other areas, such as a proof of the Riemann mapping theorem, first for smooth simply connected domains in the complex plane $\mathbb{C}$, then, after a treatment of the Dirichlet problem for the Laplace operator on domains with rough boundary, for general simply connected domains in $\mathbb{C}$. We also develop Hodge theory and apply it to DeRham cohomology, extending the study of topological applications of differential forms begun in Chap. 1.

In Chap. 6 we study linear evolution equations, in which there is a "time" variable t, and initial data are given at $t = 0$. We discuss the heat and wave equations. We also treat Maxwell's equations, for an electromagnetic field, and more general hyperbolic systems. We prove the Cauchy–Kowalewsky theorem, in the linear case, establishing local solvability of the Cauchy initial value problem for general linear PDE with analytic coefficients, and analytic data, as long as the initial surface is "noncharacteristic." The nonlinear case is treated in Chap. 16. Also in Chap. 6 we treat geometrical optics, providing approximations to solutions of wave equations whose initial data either are highly oscillatory or possess simple singularities, such as a jump across a smooth hypersurface.

Chapters 1–6, together with Appendix A and Appendix B, Manifolds, Vector Bundles, and Lie Groups, make up the first volume of this work. The second volume consists of Chaps. 7–12, covering a selection of more advanced topics in linear PDE, together with Appendix C, Connections and Curvature.

Chapter 7 deals with pseudodifferential operators (ψDOs). This class of operators includes both differential operators and parametrices of elliptic operators, that is, inverses modulo smoothing operators. There is a "symbol calculus" allowing one to analyze products of ψDOs, useful for such a parametrix construction. The L^2-boundedness of operators of order zero and the Gårding inequality for elliptic ψDOs with positive symbol provide very useful tools in linear PDE, which will be used in many subsequent chapters.

Chapter 8 is devoted to spectral theory, particularly for self-adjoint elliptic operators. First we give a proof of the spectral theorem for general self-adjoint operators on Hilbert space. Then we discuss conditions under which a differential operator yields a self-adjoint operator. We then discuss the asymptotic distribution of eigenvalues of the Laplace operator on a bounded domain, making use of a construction of a parametrix for the heat equation from Chap. 7. In the next four sections of Chap. 8 we consider the spectral behavior of various specific differential operators: the Laplace operator on a sphere, and on hyperbolic space, the "harmonic oscillator"

$$-\Delta + |x|^2, \tag{5}$$

and the operator

$$-\Delta - \frac{K}{|x|}, \tag{6}$$

which arises in the simplest quantum mechanical model of the hydrogen atom. Finally, we consider the Laplace operator on cones.

In Chap. 9 we study the scattering of waves by a compact obstacle K in $\mathbb{R}^3$. This scattering theory is to some degree an extension of the spectral theory of the

Laplace operator on $\mathbb{R}^3 \setminus K$, with the Dirichlet boundary condition. In addition to studying how a given obstacle scatters waves, we consider the *inverse* problem: how to determine an obstacle given data on how it scatters waves.

Chapter 10 is devoted to the Atiyah–Singer index theorem. This gives a formula for the index of an elliptic operator D on a compact manifold M, defined by

$$\text{Index } D = \dim \ker D - \dim \ker D^*. \tag{7}$$

We establish this formula, which is an integral over M of a certain differential form defined by a pair of "curvatures," when D is a first order differential operator of "Dirac type," a class that contains many important operators arising from differential geometry and complex analysis. Special cases of such a formula include the Chern–Gauss–Bonnet formula and the Riemann–Roch formula. We also discuss the significance of the latter formula in the study of Riemann surfaces.

In Chap. 11 we study Brownian motion, described mathematically by Wiener measure on the space of continuous paths in $\mathbb{R}^n$. This provides a probabilistic approach to diffusion and it both uses and provides new tools for the analysis of the heat equation and variants, such as

$$\frac{\partial u}{\partial t} = -\Delta u + Vu, \tag{8}$$

where V is a real-valued function. There is an integral formula for solutions to (8), known as the Feynman–Kac formula; it is an integral over path space with respect to Wiener measure, of a fairly explicit integrand. We also derive an analogous integral formula for solutions to

$$\frac{\partial u}{\partial t} = -\Delta u + Xu, \tag{9}$$

where X is a vector field. In this case, another tool is involved in constructing the integrand, the stochastic integral. We also study stochastic differential equations and applications to more general diffusion equations.

In Chap. 12 we tackle the $\bar{\partial}$-Neumann problem, a boundary problem for an elliptic operator (essentially the Laplace operator) on a domain $\Omega \subset \mathbb{C}^n$, which is very important in the theory of functions of several complex variables. From a technical point of view, it is of particular interest that this boundary problem does not satisfy the regularity criteria investigated in Chap. 5. If Ω is "strongly pseudoconvex," one has instead certain "subelliptic estimates," which are established in Chap. 12.

The third and final volume of this work contains Chaps. 13–18. It is here that we study nonlinear PDE.

We prepare the way in Chap. 13 with a further development of function space and operator theory, for use in nonlinear analysis. This includes the theory of L^p-Sobolev spaces and Hölder spaces. We derive estimates in these spaces on nonlinear functions $F(u)$, known as "Moser estimates," which are very useful. We extend the theory of pseudodifferential operators to cases where the symbols have limited smoothness, and also develop a variant of ψDO theory, the theory of "paradifferential operators," which has had a significant impact on nonlinear PDE since about 1980. We also estimate these operators, acting on the function spaces mentioned above. Other topics treated in Chap. 13 include Hardy spaces, compensated compactness, and "fuzzy functions."

Chapter 14 is devoted to nonlinear elliptic PDE, with an emphasis on second order equations. There are three successive degrees of nonlinearity: semilinear equations, such as

$$\Delta u = F(x, u, \nabla u), \tag{10}$$

quasi-linear equations, such as

$$\sum a^{jk}(x, u, \nabla u)\partial_j \partial_k u = F(x, u, \nabla u), \tag{11}$$

and completely nonlinear equations, of the form

$$G(x, D^2 u) = 0. \tag{12}$$

Differential geometry provides a rich source of such PDE, and Chap. 14 contains a number of geometrical applications. For example, to deform conformally a metric on a surface so its Gauss curvature changes from $k(x)$ to $K(x)$, one needs to solve the semilinear equation

$$\Delta u = k(x) - K(x)e^{2u}. \tag{13}$$

As another example, the graph of a function $y = u(x)$ is a minimal submanifold of Euclidean space provided u solves the quasilinear equation

$$\bigl(1 + |\nabla u|^2\bigr)\Delta u + (\nabla u) \cdot H(u)(\nabla u) = 0, \tag{14}$$

called the minimal surface equation. Here, $H(u) = (\partial_j \partial_k u)$ is the Hessian matrix of u. On the other hand, this graph has Gauss curvature $K(x)$ provided u solves the completely nonlinear equation

$$\det H(u) = K(x)\bigl(1 + |\nabla u|^2\bigr)^{(n+2)/2}, \tag{15}$$

a Monge-Ampère equation. Equations (13)–(15) are all scalar, and the maximum principle plays a useful role in the analysis, together with a number of other tools. Chapter 14 also treats nonlinear systems. Important physical examples arise in studies of elastic bodies, as well as in other areas, such as the theory of liquid crystals. Geometric examples of systems considered in Chap. 14 include equations for harmonic maps and equations for isometric imbeddings of a Riemannian manifold in Euclidean space.

In Chap. 15, we treat nonlinear parabolic equations. Partly echoing Chap. 14, we progress from a treatment of semilinear equations,

$$\frac{\partial u}{\partial t} = Lu + F(x, u, \nabla u), \tag{16}$$

where L is a linear operator, such as $L = \Delta$, to a treatment of quasi-linear equations, such as

$$\frac{\partial u}{\partial t} = \sum \partial_j a^{jk}(t, x, u)\partial_k u + X(u). \tag{17}$$

(We do very little with completely nonlinear equations in this chapter.) We study systems as well as scalar equations. The first application of (16) we consider is to the parabolic equation method of constructing harmonic maps. We also consider "reaction-diffusion" equations, $\ell \times \ell$ systems of the form (16), in which $F(x, u, \nabla u) = X(u)$, where X is a vector field on $\mathbb{R}^\ell$, and L is a diagonal operator, with diagonal elements $a_j \Delta,\ a_j \geq 0$. These equations arise in mathematical models in biology and in chemistry. For example, $u = (u_1, \dots, u_\ell)$ might represent the population densities of each of ℓ species of living creatures, distributed over an area of land, interacting in a manner described by X and diffusing in a manner described by $a_j \Delta$. If there is a nonlinear (density-dependent) diffusion, one might have a system of the form (17).

Another problem considered in Chap. 15 models the melting of ice; one has a linear heat equation in a region (filled with water) whose boundary (where the water touches the ice) is moving (as the ice melts). The nonlinearity in the problem involves the description of the boundary. We confine our analysis to a relatively simple one-dimensional case.

Nonlinear hyperbolic equations are studied in Chap. 16. Here continuum mechanics is the major source of examples, and most of them are systems, rather than scalar equations. We establish local existence for solutions to first order hyperbolic systems, which are either "symmetric" or "symmetrizable." An example of the latter class is the following system describing compressible fluid flow:

$$\frac{\partial v}{\partial t} + \nabla_v v + \frac{1}{\rho}\,\mathrm{grad}\, p = 0, \quad \frac{\partial \rho}{\partial t} + \nabla_v \rho + \rho\,\mathrm{div}\, v = 0, \tag{18}$$

for a fluid with velocity v, density ρ, and pressure p, assumed to satisfy a relation $p = p(\rho)$, called an "equation of state." Solutions to such nonlinear systems tend to break down, due to shock formation. We devote a bit of attention to the study of weak solutions to nonlinear hyperbolic systems, with shocks.

We also study second-order hyperbolic systems, such as systems for a k-dimensional membrane vibrating in $\mathbb{R}^n$, derived in Chap. 2. Another topic covered in Chap. 16 is the Cauchy–Kowalewsky theorem, in the nonlinear case. We use a method introduced by P. Garabedian to transform the Cauchy problem for an analytic equation into a symmetric hyperbolic system.

In Chap. 17 we study incompressible fluid flow. This is governed by the Euler equation

$$\frac{\partial v}{\partial t} + \nabla_v v = -\operatorname{grad} p, \qquad \operatorname{div} v = 0, \tag{19}$$

in the absence of viscosity, and by the Navier–Stokes equation

$$\frac{\partial v}{\partial t} + \nabla_v v = \nu \mathcal{L} v - \operatorname{grad} p, \qquad \operatorname{div} v = 0, \tag{20}$$

in the presence of viscosity. Here $\mathcal{L}$ is a second-order operator, the Laplace operator for a flow on flat space; the "viscosity" ν is a positive quantity. The equation (19) shares some features with quasilinear hyperbolic systems, though there are also significant differences. Similarly, (20) has a lot in common with semilinear parabolic systems.

Chapter 18, the last chapter in this work, is devoted to Einstein's gravitational equations:

$$G_{jk} = 8\pi\kappa T_{jk}. \tag{21}$$

Here G_{jk} is the Einstein tensor, given by $G_{jk} = \mathrm{Ric}_{jk} - (1/2)Sg_{jk}$, where Ric_{jk} is the Ricci tensor and S the scalar curvature, of a Lorentz manifold (or "spacetime") with metric tensor g_{jk}. On the right side of (21), T_{jk} is the stress-energy tensor of the matter in the spacetime, and κ is a positive constant, which can be identified with the gravitational constant of the Newtonian theory of gravity. In local coordinates, G_{jk} has a nonlinear expression in terms of g_{jk} and its second order derivatives. In the empty-space case, where $T_{jk} = 0$, (21) is a quasilinear second order system for g_{jk}. The freedom to change coordinates provides an obstruction to this equation being hyperbolic, but one can impose the use of "harmonic" coordinates as a constraint and transform (21) into a hyperbolic system. In the presence of matter one couples (21) to other systems, obtaining more elaborate PDE. We treat this in two cases, in the presence of an electromagnetic field, and in the presence of a relativistic fluid.

In addition to the 18 chapters just described, there are three appendices, already mentioned above. Appendix A gives definitions and basic properties of Banach and Hilbert spaces (of which L^p-spaces and Sobolev spaces are examples), Fréchet spaces (such as $C^\infty(\mathbb{R}^n)$), and other locally convex spaces (such as spaces of distributions). It discusses some basic facts about bounded linear operators, including some special properties of compact operators, and also considers certain classes of unbounded linear operators. This functional analytic material plays a major role in the development of PDE from Chap. 3 onward.

Appendix B gives definitions and basic properties of manifolds and vector bundles. It also discusses some elementary properties of Lie groups, including a little representation theory, useful in Chap. 8, on spectral theory, as well as in the Chern–Weil construction.

Appendix C, Connections and Curvature, contains material of a differential geometric nature, crucial for understanding many things done in Chaps. 10–18. We consider connections on general vector bundles, and their curvature. We discuss in detail special properties of the primary case: the Levi–Civita connection and Riemann curvature tensor on a Riemannian manifold. We discuss basic properties of the geometry of submanifolds, relating the second fundamental form to curvature via the Gauss–Codazzi equations. We describe how vector bundles arise from principal bundles, which themselves carry various connections and curvature forms. We then discuss the Chern–Weil construction, yielding certain closed differential forms associated to curvatures of connections on principal bundles. We give several proofs of the classical Gauss–Bonnet theorem and some related results on two-dimensional surfaces, which are useful particularly in Chaps. 10 and 14. We also give a geometrical proof of the Chern–Gauss–Bonnet theorem, which can be contrasted with the proof in Chap. 10, as a consequence of the Atiyah–Singer index theorem.

We mention that, in addition to these "global" appendices, there are appendices to some chapters. For example, Chap. 3 has an appendix on the gamma function. Chapter 6 has two appendices; Appendix A has some results on Banach spaces of harmonic functions useful for the proof of the linear Cauchy–Kowalewsky theorem, and Appendix B deals with the stationary phase formula, useful for the study of geometrical optics in Chap. 6 and also for results later, in Chap. 9. There are other chapters with such "local" appendices. Furthermore, there are two *sections*, both in Chap. 14, with appendices. Section 6, on minimal surfaces, has a companion, Sect. 6B, on the second variation of area and consequences, and Sect. 12, on nonlinear elliptic systems, has a companion, Sect. 12B, with complementary material.

Having described the scope of this work, we find it necessary to mention a number of topics in PDE that are not covered here, or are touched on only very briefly.

For example, we devote little attention to the real analytic theory of PDE. We note that harmonic functions on domains in $\mathbb{R}^n$ are real analytic, but we do not discuss analyticity of solutions to more general elliptic equations. We do prove the Cauchy–Kowalewsky theorem, on analytic PDE with analytic Cauchy data. We derive some simple results on unique continuation from these few analyticity results, but there is a large body of lore on unique continuation, for solutions to nonanalytic PDE, neglected here.

There is little material on numerical methods. There are a few references to applications of the FFT and of "splitting methods." Difference schemes for PDE are mentioned just once, in a set of exercises on scalar conservation laws. Finite element methods are neglected, as are many other numerical techiques.

There is a large body of work on free boundary problems, but the only one considered here is a simple one space dimensional problem, in Chap. 15.

While we have considered a variety of equations arising from classical physics and from relativity, we have devoted relatively little attention to quantum mechanics. We have considered one quantum mechanical operator, given in formula (6) above. Also, there are some exercises on potential scattering mentioned in Chap. 9. However, the physical theories behind these equations are not discussed here.

There are a number of nonlinear evolution equations, such as the Korteweg–deVries equation, that have been perceived to provide infinite dimensional analogues of completely integrable Hamiltonian systems, and to arise "universally" in asymptotic analyses of solutions to various nonlinear wave equations. They are not here. Nor is there a treatment of the Yang–Mills equations for gauge fields, with their wonderful applications to the geometry and topology of four dimensional manifolds.

Of course, this is not a complete list of omitted material. One can go on and on listing important topics in this vast subject. The author can at best hope that the reader will find it easier to understand many of these topics with this book, than without it.

Acknowledgments

I have had the good fortune to teach at least one course relevant to the material of this book, almost every year since 1976. These courses led to many course notes, and I am grateful to many colleagues at Rice University, SUNY at Stony Brook, the California Institute of Technology, and the University of North Carolina, for the supportive atmospheres at these institutions. Also, a number of individuals provided valuable advice on various portions of the manuscript, as it grew over the years. I particularly want to thank Florin David, David Ebin, Frank Jones, Anna Mazzucato, Richard Melrose, James Ralston, Jeffrey Rauch, Santiago Simanca, and James York. The final touches were put on the manuscript while I was visiting the Institute for Mathematics and its Applications, at the University of Minnesota, which I thank for its hospitality and excellent facilities.

Finally, I would like to acknowledge the impact on my studies of my senior thesis and Ph.D. thesis advisors, Edward Nelson and Heinz Cordes.

Introduction to the Second Edition

In addition to making numerous small corrections to this work, collected over the past dozen years, I have taken the opportunity to make some very significant changes, some of which broaden the scope of the work, some of which clarify previous presentations, and a few of which correct errors that have come to my attention.

There are seven additional sections in this edition, two in Volume 1, two in Volume 2, and three in Volume 3. Chapter 4 has a new section, "Sobolev spaces on rough domains," which serves to clarify the treatment of the Dirichlet problem on rough domains in Chap. 5. Chapter 6 has a new section, "Boundary layer phenomena for the heat equation," which will prove useful in one of the new sections in Chap. 17. Chapter 7 has a new section, "Operators of harmonic oscillator type," and Chap. 10 has a section that presents an index formula for elliptic systems of operators of harmonic oscillator type. Chapter 13 has a new appendix, "Variations on complex interpolation," which has material that is useful in the study of Zygmund spaces. Finally, Chap. 17 has two new sections, "Vanishing viscosity limits" and "From velocity convergence to flow convergence."

In addition, several other sections have been substantially rewritten, and numerous others polished to reflect insights gained through the use of these books over time.

13

Function Space and Operator Theory for Nonlinear Analysis

Introduction

This chapter examines a number of analytical techniques, which will be applied to diverse nonlinear problems in the remaining chapters. For example, we study Sobolev spaces based on L^p, rather than just L^2. Sections 1 and 2 discuss the definition of Sobolev spaces $H^{k,p}$, for $k \in \mathbb{Z}^+$, and inclusions of the form $H^{k,p} \subset L^q$. Estimates based on such inclusions have refined forms, due to E. Gagliardo and L. Nirenberg. We discuss these in § 3, together with results of J. Moser on estimates on nonlinear functions of an element of a Sobolev space, and on commutators of differential operators and multiplication operators. In § 4 we establish some integral estimates of N. Trudinger, on functions in Sobolev spaces for which L^∞-bounds just fail. In these sections we use such basic tools as Hölder's inequality and integration by parts.

The Fourier transform is not as effective for analysis on L^p as on L^2. One result that does often serve when, in the L^2-theory, one could appeal to the Plancherel theorem, is Mikhlin's Fourier multiplier theorem, established in § 5. This enables interpolation theory to be applied to the study of the spaces $H^{s,p}$, for noninteger s, in § 6. In § 7 we apply some of this material to the study of L^p-spectral theory of the Laplace operator, on compact manifolds, possibly with boundary.

In § 8 we study spaces C^r of Hölder continuous functions, and their relation with Zygmund spaces C^r_*. We derive estimates in these spaces for solutions to elliptic boundary problems.

The next two sections extend results on pseudodifferential operators, introduced in Chap. 7. Section 9 considers symbols $p(x,\xi)$ with minimal regularity in x. We derive both L^p- and Hölder estimates. Section 10 considers paradifferential operators, a variant of pseudodifferential operator calculus particularly well suited to nonlinear analysis. Sections 9 and 10 are largely taken from [T2].

In § 11 we consider "fuzzy functions," consisting of a pair (f, λ), where f is a function on a space Ω and λ is a measure on $\Omega \times \mathbb{R}$, with the property that $\iint y\varphi(x)\, d\lambda(x,y) = \int \varphi(x) f(x)\, dx$. The measure λ is known as a Young measure. It incorporates information on how f may have arisen as a weak limit of

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smooth ("sharply defined") functions, and it is useful for analyses of nonlinear maps that do not generally preserve weak convergence.

In § 12 there is a brief discussion of Hardy spaces, subspaces of $L^1(\mathbb{R}^n)$ with many desirable properties, only a few of which are discussed here. Much more on this topic can be found in [S3], but material covered here will be useful for some elliptic regularity results in § 12B of Chap. 14.

We end this chapter with Appendix A, discussing variants of the complex interpolation method introduced in Chap. 4 and used a lot in the early sections of this chapter. It turns out that slightly different complex interpolation functors are better suited to the scale of Zygmund spaces.

1. L^p-Sobolev spaces

Let $p \in [1, \infty)$. In analogy with the definition of the Sobolev spaces in Chap. 4, we set, for $k = 0, 1, 2, \dots,$

$$H^{k,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n) \text{ for } |\alpha| \le k\}. \tag{1.1}$$

It is easy to see that $\mathcal{S}(\mathbb{R}^n)$ is dense in each space $H^{k,p}(\mathbb{R}^n)$, with its natural norm

$$\|u\|_{H^{k,p}} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p}. \tag{1.2}$$

For $p \neq 2$, we cannot characterize the spaces $H^{k,p}(\mathbb{R}^n)$ conveniently in terms of the Fourier transform. It is still possible to define spaces $H^{s,p}(\mathbb{R}^n)$ by interpolation; we will examine this in § 6. Here we will consider only the spaces $H^{k,p}(\mathbb{R}^n)$ with k a nonnegative integer.

The chain rule allows us to say that if $\chi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism that is linear outside a compact set, then $\chi^* : H^{k,p}(\mathbb{R}^n) \to H^{k,p}(\mathbb{R}^n)$. Also multiplication by an element $\varphi \in C_0^\infty(\mathbb{R}^n)$ maps $H^{k,p}(\mathbb{R}^n)$ to itself. This allows us to define $H^{k,p}(M)$ for a compact manifold M via a partition of unity subordinate to a coordinate chart. Also, for compact M, if we define $\text{Diff}^k(M)$ to be the set of differential operators of order $\le k$ on M, with smooth coefficients, then

$$H^{k,p}(M) = \{u \in L^p(M) : Pu \in L^p(M) \text{ for all } P \in \text{Diff}^k(M)\}. \tag{1.3}$$

We can define $H^{k,p}(\mathbb{R}^n_+)$ as in (1.1), with $\mathbb{R}^n$ replaced by $\mathbb{R}^n_+$. The extension operator defined by (4.2)–(4.4) of Chap. 4 also works to produce extension maps $E : H^{k,p}(\mathbb{R}^n_+) \to H^{k,p}(\mathbb{R}^n)$. Similarly, if M is a compact manifold with smooth boundary, with double N, we can define $H^{k,p}(M)$ via coordinate charts and the notion of $H^{k,p}(\mathbb{R}^n_+)$, or by (1.3), and we have extension operators $E : H^{k,p}(M) \to H^{k,p}(N)$.

We also note the obvious fact that

$$D^\alpha : H^{k,p}(\mathbb{R}^n) \longrightarrow H^{k-|\alpha|,p}(\mathbb{R}^n), \tag{1.4}$$

for $|\alpha| \le k$, and

$$P : H^{k,p}(M) \longrightarrow H^{k-\ell,p}(M) \ \text{ if } P \in \mathrm{Diff}^\ell(M), \tag{1.5}$$

provided $\ell \le k$.

Exercises

1. A Friedrichs mollifier on $\mathbb{R}^n$ is a family of smoothing operators $J_\varepsilon u(x) = j_\varepsilon * u(x)$ where

$$j_\varepsilon(x) = \varepsilon^{-n} j(\varepsilon^{-1}x), \quad \int j(x)dx = 1, \quad j \in \mathcal{S}(\mathbb{R}^n).$$

Equivalently, $J_\varepsilon u(x) = \varphi(\varepsilon D)u(x)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi(0) = 1$. Show that, for each $p \in [1, \infty)$, $k \in \mathbb{Z}^+$,

$$J_\varepsilon : H^{k,p}(\mathbb{R}^n) \longrightarrow \bigcap_{\ell<\infty} H^{\ell,p}(\mathbb{R}^n),$$

for each $\varepsilon > 0$, and

$$J_\varepsilon u \to u \quad \text{in } H^{k,p}(\mathbb{R}^n)$$

as $\varepsilon \to 0$ if $u \in H^{k,p}(\mathbb{R}^n)$.

2. Suppose $A \in C^1(\mathbb{R}^n)$, with $\|A\|_{C^1} = \sup_{|\alpha|\le 1} \|D^\alpha A\|_{L^\infty}$. Show that when J_ε is a Friedrichs mollifier as above, then

$$\|[A, J_\varepsilon]v\|_{H^{1,p}} \le C\|A\|_{C^1}\|v\|_{L^p},$$

with C independent of $\varepsilon \in (0, 1]$. (*Hint*: Write $A(x) - A(y) = \sum B_k(x, y)(x_k - y_k)$, $|B_k(x, y)| \le K$, and, with $q_\ell(x) = \partial j/\partial x_\ell$,

$$\partial_\ell[A, J_\varepsilon]v(x) = \sum \int B_k(x, y)\Big[\varepsilon^{-n} q_\ell\Big(\frac{x-y}{\varepsilon}\Big) \cdot \frac{x_k - y_k}{\varepsilon}\Big]v(y)\, dy,$$

with absolute value bounded by

$$K\,\varepsilon^{-n} \sum \int \big|\varphi_{k\ell}\big(\varepsilon^{-1}(x-y)\big)\big| \cdot |v(y)|\, dy,$$

where $\varphi_{k\ell}(x) = x_k q_\ell(x)$.)

3. Using Exercise 2, show that

$$\|[A, J_\varepsilon]\partial_j v\|_{L^p} \le C\|A\|_{C^1}\|v\|_{L^p}.$$

2. Sobolev imbedding theorems

We will derive various inclusions of the type $H^{k,p}(M) \subset H^{\ell,q}(M)$. We will concentrate on the case $M = \mathbb{R}^n$. The discussion of § 1 will give associated results when M is a compact manifold, possibly with (smooth) boundary.

One technical tool useful for our estimates is the following generalized Hölder inequality:

Lemma 2.1. *If $p_j \in [1,\infty]$, $\sum p_j^{-1} = 1$, then*

$$\int_M |u_1 \cdots u_m|\, dx \le \|u_1\|_{L^{p_1}(M)} \cdots \|u_m\|_{L^{p_m}(M)}. \tag{2.1}$$

The proof follows by induction from the case $m = 2$, which is the usual Hölder inequality.

Our first Sobolev imbedding theorem is the following:

Proposition 2.2. *For $p \in [1, n)$,*

$$H^{1,p}(\mathbb{R}^n) \subset L^{np/(n-p)}(\mathbb{R}^n). \tag{2.2}$$

In fact, there is an estimate

$$\|u\|_{L^{np/(n-p)}} \le C\|\nabla u\|_{L^p}, \tag{2.3}$$

for $u \in H^{1,p}(\mathbb{R}^n)$, with $C = C(p,n)$.

Proof. It suffices to establish (2.3) for $u \in C_0^\infty(\mathbb{R}^n)$. Clearly,

$$|u(x)| \le \int_{-\infty}^{\infty} |D_j u|\, dx_j, \tag{2.4}$$

so

$$|u(x)|^{n/(n-1)} \le \left\{ \prod_{j=1}^{n} \int_{-\infty}^{\infty} |D_j u|\, dx_j \right\}^{1/(n-1)}. \tag{2.5}$$

We can integrate (2.5) successively over each variable x_j, $j = 1, \dots, n$, and apply the generalized Hölder inequality (2.1) with $m = p_1 = \cdots = p_m = n-1$ after each integration. We get

$$\|u\|_{L^{n/(n-1)}} \le \left\{ \prod_{j=1}^{n} \int_{\mathbb{R}^n} |D_j u|\, dx \right\}^{1/n} \le C\|\nabla u\|_{L^1}. \tag{2.6}$$

This establishes (2.3) in the case $p = 1$. We can apply this to $v = |u|^\gamma,\ \gamma > 1$, obtaining

$$\big\| |u|^\gamma \big\|_{L^{n/(n-1)}} \le C \big\| |u|^{\gamma-1} |\nabla u| \big\|_{L^1} \le C \big\| |u|^{\gamma-1} \big\|_{L^{p'}} \big\| \nabla u \big\|_{L^p}. \tag{2.7}$$

For $p < n$, pick $\gamma = (n-1)p/(n-p)$. Then (2.7) gives (2.3) and the proposition is proved.

Given $u \in H^{k,p}(\mathbb{R}^n)$, we can apply Proposition 2.2 to estimate the $L^{np/(n-p)}$-norm of $D^{k-1}u$ in terms of $\|D^k u\|_{L^p}$, where we use the notation

$$D^k u = \{D^\alpha u : |\alpha| = k\}, \quad \|D^k u\|_{L^p} = \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p}, \tag{2.8}$$

and proceed inductively, obtaining the following corollary.

Proposition 2.3. *For $kp < n$,*

$$H^{k,p}(\mathbb{R}^n) \subset L^{np/(n-kp)}(\mathbb{R}^n). \tag{2.9}$$

The same result holds with $\mathbb{R}^n$ replaced by a compact manifold of dimension n. If we take $p = 2$, then for the Sobolev spaces $H^k(\mathbb{R}^n) = H^{k,2}(\mathbb{R}^n)$, we have

$$H^k(\mathbb{R}^n) \subset L^{2n/(n-2k)}(\mathbb{R}^n), \quad k < \frac{n}{2}. \tag{2.10}$$

Consequently, the interpolation theory developed in Chap. 4 implies

$$H^s(\mathbb{R}^n) \subset L^{2n/(n-2s)}(\mathbb{R}^n), \tag{2.11}$$

for any real $s \in [0,k],\ k < n/2$ an integer. Actually, (2.11) holds for any real $s \in [0, n/2)$, as will be shown in § 6. We write down some particular examples, for $n = 2, 3, 4$, which will play a role later in various nonlinear evolution equations, such as the Navier–Stokes equations. The cases $n = 3, 4$ follow from the results proved above, while the case $n = 2$ follows from the general case of (2.11) established in § 6.

$$\begin{array}{ccc} & H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3) & H^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4) \\ & H^{3/4}(\mathbb{R}^3) \subset L^4(\mathbb{R}^3) & \\ H^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2) & H^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) & \end{array} \tag{2.12}$$

Note that interpolation of the $\mathbb{R}^2$-result with $L^2(\mathbb{R}^2) = L^2(\mathbb{R}^2)$ yields

$$H^{1/3}(\mathbb{R}^2) \subset L^3(\mathbb{R}^2).$$

The next result provides a partial generalization of the Sobolev imbedding theorem,

$$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n), \quad s > \frac{n}{2},$$

proved in Chap. 4. A more complete generalization is given in § 6.

Proposition 2.4. *We have*

$$H^{k,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \ \ \textit{for } kp > n. \tag{2.13}$$

Proof. It suffices to obtain a bound on $\|u\|_{L^\infty(\mathbb{R}^n)}$ for $u \in H^{k,p}(\mathbb{R}^n)$, if $kp > n$. In turn, it suffices to bound $u(0)$ appropriately, for $u \in C_0^\infty(\mathbb{R}^n)$. Use polar coordinates, $x = r\omega$, $\omega \in S^{n-1}$. Let $g \in C^\infty(\mathbb{R})$ have the property that $g(r) = 1$ for $r < 1/2$ and $g(r) = 0$ for $r > 3/4$. Then, for each ω, we have

$$\begin{aligned} u(0) &= -\int_0^1 \frac{\partial}{\partial r}\,[g(r)u(r,\omega)]\;dr \\ &= \frac{(-1)^k}{(k-1)!}\int_0^1 r^{k-n}\left\{\left(\frac{\partial}{\partial r}\right)^k [g(r)u(r,\omega)]\right\}\; r^{n-1}dr, \end{aligned}$$

upon integrating by parts $k-1$ times. Integrating over $\omega \in S^{n-1}$ gives

$$|u(0)| \le C\int\limits_B r^{k-n}\left|\left(\frac{\partial}{\partial r}\right)^k [g(r)u(x)]\right|\;dx,$$

where B is the unit ball centered at 0. Hölder's inequality gives

$$|u(0)| \le C\,\|r^{k-n}\|_{L^{p'}(B)}\,\big\|\partial_r^k[g(r)u(x)]\big\|_{L^p(B)}, \tag{2.14}$$

with $1/p + 1/p' = 1$. We claim that $(\partial/\partial r)^k$ is a linear combination of D^α, $|\alpha| = k$, with L^∞-coefficients. To see this, note that ∂_r^k annihilates x^α for $|\alpha| < k$, so we get

$$\left(\frac{\partial}{\partial r}\right)^k = \sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha, \tag{2.15}$$

with $a_\alpha(x) = (1/\alpha!)\partial_r^k x^\alpha$, for $|\alpha| = k$, or

$$a_\alpha(r\omega) = \frac{k!}{\alpha!}\,\omega^\alpha,$$

so $a_\alpha(x)$ is homogeneous of degree 0 in x and smooth on $\mathbb{R}^n \setminus 0$.

Returning to the estimate of (2.14), our information on $(\partial/\partial r)^k$ implies that the last factor on the right side is bounded by the $H^{k,p}$-norm of u. The factor $\|r^{k-n}\|_{L^{p'}(B)}$ is finite provided $kp > n$, so the proposition is proved.

To close this section, we note the following simple consequence of Proposition 2.2, of occasional use in analysis. Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of locally finite Borel measures (not necessarily positive) on $\mathbb{R}^n$. Let us assume that $n \geq 2$.

Proposition 2.5. *If we have $u \in \mathcal{M}(\mathbb{R}^n)$ and $\nabla u \in \mathcal{M}(\mathbb{R}^n)$, then it follows that $u \in L^{n/(n-1)}_{loc}(\mathbb{R}^n)$.*

Proof. Using a cut-off in C_0^∞, we can assume u has compact support. Applying a mollifier, we get $u_j = \chi_j * u \in C_0^\infty(\mathbb{R}^n)$ such that $u_j \to u$ and $\nabla u_j \to \nabla u$ in $\mathcal{M}(\mathbb{R}^n)$. In particular, we have a uniform L^1-norm estimate on ∇u_j. By (2.3) we have a uniform $L^{n/(n-1)}$-norm estimate on u_j, which gives the result, since $L^{n/(n-1)}(\mathbb{R}^n)$ is reflexive.

Exercises

1. If $p_j \in [1,\infty]$ and $u_j \in L^{p_j}$, show that $u_1u_2 \in L^r$ provided $r^{-1} = {p_1}^{-1} + {p_2}^{-1} \in [0,1]$. Show that this implies Lemma 2.1.
2. Use the containment (which follows from Proposition 2.2)

$$H^{k,p}(\mathbb{R}^n) \subset H^{1,np/(n-(k-1)p)}(\mathbb{R}^n) \quad \text{if } (k-1)p < n$$

to show that if Proposition 2.4 is proved in the case $k = 1$, then it follows in general. Note that the proof in the text of Proposition 2.4 is slightly simpler in the case $k = 1$ than for $k \geq 2$.

3. Suppose $k = 2\ell$ is even. Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ and

$$(-\Delta + 1)^\ell u = f \in L^p(\mathbb{R}^n).$$

Show that

$$u = \mathcal{J}_k * f, \quad \widehat{\mathcal{J}}_k(\xi) = \langle \xi \rangle^{-k}.$$

Using estimates on $\mathcal{J}_k(x)$ established in Chap. 3, § 8, show that

$$kp > n \Longrightarrow u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Show that this gives an alternative proof of Proposition 2.4 in case k is even.

4. Suppose $k = 2\ell + 1$ is odd, $kp > 1$. Use the containment

$$H^{k,p}(\mathbb{R}^n) \subset H^{k-1,np/(n-p)}(\mathbb{R}^n) \quad \text{if } p < n,$$

which follows from Proposition 2.2, to deduce from Exercise 3 that Proposition 2.4 holds for all integers $k \geq 2$.

5. Establish the following variant of the $k = 1$ case of (2.14):

$$|u(0) - u(x)| \leq C\|\nabla u\|_{L^p(B)}, \quad p > n,\ x \in \partial B. \tag{2.16}$$

(*Hint*: Suppose $x = e_1$. If γ_z is the line segment from 0 to z, followed by the line segment from z to e_1, write

$$u(e_1) - u(0) = \int_\Sigma \Big(\int_{\gamma_z} du \Big)\, dS(z), \quad \Sigma = \Big\{ x \in B : x_1 = \frac{1}{2} \Big\}.$$

Show that this gives $u(e_1) - u(0) = \int_B \nabla u(z) \cdot \varphi(z)\, dz$, with $\varphi \in L^q(B),\ \forall\, q < n/(n-1)$.)

6. Show that $H^{n,1}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
(*Hint*: $u(x) = \int_{-\infty}^0 \cdots \int_{-\infty}^0 D_1 \cdots D_n u(x+y)\, dy_1 \cdots dy_n$.)

3. Gagliardo–Nirenberg–Moser estimates

In this section we establish further estimates on various L^p-norms of derivatives of functions, which are very useful in nonlinear PDE. Estimates of this sort arose in work of Gagliardo [Gag], Nirenberg [Ni], and Moser [Mos]. Our first such estimate is the following. We keep the convention (2.8).

Proposition 3.1. *For real $k \ge 1,\ 1 \le p \le k$, we have*

$$\|D_j u\|^2_{L^{2k/p}} \le C\, \|u\|_{L^{2k/(p-1)}} \cdot \|D_j^2 u\|_{L^{2k/(p+1)}}, \tag{3.1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, hence for all $u \in L^{q_2}(\mathbb{R}^n) \cap H^{2,q_1}$, where

$$q_1 = \frac{2k}{p+1}, \quad q_2 = \frac{2k}{p-1}. \tag{3.2}$$

Proof. Given $v \in C_0^\infty(\mathbb{R}^n),\ q \ge 2$, we have $v|v|^{q-2} \in C_0^1(\mathbb{R}^n)$ and

$$D_j(v|v|^{q-2}) = (q-1)(D_j v)|v|^{q-2}.$$

Letting $v = D_j u$, we have

$$|D_j u|^q = D_j(u\, D_j u |D_j u|^{q-2}) - (q-1) u\, D_j^2 u |D_j u|^{q-2}.$$

Integrating this, we have, by the generalized Hölder inequality (2.1),

$$\|D_j u\|^q_{L^q} \le |q-1| \cdot \|u\|_{L^{q_2}}\, \|D_j^2 u\|_{L^{q_1}}\, \|D_j u\|^{q-2}_{L^q}, \tag{3.3}$$

where $q = 2k/p$ and q_1 and q_2 are given by (3.2). Dividing by $\|D_j u\|^{q-2}_{L^q}$ gives the estimate (3.1) for $u \in C_0^\infty(\mathbb{R}^n)$, and the proposition follows.

If we apply (3.1) to $D^{\ell-1}u$, we get

$$\|D^\ell u\|^2_{L^{2k/p}} \le C\, \|D^{\ell-1} u\|_{L^{2k/(p-1)}}\, \|D^{\ell+1} u\|_{L^{2k/(p+1)}}, \tag{3.4}$$

for real $k \geq 1,\ p \in [1,k],\ \ell \geq 1$. Consequently, for any $\varepsilon > 0$,

$$\|D^\ell u\|_{L^{2k/p}} \leq C\varepsilon \|D^{\ell-1}\|_{L^{2k/(p-1)}} + C(\varepsilon)\|D^{\ell+1}u\|_{L^{2k/(p+1)}}. \tag{3.5}$$

If $p \in [2,k]$ and $\ell \geq 2$, we can apply (3.5) with p replaced by $p-1$ and $D^{\ell-1}u$ replaced by $D^{\ell-2}u$, to get, for any $\varepsilon_1 > 0$,

$$\|D^{\ell-1}u\|_{L^{2k/(p-1)}} \leq C\varepsilon_1 \|D^{\ell-2}u\|_{L^{2k/(p-2)}} + C(\varepsilon_1)\|D^\ell u\|_{L^{2k/p}}. \tag{3.6}$$

Now we can plug (3.6) into (3.5); fix ε_1 (e.g., $\varepsilon_1 = 1$), and pick ε so small that $C\varepsilon C(\varepsilon_1) \leq 1/2$, so the term $C\varepsilon C(\varepsilon_1)\|D^\ell u\|_{L^{2k/p}}$ can be absorbed on the left, to yield

$$\|D^\ell u\|_{L^{2k/p}} \leq C\varepsilon \|D^{\ell-2}u\|_{L^{2k/(p-2)}} + C(\varepsilon)\|D^{\ell+1}u\|_{L^{2k/(p+1)}}, \tag{3.7}$$

for real $k \geq 2,\ p \in [2,k],\ \ell \geq 2$. Continuing in this fashion, we get

$$\|D^\ell u\|_{L^{2k/p}} \leq C\varepsilon \|D^{\ell-j}u\|_{L^{2k/(p-j)}} + C(\varepsilon)\|D^{\ell+1}u\|_{L^{2k/(p+1)}}, \tag{3.8}$$

$j \leq p \leq k,\ \ell \geq j$. Similarly working on the last term in (3.8), we have the following:

Proposition 3.2. *If $j \leq p \leq k+1-m,\ \ell \geq j$, then (for sufficiently small $\varepsilon > 0$)*

$$\|D^\ell u\|_{L^{2k/p}} \leq C\varepsilon \|D^{\ell-j}u\|_{L^{2k/(p-j)}} + C(\varepsilon)\|D^{\ell+m}u\|_{L^{2k/(p+m)}}. \tag{3.9}$$

Here, j, ℓ, and m must be positive integers, but p and k are real. Of course, the full content of (3.9) is represented by the case $\ell = j$, which reads

$$\|D^\ell u\|_{L^{2k/p}} \leq C\varepsilon \|u\|_{L^{2k/(p-\ell)}} + C(\varepsilon)\|D^{\ell+m}u\|_{L^{2k/(p+m)}}, \tag{3.10}$$

for $\ell \leq p \leq k+1-m$. Taking $p+m = k$, we note the following important special case.

Corollary 3.3. *If ℓ, p, and k are positive integers satisfying $\ell \leq p \leq k-1$, then*

$$\|D^\ell u\|_{L^{2k/p}} \leq C\varepsilon \|u\|_{L^{2k/(p-\ell)}} + C(\varepsilon)\|D^{k+\ell-p}u\|_{L^2}. \tag{3.11}$$

In particular, taking $p = \ell$, if $\ell < k$, then

$$\|D^\ell u\|_{L^{2k/\ell}} \leq C\varepsilon \|u\|_{L^\infty} + C(\varepsilon)\|D^k u\|_{L^2}, \tag{3.12}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

We want estimates for the left sides of (3.11) and (3.12) which involve products, as in (3.1), rather than sums. The following simple general result produces such estimates.

Proposition 3.4. *Let ℓ, μ, and m be nonnegative integers satisfying $\ell \le \max(\mu, m)$, and let q, r, and ρ belong to $[1, \infty]$. Suppose the estimate*

$$\|D^\ell u\|_{L^q} \le C_1 \|D^\mu u\|_{L^r} + C_2 \|D^m u\|_{L^\rho} \tag{3.13}$$

is valid for all $u \in C_0^\infty(\mathbb{R}^n)$. Then

$$\|D^\ell u\|_{L^q} \le (C_1 + C_2) \|D^\mu u\|_{L^r}^{\beta/(\alpha+\beta)} \cdot \|D^m u\|_{L^\rho}^{\alpha/(\alpha+\beta)}, \tag{3.14}$$

with

$$\alpha = \frac{n}{q} - \frac{n}{r} + \mu - \ell, \quad \beta = -\frac{n}{q} + \frac{n}{\rho} - m + \ell, \tag{3.15}$$

provided these quantities are not both zero. If (3.13) is valid and the quantities (3.15) are both nonzero, then they have the same sign.

Proof. Replacing $u(x)$ in (3.13) by $u(sx)$ produces from (3.13), which we write schematically as $Q \le C_1 R + C_2 P$, the estimate

$$s^{\ell - n/q} Q \le C_1 s^{\mu - n/r} R + C_2 s^{m - n/\rho} P, \quad \text{for all } s > 0,$$

or equivalently,

$$Q \le C_1 s^\alpha R + C_2 s^{-\beta} P, \quad \text{for all } s > 0,$$

with α and β given by (3.15). If α and β have opposite signs, one can take $s \to 0$ or $s \to \infty$ to produce the absurd conclusion $Q = 0$. If they have the same sign, one can take s so that $s^\alpha R = s^{-\beta} P = P^a R^b$, which can be done with $a = \alpha/(\alpha + \beta),\ b = \beta/(\alpha + \beta)$, and the estimate (3.14) results.

Applying Proposition 3.4 to the estimate (3.11), we find $\alpha = (n - 2k)\ell/2k,\ \beta = (n - 2k)(k - p)/2k$, which gives the following:

Proposition 3.5. *If ℓ, p, and k are positive integers satisfying $\ell \le p \le k - 1$, then*

$$\|D^\ell u\|_{L^{2k/p}} \le C \|u\|_{L^{2k/(p-\ell)}}^{(k-p)/(k+\ell-p)} \cdot \|D^{k+\ell-p} u\|_{L^2}^{\ell/(k+\ell-p)}. \tag{3.16}$$

In particular, taking $p = \ell$, if $\ell < k$, then

$$\|D^\ell u\|_{L^{2k/\ell}} \le C \|u\|_{L^\infty}^{1-\ell/k} \cdot \|D^k u\|_{L^2}^{\ell/k}. \tag{3.17}$$

One of the principal applications of such an inequality as (3.17) is to bilinear estimates, such as the following.

Proposition 3.6. *If $|\beta| + |\gamma| = k$, then*

$$\|(D^\beta f)(D^\gamma g)\|_{L^2} \le C\|f\|_{L^\infty}\|g\|_{H^k} + C\|f\|_{H^k}\|g\|_{L^\infty}, \tag{3.18}$$

for all $f, g \in C_o(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$.

Proof. With $|\beta| = \ell, |\gamma| = m$, and $\ell + m = k$, we have

$$\begin{aligned}\|(D^\beta f)(D^\gamma g)\|_{L^2} &\le \|D^\beta f\|_{L^{2k/\ell}} \cdot \|D^\gamma g\|_{L^{2k/m}} \\ &\le C\|f\|_{L^\infty}^{1-\ell/k} \cdot \|f\|_{H^k}^{\ell/k} \cdot \|g\|_{L^\infty}^{1-m/k} \cdot \|g\|_{H^k}^{m/k},\end{aligned} \tag{3.19}$$

using Hölder's inequality and (3.17). We can write the right side of (3.19) as

$$C\left(\|f\|_{L^\infty}\|g\|_{H^k}\right)^{m/k}\left(\|f\|_{H^k}\|g\|_{L^\infty}\right)^{\ell/k}, \tag{3.20}$$

and this is readily dominated by the right side of (3.18).

The two estimates of the next proposition are major implications of (3.18).

Proposition 3.7. *We have the estimates*

$$\|f \cdot g\|_{H^k} \le C\|f\|_{L^\infty}\|g\|_{H^k} + C\|f\|_{H^k}\|g\|_{L^\infty} \tag{3.21}$$

and, for $|\alpha| \le k$,

$$\|D^\alpha(f \cdot g) - fD^\alpha g\|_{L^2} \le C\|f\|_{H^k}\|g\|_{L^\infty} + C\|\nabla f\|_{L^\infty}\|g\|_{H^{k-1}}. \tag{3.22}$$

Proof. The estimate (3.21) is an immediate consequence of (3.18). To prove (3.22), write

$$D^\alpha(f \cdot g) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \left(D^\beta f\right)\left(D^\gamma g\right), \tag{3.23}$$

so, if $|\alpha| = k$,

$$\begin{aligned}D^\alpha(f \cdot g) - fD^\alpha g &= \sum_{\beta+\gamma=\alpha, \beta>0} \binom{\alpha}{\beta} \left(D^\beta f\right)\left(D^\gamma g\right) \\ &= \sum_{|\beta|+|\gamma|=k-1} C_{j\beta\gamma}(D^\beta D_j f)(D^\gamma g).\end{aligned} \tag{3.24}$$

Hence, with $u_j = D_j f$,

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \le C \sum_{|\beta|+|\gamma|=k-1} \|(D^\beta u_j)(D^\gamma g)\|_{L^2}. \tag{3.25}$$

From here, the estimate (3.22) follows immediately from (3.18), and Proposition 3.7 is proved. Note that on the right side of (3.22), we can replace $\|f\|_{H^k}$ by $\|\nabla f\|_{H^{k-1}}$.

From Proposition 3.4 there follow further estimates involving products of norms, which can be quite useful. We record a few here.

Proposition 3.8. *We have the estimates*

$$\|u\|_{L^\infty} \le C\,\|D^{m+1}u\|_{L^2}^{1/2} \cdot \|D^{m-1}u\|_{L^2}^{1/2}, \quad \textit{for } u \in C_0^\infty(\mathbb{R}^{2m}), \tag{3.26}$$

and

$$\|u\|_{L^\infty} \le C\,\|D^{m+1}u\|_{L^2}^{1/2} \cdot \|D^{m}u\|_{L^2}^{1/2}, \quad \textit{for } u \in C_0^\infty(\mathbb{R}^{2m+1}). \tag{3.27}$$

Proof. It is easy to see that

$$\|u\|_{L^\infty}^2 \le C\,\|D^{m+1}u\|_{L^2}^2 + C\,\|D^{m-1}u\|_{L^2}^2, \quad \text{for } u \in C_0^\infty(\mathbb{R}^{2m}), \tag{3.28}$$

and

$$\|u\|_{L^\infty}^2 \le C\,\|D^{m+1}u\|_{L^2}^2 + C\,\|D^{m}u\|_{L^2}^2, \quad \text{for } u \in C_0^\infty(\mathbb{R}^{2m+1}). \tag{3.29}$$

Proposition 3.4 then yields $\alpha = \beta = 1$ in case (3.28) and $\alpha = \beta = 1/2$ in case (3.29), proving (3.26) and (3.27).

A more delicate L^∞-estimate will be proved in § 8.
It is also useful to have the following estimates on compositions.

Proposition 3.9. *Let F be smooth, and assume $F(0) = 0$. Then, for $u \in H^k \cap L^\infty$,*

$$\|F(u)\|_{H^k} \le C_k(\|u\|_{L^\infty})\,(1 + \|u\|_{H^k}). \tag{3.30}$$

Proof. The chain rule gives

$$D^\alpha F(u) = \sum_{\beta_1 + \cdots \beta_\mu = \alpha} C_\beta u^{(\beta_1)} \cdots u^{(\beta_\mu)} F^{(\mu)}(u),$$

hence

$$\|D^k F(u)\|_{L^2} \le C_k(\|u\|_{L^\infty}) \sum \big\| u^{(\beta_1)} \cdots u^{(\beta_\mu)} \big\|_{L^2}. \tag{3.31}$$

From here, (3.30) is obtained via the following simple generalization of Proposition 3.6:

Lemma 3.10. *If* $|\beta_1| + \cdots + |\beta_\mu| = k$, *then*

(3.32)
$$\|f_1^{(\beta_1)} \cdots f_\mu^{(\beta_\mu)}\|_{L^2} \le C \sum_\nu \Big[\|f_1\|_{L^\infty} \cdots \widehat{\|f_\nu\|_{L^\infty}} \cdots \|f_\mu\|_{L^\infty}\Big] \|f\|_{H^k}.$$

Proof. The generalized Hölder inequality dominates the left side of (3.32) by

$$\|f_1^{(\beta_1)}\|_{L^{2k/|\beta_1|}} \cdots \|f_\mu^{(\beta_\mu)}\|_{L^{2k/|\beta_\mu|}}. \tag{3.33}$$

Then applying (3.17) dominates this by

$$C\|f_1\|_{L^\infty}^{1-|\beta_1|/k} \cdot \|f_1\|_{H^k}^{|\beta_1|/k} \cdots \|f_\mu\|_{L^\infty}^{1-|\beta_\mu|/k} \cdot \|f_\mu\|_{H^k}^{|\beta_\mu|/k}, \tag{3.34}$$

which in turn is easily bounded by the right side of (3.32) (with $f = (f_1, \dots, f_\mu)$).

We remark that Proposition 3.9 also works if u takes values in $\mathbb{R}^L$. The estimates in Propositions 3.7 and 3.8 are called *Moser estimates*, and are very useful in nonlinear PDE. Some extensions will be given in (10.20) and (10.52).

Exercises

1. Show that the proof of Proposition 3.1 yields

$$\|D_j u\|_{L^q}^2 \le C\|u\|_{L^{q_1}} \cdot \|D^2 u\|_{L^{q_2}} \tag{3.35}$$

   whenever $2 \le q < \infty$, $1 \le q_j \le \infty$, and $1/q_1 + 1/q_2 = 2/q$. Show that if $q_2 < q < q_1$, then (3.35) and (3.1) are equivalent. Is (3.35) valid if the hypothesis $q \ge 2$ is relaxed to $q \ge 1$?
2. Show directly that (3.35) holds with $q_1 = q_2 = q \in [1, \infty]$. (*Hint*: Do the next exercise.)
3. Let A generate a contraction semigroup on a Banach space B. Show that

$$\|Au\|^2 \le 8\|u\| \cdot \|A^2 u\|, \quad \text{for } u \in \mathcal{D}(A^2). \tag{3.36}$$

   (*Hint*: Use the identity $-tAu = t(t-A)^{-1}A^2u + t^2u - t^2 t(t-A)^{-1}u$ together with the estimate $\|t(t-A)^{-1}\| \le 1$, for $t > 0$, to obtain the estimate $t\|Au\| \le \|A^2u\| + 2t^2\|u\|$, for $t > 0$.) Try to improve the 8 to a 4 in (3.36), in case B is a *Hilbert* space.
4. Show that (3.10) implies

$$\|D^\ell u\|_{L^q} \le C_1\|u\|_{L^r} + C_2\|D^{\ell+m}u\|_{L^\rho} \tag{3.37}$$

   when $\rho < q < r$ are related by

$$\frac{1}{q} = \frac{m}{m+\ell}\frac{1}{r} + \frac{\ell}{m+\ell}\frac{1}{\rho}, \tag{3.38}$$

as long as we require furthermore that $q > 2$, in order to satisfy the hypothesis $p/k \leq 1-(m-1)/k$ used for (3.10). In how much greater generality can you establish (3.37)? Note that if Proposition 3.4 is applied to (3.37), one gets

$$\|D^\ell u\|_{L^q} \leq C\|u\|_{L^r}^{m/(m+\ell)} \cdot \|D^{\ell+m}u\|_{L^\rho}^{\ell/(m+\ell)}, \tag{3.39}$$

provided (3.38) holds.

5. Generalize Propositions 3.6 and 3.7, replacing L^2 and H^k by L^p and $H^{k,p}$. Use (3.10) to do this for $p \geq 2$. Can you also treat the case $1 \leq p < 2$?
6. Show that in (3.30) you can use $C_k(\|u\|_{L^\infty})$ with

$$C_k(\lambda) = \sup_{|x|\leq\lambda,\mu\leq k} |F^{(\mu)}(x)|. \tag{3.40}$$

7. Extend the Moser estimates in Propositions 3.7 and 3.9 to estimates in $H^{k,p}$-norms.

4. Trudinger's inequalities

The space $H^{n/2}(\mathbb{R}^n)$ does not quite belong to $L^\infty(\mathbb{R}^n)$, although $H^{n/2}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $p \in [2,\infty)$. In fact, quite a bit more is true; exponential functions of $u \in H^{n/2}(\mathbb{R}^n)$ are locally integrable. The proof of this starts with the following estimate of $\|u\|_{L^p(\mathbb{R}^n)}$ as $p \to \infty$.

Proposition 4.1. *If $u \in H^{n/2}(\mathbb{R}^n)$, then, for $p \in [2,\infty)$,*

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C_n p^{1/2}\|u\|_{H^{n/2}(\mathbb{R}^n)}. \tag{4.1}$$

Proof. We have $u = \Lambda^{-n/2}v$ for $v \in L^2(\mathbb{R}^n)$, where, recall,

$$\big(\Lambda^{-s}v\big)\hat{}(\xi) = \langle\xi\rangle^{-s}\hat{v}(\xi). \tag{4.2}$$

Hence, with $v \in L^2(\mathbb{R}^n)$,

$$u = \mathcal{J}_{n/2} * v, \tag{4.3}$$

where

$$\widehat{\mathcal{J}}_{n/2}(\xi) = \langle\xi\rangle^{-n/2}. \tag{4.4}$$

The behavior of $\mathcal{J}_{n/2}(x)$ follows results of Chap. 3. By Proposition 8.2 of Chap. 3, $\mathcal{J}_{n/2}(x)$ is C^∞ on $\mathbb{R}^n \setminus 0$ and vanishes rapidly as $|x| \to \infty$. By Proposition 9.2 of Chap. 3, we have

$$\mathcal{J}_{n/2}(x) \leq C|x|^{-n/2}, \quad \text{for } |x| \leq 1. \tag{4.5}$$

Consequently, $\mathcal{J}_{n/2}$ just misses being in $L^2(\mathbb{R}^n)$; we have, for $\delta \in (0,1]$,

$$\|\mathcal{J}_{n/2}\|^{2-\delta}_{L^{2-\delta}(\mathbb{R}^n)} \le C + C\int_0^1 r^{n\delta/2-1}\,dr \le \frac{C_n}{\delta}. \tag{4.6}$$

Now the map K_v defined by $K_v f = v * f$, with v given in $L^2(\mathbb{R}^n)$, satisfies

$$K_v : L^2 \to L^\infty, \quad K_v : L^1 \to L^2, \tag{4.7}$$

both maps having operator norm $\|v\|_{L^2}$. By interpolation,

$$\|K_v f\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^q(\mathbb{R}^n)} \cdot \|v\|_{L^2(\mathbb{R}^n)}, \quad \text{for } q \in [1,2], \tag{4.8}$$

where p is defined by $1/q - 1/p = 1/2$. Taking $f = \mathcal{J}_{n/2},\ q = 2-\delta$, we have, for $v \in L^2(\mathbb{R}^n)$,

$$\|\mathcal{J}_{n/2} * v\|_{L^p} \le \Big(\frac{C_n}{\delta}\Big)^{1/(2-\delta)} \|v\|_{L^2}, \quad p = \frac{2(2-\delta)}{\delta}, \tag{4.9}$$

which gives (4.1).

The following result, known as Trudinger's inequality, is a direct consequence of (4.1):

Proposition 4.2. *If $u \in H^{n/2}(\mathbb{R}^n)$, there is a constant $\gamma = \gamma(u) > 0$, of the form*

$$\gamma(u) = \frac{\gamma_n}{\|u\|^2_{H^{n/2}}}, \tag{4.10}$$

such that

$$\int_{\mathbb{R}^n} \Big(e^{\gamma|u(x)|^2} - 1\Big)\,dx < \infty. \tag{4.11}$$

If M is a compact manifold, possibly with boundary, of dimension n, and if $u \in H^{n/2}(M)$, then there exists $\gamma = \gamma(M)/\|u\|^2_{H^{n/2}(M)}$ such that

$$\int_M e^{\gamma|u(x)|^2}\,dV(x) < \infty. \tag{4.12}$$

Proof. We have

$$e^{\gamma|u(x)|^2} - 1 = \gamma|u(x)|^2 + \frac{\gamma^2}{2}|u(x)|^4 + \cdots + \frac{\gamma^m}{m!}|u(x)|^{2m} + \cdots.$$

By (4.1),

$$\frac{\gamma^m}{m!} \int |u(x)|^{2m}\, dV(x) \le C_n^{2m} \frac{\gamma^m}{m!} (2m)^m \|u\|_{H^{n/2}}^{2m}, \tag{4.13}$$

which is bounded by $C' \kappa^m$, for some $\kappa < 1$, if γ has the form (4.10), with $\gamma_n < 1/(2eC_n^2)$, as can be seen via Stirling's formula for $m!$. This proves the proposition.

We note that the same argument involving (4.2)–(4.8) also shows that, for any $p \in [2, \infty)$, there is an $\varepsilon > 0$ such that

$$H^{n/2-\varepsilon}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n). \tag{4.14}$$

Similarly, we have $H^{n/2-\varepsilon}(M) \subset L^p(M)$, when M is a compact manifold, perhaps with boundary, of dimension n. By virtue of Rellich's theorem, we have for such M that the natural inclusion

$$\iota : H^{n/2}(M) \hookrightarrow L^p(M) \text{ is compact, for all } p < \infty. \tag{4.15}$$

Using this, we obtain the following result:

Proposition 4.3. *If M is a compact manifold (with boundary) of dimension n, $\alpha \in \mathbb{R}$, then*

$$u_j \to u \text{ weakly in } H^{n/2}(M) \Longrightarrow e^{\alpha u_j} \to e^{\alpha u} \text{ in } L^1(M)\text{-norm.} \tag{4.16}$$

Proof. We have

$$\left| e^{\alpha u_j} - e^{\alpha u} \right| \le \sum_{m \ge 1} \frac{|\alpha|^m}{m!} \Big| |u_j(x)|^m - |u(x)|^m \Big|.$$

If $\|u_j\|_{H^{n/2}(M)} \le A$, we obtain

$$\begin{aligned} \|e^{\alpha u_j} - e^{\alpha u}\|_{L^1} \le \sum_{m \le k} \frac{|\alpha|^m}{m!} \|u_j - u\|_{L^m} \cdot m \Big[\|u_j\|_{L^m}^{m-1} + \|u\|_{L^m}^{m-1} \Big] \\ + C \sum_{m > k} \frac{m^{m/2}}{m!} |AC_n \alpha|^m, \end{aligned} \tag{4.17}$$

where we use

$$\Big| |u_j|^m - |u|^m \Big| \le m|u_j - u| \big(|u_j|^{m-1} + |u|^{m-1} \big)$$

to estimate the sum over $m \le k$, and we use (4.1) to estimate the sum over $m > k$. By (4.15), for any k, the first sum on the right side of (4.17) goes to 0 as $j \to \infty$. Meanwhile the second sum vanishes as $k \to \infty$, so (4.16) follows.

Exercises

1. Partially generalizing (4.10), let $p \in (1, \infty)$, and let $u \in H^{k,p}(\mathbb{R}^n)$, with $kp = n,\ k \in \mathbb{Z}^+$. Show that there exists $\gamma = \gamma_p(u)$ such that

$$\int_{|x| \le R} e^{\gamma |u(x)|^{p/(p-1)}}\, dx \le C_{pR}. \tag{4.18}$$

   For a more complete generalization, see Exercise 5 of § 6.
   Note: Finding the best constant γ in (4.18) is subtle and has some important uses; see [Mos2], [Au], particularly for the case $k = 1,\ p = n$.

5. Singular integral operators on L^p

One way the Fourier transform makes analysis on $L^2(\mathbb{R}^n)$ easier than analysis on other L^p-spaces is by the definitive result the Plancherel theorem gives as a condition that a convolution operator $k * u = P(D)u$ be L^2-bounded, namely that $\hat{k}(\xi) = P(\xi)$ be a bounded function of ξ. A replacement for this that advances our ability to pursue analysis on L^p is the next result, established by S. Mikhlin, following related work for $L^p(\mathbb{T}^n)$ by J. Marcinkiewicz.

Theorem 5.1. *Suppose $P(\xi)$ satisfies*

$$|D^\alpha P(\xi)| \le C_\alpha \langle \xi \rangle^{-|\alpha|}, \tag{5.1}$$

for $|\alpha| \le n + 1$. Then

$$P(D) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \ \textit{for } 1 < p < \infty. \tag{5.2}$$

Stronger results have been proved; one needs (5.1) only for $|\alpha| \le [n/2]+1$, and one can use certain L^2-estimates on the derivatives of $P(\xi)$. These sharper results can be found in [H1] and [S1]. Note that the characterization of $P(\xi) \in S^0_1(\mathbb{R}^n)$ is that (5.1) hold for all α.

The theorem stated above is a special case of a result that applies to pseudodifferential operators with symbols in $S^0_{1,\delta}(\mathbb{R}^n)$. As shown in § 2 of Chap. 7, if $p(x, \xi)$ satisfies the estimates

$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \le C_{\alpha\beta} \langle \xi \rangle^{-|\alpha|+|\beta|}, \tag{5.3}$$

for

$$|\beta| \le 1, \quad |\alpha| \le n + 1 + |\beta|, \tag{5.4}$$

then the Schwartz kernel $K(x, y)$ of $P = p(x, D)$ satisfies the estimates

$$|K(x, y)| \le C|x - y|^{-n} \tag{5.5}$$

and

$$|\nabla_{x,y} K(x, y)| \le C|x - y|^{-n-1}. \tag{5.6}$$

Furthermore, at least when $\delta < 1$, we have an L^2-bound:

$$\|Pu\|_{L^2} \le K\|u\|_{L^2}, \tag{5.7}$$

and smoothings of such an operator have smooth Schwartz kernels satisfying (5.5)–(5.7) for fixed C, K. (Results in § 9 of this chapter will contain another proof of this L^2-estimate. Note that when $p(x,\xi) = p(\xi)$ the estimate (5.7) follows from the Plancherel theorem.) Our main goal here is to give a proof of the following fundamental result of A. P. Calderon and A. Zygmund:

Theorem 5.2. *Suppose $P : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a weak limit of operators with smooth Schwartz kernels satisfying (5.5)–(5.7) uniformly. Then*

$$P : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 1 < p < \infty. \tag{5.8}$$

In particular, this holds when $P \in OPS^0_{1,\delta}(\mathbb{R}^n),\ \delta \in [0, 1)$.

The hypotheses do not imply boundedness on $L^1(\mathbb{R}^n)$ or on $L^\infty(\mathbb{R}^n)$. They will imply that P is of weak type $(1, 1)$. By definition, an operator P is of weak type (q, q) provided that, for any $\lambda > 0$,

$$\text{meas}\,\{x : |Pu(x)| > \lambda\} \le C\lambda^{-q}\|u\|^q_{L^q}. \tag{5.9}$$

Any bounded operator on L^q is a fortiori of weak type (q, q), in view of the simple inequality

$$\text{meas}\,\{x : |u(x)| > \lambda\} \le \lambda^{-1}\|u\|_{L^1}. \tag{5.10}$$

A key ingredient in proving Theorem 5.2 is the following result:

Proposition 5.3. *Under the hypotheses of Theorem 5.2, P is of weak type* $(1, 1)$.

Once this is established, Theorem 5.2 will then follow from the next result, known as the *Marcinkiewicz interpolation theorem.*

Proposition 5.4. *If $r < p < q$ and if T is both of weak type (r, r) and of weak type (q, q), then $T : L^p \to L^p$.*

Proof. Write $u = u_1 + u_2$, with $u_1(x) = u(x)$ for $|u(x)| > \lambda$ and $u_2(x) = u(x)$ for $|u(x)| \le \lambda$. With the notation

$$\mu_f(\lambda) = \text{ meas}\,\{x : |f(x)| \ge \lambda\}, \tag{5.11}$$

we have

$$\begin{aligned}\mu_{Tu}(2\lambda) &\leq \mu_{Tu_1}(\lambda) + \mu_{Tu_2}(\lambda) \\ &\leq C_1\lambda^{-r}\|u_1\|_{L^r}^r + C_2\lambda^{-q}\|u_2\|_{L^q}^q.\end{aligned} \tag{5.12}$$

Also, there is the formula

$$\int |f(x)|^p\, dx = p\int_0^\infty \mu_f(\lambda)\lambda^{p-1}\, d\lambda.$$

Hence

$$\begin{aligned}\int |Tu(x)|^p\, dx &= p\int_0^\infty \mu_{Tu}(\lambda)\lambda^{p-1}\, d\lambda \\ &\leq C_1 p\int_0^\infty \lambda^{p-1-r}\Big(\int\limits_{|u|>\lambda} |u(x)|^r\, dx\Big)\, d\lambda \\ &\quad + C_2 p\int_0^\infty \lambda^{p-1-q}\Big(\int\limits_{|u|\leq\lambda} |u(x)|^q\, dx\Big)\, d\lambda.\end{aligned} \tag{5.13}$$

Now

$$\int_0^\infty \lambda^{p-1-r}\Big(\int\limits_{|u|>\lambda} |u(x)|^r\, dx\Big)\, d\lambda = \frac{1}{p-r}\int |u(x)|^p\, dx \tag{5.14}$$

and, similarly,

$$\int_0^\infty \lambda^{p-1-q}\Big(\int\limits_{|u|\leq\lambda} |u(x)|^q\, dx\Big)\, d\lambda = \frac{1}{q-p}\int |u(x)|^p\, dx. \tag{5.15}$$

Combining these gives the desired estimate on $\|Tu\|_{L^p}^p$.

We will apply Proposition 5.4 in conjunction with the following covering lemma of Calderon and Zygmund:

Lemma 5.5. *Let $u \in L^1(\mathbb{R}^n)$ and $\lambda > 0$ be given. Then there exist $v, w_k \in L^1(\mathbb{R}^n)$ and disjoint cubes Q_k, $1 \leq k < \infty$, with centers x_k, such that*

$$u = v + \sum_k w_k, \quad \|v\|_{L^1} + \sum_k \|w_k\|_{L^1} \leq 3\|u\|_{L^1}, \tag{5.16}$$

$$|v(x)| \leq 2^n\lambda, \tag{5.17}$$

$$\int\limits_{Q_k} w_k(x)\, dx = 0 \ \text{ and supp } w_k \subset Q_k, \tag{5.18}$$

$$\sum_k \mathit{meas}(Q_k) \le \lambda^{-1} \|u\|_{L^1}. \tag{5.19}$$

Proof. Tile $\mathbb{R}^n$ with cubes of volume greater than $\lambda^{-1}\|u\|_{L^1}$. The mean value of $|u(x)|$ over each such cube is $< \lambda$. Divide each of these cubes into 2^n equal cubes, and let $I_{11}, I_{12}, I_{13}, \dots$ be those so obtained over which the mean value of $|u(x)|$ is $\ge \lambda$. Note that

$$\lambda \text{ meas}(I_{1k}) \le \int_{I_{1k}} |u(x)|\, dx \le 2^n \lambda \text{ meas}(I_{1k}). \tag{5.20}$$

Now set

$$v(x) = \frac{1}{\text{meas}(I_{1k})} \int_{I_{1k}} u(y)\, dy, \quad \text{for } x \in I_{1k}, \tag{5.21}$$

and

$$\begin{aligned} w_{1k}(x) = u(x) - v(x), &\quad \text{for } x \in I_{1k}, \\ 0, &\quad \text{for } x \notin I_{1k}. \end{aligned} \tag{5.22}$$

Next take all the cubes that are not among the I_{1k}, subdivide each into 2^n equal parts, select those new cubes $I_{21}, I_{22}, \dots,$ over which the mean value of $|u(x)|$ is $\ge \lambda$, and extend the definitions (5.21)–(5.22) to these cubes, in the natural fashion. Continue in this way, obtaining disjoint cubes I_{jk} and functions w_{jk}. Then reorder these cubes and functions as $Q_1, Q_2, \dots,$ and $w_1, w_2, \dots$. Complete the definition of v by setting $v(x) = u(x)$, for $x \notin \cup Q_k$. Then we have the first part of (5.16). Since

$$\int_{Q_k} \bigl(|v(x)| + |w_k(x)|\bigr)\, dx \le 3 \int_{Q_k} |u(x)|\, dx, \tag{5.23}$$

and since the cubes are disjoint, w_k is supported in Q_k, and $v = u$ on $\mathbb{R}^n \setminus \cup Q_k$, we obtain the rest of (5.16).

Next, (5.17) follows from (5.20) if $x \in \cup Q_k$. But if $x \notin \cup Q_k$, there are arbitrarily small cubes containing x over which the mean value of $|u(x)|$ is $< \lambda$, so (5.17) holds almost everywhere on $\mathbb{R}^n \setminus \cup Q_k$ as well. The assertion (5.18) is obvious from the construction, and (5.19) follows by summing (5.20). The lemma is proved.

One thinks of v as the "good" piece and $w = \sum w_k$ as the "bad" piece. What is "good" about v is that $\|v\|_{L^2}^2 \le 2^n \lambda \|u\|_{L^1}$, so

$$\|Pv\|_{L^2}^2 \le K^2 \|v\|_{L^2}^2 \le 4^n K^2 \lambda \|u\|_{L^1}. \tag{5.24}$$

Hence

$$\left(\frac{\lambda}{2}\right)^2 \operatorname{meas}\left\{x : |Pv(x)| > \frac{\lambda}{2}\right\} \le C\lambda\|u\|_{L^1}. \tag{5.25}$$

To treat the action of P on the "bad" term w, we make use of the following essentially elementary estimate on the Schwartz kernel K. The proof is an exercise.

Lemma 5.6. *There is a $C_0 < \infty$ such that, for any $t > 0$, if $|y| \le t, x_0 \in \mathbb{R}^n$,*

$$\int\limits_{|x-x_0|\ge 2t} \big|K(x, x_0 + y) - K(x, x_0)\big|\, dx \le C_0. \tag{5.26}$$

To estimate Pw, we have

$$\begin{aligned} Pw_k(x) &= \int K(x,y) w_k(y)\, dy \\ &= \int\limits_{Q_k} \big[K(x,y) - K(x,x_k)\big] w_k(y)\, dy. \end{aligned} \tag{5.27}$$

Before we make further use of this, a little notation: Let Q_k^* be the cube concentric with Q_k, enlarged by a linear factor of $2n^{1/2}$, so meas $Q_k^* = (4n)^{n/2}$ meas Q_k. For some $t_k > 0$, we can arrange that

$$\begin{aligned} Q_k &\subset \{x : |x - x_k| \le t_k\}, \\ Y_k = \mathbb{R}^n \setminus Q_k^* &\subset \{x : |x - x_k| > 2t_k\}. \end{aligned} \tag{5.28}$$

Furthermore, set $\mathcal{O} = \cup Q_k^*$, and note that

$$\operatorname{meas} \mathcal{O} \le L\lambda^{-1}\|u\|_{L^1}, \tag{5.29}$$

with $L = (4n)^{n/2}$. Now, from (5.27), we have

$$\begin{aligned} &\int\limits_{Y_k} |Pw_k(x)|\, dx \\ &\le \int\limits_{|y|\le t_k}\ \int\limits_{|x|\ge 2t_k} \big|K(x + x_k, x_k + y) - K(x + x_k, x_k)\big| \\ &\qquad\qquad \cdot |w_k(y + x_k)|\, dx\, dy \\ &\le C_0\|w_k\|_{L^1}, \end{aligned} \tag{5.30}$$

the last estimate using Lemma 5.6. Thus

$$\int\limits_{\mathbb{R}^n\setminus\mathcal{O}} |Pw(x)|\, dx \le 3C_0\|u\|_{L^1}. \tag{5.31}$$

Together with (5.29), this gives

$$\text{meas}\Big\{x : |Pw(x)| > \frac{\lambda}{2}\Big\} \le \frac{C_1}{\lambda}\|u\|_{L^1}, \tag{5.32}$$

and this estimate together with (5.25) yields the desired weak (1,1)-estimate:

$$\text{meas}\{x : |Pu(x)| > \lambda\} \le \frac{C_2}{\lambda}\|u\|_{L^1}. \tag{5.33}$$

This proves Proposition 5.3.

To complete the proof of Theorem 5.2, we apply Marcinkiewicz interpolation to obtain (5.8) for $p \in (1,2]$. Note that the Schwartz kernel of P^* also satisfies the hypotheses of Theorem 5.2, so we have $P^* : L^p \to L^p$, for $1 < p \le 2$. Thus the result (5.8) for $p \in [2,\infty)$ follows by duality.

We remark that if (5.6) is weakened to $|\nabla_y K(x,y)| \le C|x-y|^{-n-1}$, while the hypotheses (5.5) and (5.7) are retained, then Lemma 5.6 still holds, and hence so does Proposition 5.3. Thus, we still have $P : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 < p \le 2$, but the duality argument gives only $P^* : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $2 \le p < \infty$.

We next describe an important generalization to operators acting on Hilbert space-valued functions. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and suppose

$$P : L^2(\mathbb{R}^n, \mathcal{H}_1) \longrightarrow L^2(\mathbb{R}^n, \mathcal{H}_2). \tag{5.34}$$

Then P has an $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$-operator-valued Schwartz kernel K. Let us impose on K the hypotheses of Theorem 5.2, where now $|K(x,y)|$ stands for the $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$-norm of $K(x,y)$. Then all the steps in the proof of Theorem 5.2 extend to this case. Rather than formally state this general result, we will concentrate on an important special case.

Proposition 5.7. *Let $P(\xi) \in C^\infty(\mathbb{R}^n, \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2))$ satisfy*

$$\|D_\xi^\alpha P(\xi)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \le C_\alpha \langle\xi\rangle^{-|\alpha|}, \tag{5.35}$$

for all $\alpha \ge 0$. Then

$$P(D) : L^p(\mathbb{R}^n, \mathcal{H}_1) \longrightarrow L^p(\mathbb{R}^n, \mathcal{H}_2), \quad \text{for } 1 < p < \infty. \tag{5.36}$$

This leads to an important circle of results known as *Littlewood–Paley theory*. To obtain this, start with a partition of unity

$$1 = \sum_{j=0}^{\infty} \varphi_j(\xi)^2, \tag{5.37}$$

where $\varphi_j \in C^\infty$, $\varphi_0(\xi)$ is supported on $|\xi| \le 1$, $\varphi_1(\xi)$ is supported on $1/2 \le |\xi| \le 2$, and $\varphi_j(\xi) = \varphi_1(2^{1-j}\xi)$ for $j \ge 2$. We take $\mathcal{H}_1 = \mathbb{C}$, $\mathcal{H}_2 = \ell^2$, and look at

$$\Phi : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n, \ell^2) \tag{5.38}$$

given by

$$\Phi(f) = (\varphi_0(D)f, \varphi_1(D)f, \varphi_2(D)f, \dots). \tag{5.39}$$

This is clearly an isometry, though of course it is not surjective. The adjoint

$$\Phi^* : L^2(\mathbb{R}^n, \ell^2) \longrightarrow L^2(\mathbb{R}^n), \tag{5.40}$$

given by

$$\Phi^*(g_0, g_1, g_2, \dots) = \sum \varphi_j(D) g_j, \tag{5.41}$$

satisfies

$$\Phi^*\Phi = I \tag{5.42}$$

on $L^2(\mathbb{R}^n)$. Note that $\Phi = \Phi(D)$, where

$$\Phi(\xi) = (\varphi_0(\xi), \varphi_1(\xi), \varphi_2(\xi), \dots). \tag{5.43}$$

It is easy to see that the hypothesis (5.35) is satisfied by both $\Phi(\xi)$ and $\Phi^*(\xi)$. Hence, for $1 < p < \infty$,

$$\begin{aligned} \Phi &: L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n, \ell^2), \\ \Phi^* &: L^p(\mathbb{R}^n, \ell^2) \longrightarrow L^p(\mathbb{R}^n). \end{aligned} \tag{5.44}$$

In particular, Φ maps $L^p(\mathbb{R}^n)$ isomorphically onto a closed subspace of $L^p(\mathbb{R}^n, \ell^2)$, and we have compatibility of norms:

$$\|u\|_{L^p} \approx \|\Phi u\|_{L^p(\mathbb{R}^n, \ell^2)}. \tag{5.45}$$

In other words,

$$C_p' \|u\|_{L^p} \le \Big\| \Big\{ \sum_{j=0}^{\infty} |\varphi_j(D)u|^2 \Big\}^{1/2} \Big\|_{L^p} \le C_p \|u\|_{L^p}, \tag{5.46}$$

for $1 < p < \infty$.

Exercises

1. Estimate the family of symbols $a_y(\xi) = \langle\xi\rangle^{iy}$, $y \in \mathbb{R}$. Show that if $\Lambda^{iy} = a_y(D)$, then

(5.47)
$$\|\Lambda^{iy} u\|_{L^p(\mathbb{R}^n)} \le C_p \langle y\rangle^{n+1} \|u\|_{L^p(\mathbb{R}^n)}.$$

This estimate will be useful for the development of the Sobolev spaces $H^{s,p}$ in the next section.

2. Let $\tilde{\psi}_1(\xi)$ be supported on $1/4 \le |\xi| \le 4$, $\tilde{\psi}_1(\xi) = 1$ for $1/2 \le |\xi| \le 2$, and $\tilde{\psi}_j(\xi) = \tilde{\psi}_1(2^{1-j}\xi)$ for $j \ge 2$. Let $s \in \mathbb{R}$. Show that

$$A(D), B(D) : L^p(\mathbb{R}^n, \ell^2) \longrightarrow L^p(\mathbb{R}^n, \ell^2), \quad 1 < p < \infty,$$

for

$$\begin{aligned} A_{jk}(\xi) &= 2^{ks} \langle\xi\rangle^{-s} \tilde{\psi}_j(\xi) \delta_{jk}, \\ B_{jk}(\xi) &= 2^{-ks} \langle\xi\rangle^{s} \tilde{\psi}_j(\xi) \delta_{jk}, \end{aligned}$$

by applying Proposition 5.7.

3. Give a proof that

(5.48)
$$\int |f(x)|^p \, dx = p \int_0^\infty \mu_f(\lambda)\, \lambda^{p-1} \, d\lambda,$$

used in (5.13). Also, demonstrate (5.14) and (5.15). (*Hint*: After doing (5.48), get an analogous identity for the integral of $|f(x)|^p$ over the set $\{x : |f(x)| > \lambda\}$, resp., $\le \lambda$.)

4. Give a detailed proof of Lemma 5.6.

5. Let $A \in OPS^1_{1,0}(\mathbb{R}^n)$, and suppose $A(x,\xi) = 0$ for $x_n = 0$. Define $Tf = Af\big|_{\mathbb{R}^n_-}$, where $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n : \pm x_n \ge 0\}$. Show that, for $1 \le p \le \infty$,

(5.49)
$$f \in L^p(\mathbb{R}^n), \ \operatorname{supp} f \subset \mathbb{R}^n_+ \Longrightarrow Tf \in L^p(\mathbb{R}^n_-).$$

(*Hint*: Apply Proposition 5.1 of Appendix A. Compare with Exercise 3 in § 5 of Appendix A.)

6. The spaces $H^{s,p}$

Here we define and study $H^{s,p}$ for any $s \in \mathbb{R}$, $p \in (1,\infty)$. In analogy with the characterization of $H^s(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$ given in § 1 of Chap. 4, we set

(6.1)
$$H^{s,p}(\mathbb{R}^n) = \Lambda^{-s} L^p(\mathbb{R}^n).$$

Given the results of § 5, we can establish the following.

Proposition 6.1. *When $s = k$ is a positive integer, $p \in (1,\infty)$, the spaces $H^{k,p}(\mathbb{R}^n)$ of § 1 coincide with (6.1).*

Proof. For $|\alpha| \le k,\ \xi^\alpha \langle\xi\rangle^{-k}$ belongs to $S^0_1(\mathbb{R}^n)$. Thus, by Theorem 5.1, $D^\alpha \Lambda^{-k}$ maps $L^p(\mathbb{R}^n)$ to itself. Thus any $u \in \Lambda^{-k} L^p(\mathbb{R}^n)$ satisfies the definition of $H^{k,p}(\mathbb{R}^n)$ given in § 1. For the converse, note that one can write

$$\langle\xi\rangle^k = \sum_{|\alpha|\le k} q_\alpha(\xi)\xi^\alpha, \tag{6.2}$$

with coefficients $q_\alpha \in S^0_1(\mathbb{R}^n)$. Thus if $D^\alpha u \in L^p(\mathbb{R}^n)$ for all $|\alpha| \le k$, it follows that $\Lambda^k u \in L^p(\mathbb{R}^n)$.

We next prove an interpolation theorem generalizing the identity

$$\big[L^2(\mathbb{R}^n), H^s(\mathbb{R}^n)\big]_\theta = H^{\theta s}(\mathbb{R}^n), \quad \text{for } \theta \in [0,1],$$

proven in § 2 of Chap. 4.

Proposition 6.2. *For $s \in \mathbb{R},\ \theta \in (0,1)$, and $p \in (1,\infty)$,*

$$\big[L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)\big]_\theta = H^{\theta s,p}(\mathbb{R}^n). \tag{6.3}$$

Proof. The proof is parallel to that of Proposition 2.2 of Chap. 4, except that we use the estimate (5.47) of the last section in place of the obvious identity $\|A^{iy}\| = 1$ for a unitary operator A^{iy} on a Hilbert space. Thus, if $v \in H^{\theta s,p}(\mathbb{R}^n)$, let

$$u(z) = e^{z^2} \Lambda^{(\theta - z)s} v. \tag{6.4}$$

Then $u(\theta) = e^{\theta^2} v,\ u(iy) = e^{-y^2} \Lambda^{-iys}\big(\Lambda^{s\theta} v\big)$ is bounded in $L^p(\mathbb{R}^n)$, by (5.47), and also $u(1+iy) = e^{-(y-i)^2} \Lambda^{-s} \Lambda^{-iys}\big(\Lambda^{s\theta} v\big)$ is bounded in the space $H^{s,p}(\mathbb{R}^n)$. Therefore, such a function v belongs to the left side of (6.3). The reverse containment is similarly established as in the proof of Proposition 2.2 of Chap. 4.

This sort of argument yields more generally that, for $\sigma, s \in \mathbb{R},\ \theta \in (0,1)$, and $p \in (1,\infty)$,

$$[H^{\sigma,p}(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)]_\theta = H^{\theta s + (1-\theta)\sigma, p}(\mathbb{R}^n). \tag{6.5}$$

With Proposition 6.2 established, we can define and analyze spaces $H^{s,p}$ on compact manifolds in the same way as we did for $p = 2$ in Chap. 4. If M is a compact manifold without boundary, one defines $H^{s,p}(M)$ in analogy with $H^s(M)$, via coordinate charts, and proves

$$[H^{\sigma,p}(M), H^{s,p}(M)]_\theta = H^{\theta s + (1-\theta)\sigma, p}(M), \tag{6.6}$$

for $p \in (1,\infty),\ \theta \in (0,1)$. If Ω is a compact subdomain of M with smooth boundary, we define $H^{k,p}(\Omega)$ as in § 1, and recall the extension operator $E : H^{k,p}(\Omega) \to H^{k,p}(M)$. If we define $H^{s,p}(\Omega)$ for $s > 0$ by

$$H^{s,p}(\Omega) = [L^p(\Omega), H^{k,p}(\Omega)]_\theta, \quad \theta \in (0,1), \ \ s = k\theta, \tag{6.7}$$

it follows that $E : H^{s,p}(\Omega) \to H^{s,p}(M)$ and hence

$$H^{s,p}(\Omega) \approx H^{s,p}(M)/\{u : u = 0 \text{ on } \Omega\}. \tag{6.8}$$

Also, of course, $H^{s,p}(\Omega)$ agrees with the characterization of § 1 when $s = k$ is a positive integer. Generalizing the theorem of Rellich, Proposition 4.4 of Chap. 4, one has, for $s \geq 0, \ 1 < p < \infty$,

$$\iota : H^{s+\sigma,p}(\Omega) \hookrightarrow H^{s,p}(\Omega) \text{ is compact for } \sigma > 0. \tag{6.9}$$

By the arguments used in Chap. 4, we easily reduce this to showing that, for $\sigma > 0, 1 < p < \infty$,

$$\Lambda^{-\sigma} : L^p(\mathbb{T}^n) \longrightarrow L^p(\mathbb{T}^n) \text{ is compact.} \tag{6.10}$$

Indeed, the operator (6.10) is of the form $\Lambda^{-\sigma} u = k_\sigma * u$, with $k_\sigma \in L^1(\mathbb{T}^n)$ for any $\sigma > 0$. Thus k_σ is an L^1-norm limit of $k_{\sigma,j} \in C^\infty(\mathbb{T}^n)$, so $\Lambda^{-\sigma}$ is an operator norm limit of convolution maps $L^p(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$, which are clearly compact on $L^p(\mathbb{T}^n)$.

We now extend some of the Sobolev imbedding theorems of § 2. Once they are obtained on $\mathbb{R}^n$, they easily yield similar results for functions on compact manifolds, perhaps with boundary.

Proposition 6.3. *If $s > n/p$, then $H^{s,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.*

Proof. $\Lambda^{-s} u = \mathcal{J}_s * u$, where $\widehat{\mathcal{J}}_s(\xi) = \langle \xi \rangle^{-s}$. It suffices to show that

$$\mathcal{J}_s \in L^{p'}(\mathbb{R}^n), \quad \text{for } s > \frac{n}{p}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{6.11}$$

Indeed, estimates established in § 8 of Chap. 3 imply that $\mathcal{J}_s(x)$ is smooth on $\mathbb{R}^n \setminus 0$, rapidly decreasing as $|x| \to \infty$, and

$$|\mathcal{J}_s(x)| \leq C|x|^{-n+s}, \quad |x| \leq 1, \ \ s < n, \tag{6.12}$$

which is sufficient. Compare estimates for $s = n/2$ in (4.4)–(4.9).

Next we generalize (2.9).

Proposition 6.4. *For $sp < n, \ p \in (1,\infty)$, we have*

$$H^{s,p}(\mathbb{R}^n) \subset L^{np/(n-sp)}(\mathbb{R}^n). \tag{6.13}$$

Proof. Suppose $s = k + \sigma, \ k \in \mathbb{Z}^+, \ \sigma \in [0,1)$. Then $u \in H^{s,p} \Rightarrow \Lambda^\sigma u \in H^{k,p}$, and by (2.9) this gives $\Lambda^\sigma u \in L^q(\mathbb{R}^n)$, with $q = np/(n-kp)$. Note that

$q \in (1,\infty)$ and $np/(n-sp) = nq/(n-\sigma q)$, so also $\sigma q < n$. Hence it suffices to show that

$$\Lambda^{-\sigma} : L^q(\mathbb{R}^n) \longrightarrow L^{nq/(n-\sigma q)}(\mathbb{R}^n), \tag{6.14}$$

when $\sigma \in (0,1)$, $q \in (1,\infty)$, and $\sigma q < n$. We divide the analysis into cases.

Case I: $1 < q < n$. In this case, we have, by (2.2),

$$H^{1,q}(\mathbb{R}^n) \subset L^{nq/(n-q)}(\mathbb{R}^n). \tag{6.15}$$

Fixing $v \in L^q(\mathbb{R}^n)$, consider $\Lambda^{-z}v$ for $z \in \overline{\Omega} = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$. Note that Proposition 5.7 implies

$$\|\Lambda^{iy}v\|_{L^q} \le Ae^{B|y|}\|v\|_{L^q}, \tag{6.16}$$

for $y \in \mathbb{R}$. Making use also of (6.15), we have

$$\|\Lambda^{-(1+iy)}v\|_{L^{nq/(n-q)}} \le Ae^{B|y|}\|v\|_{L^q}. \tag{6.17}$$

From here a complex interpolation argument gives (6.14) in this case.

Case II: $2 \le n \le q < \infty$. In this case, set $r = nq/(n-\sigma q)$. Note that

$$\frac{1}{r} = \frac{1}{q} - \frac{\sigma}{n} \quad \text{and} \quad \frac{1}{r'} = \frac{1}{q'} + \frac{\sigma}{n}, \tag{6.18}$$

where r' is the dual exponent to r. We have $r > q \ge n \ge 2$, so $r' < 2 \le n$, and Case I gives

$$\Lambda^{-\sigma} : L^{r'}(\mathbb{R}^n) \longrightarrow L^{q'}(\mathbb{R}^n). \tag{6.19}$$

Then (6.14) follows by duality.

Case III: $n = 1$. Here one needs a different approach. Since this case is not so crucial for PDE, we omit it. Various proofs that include this case can be found in [S1], [S3], and [BL].

The following result is an immediate consequence of the definition (6.1), the pseudodifferential operator calculus, and the L^p-boundedness result of Theorem 5.2.

Proposition 6.5. *If $P \in OPS^m_{1,\delta}(\mathbb{R}^n)$, $0 \le \delta < 1$, and $1 < p < \infty$, then*

$$P : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n). \tag{6.20}$$

In view of the construction of parametrices for elliptic operators, we deduce various $H^{s,p}$-regularity results for solutions to linear elliptic equations. A sequence of exercises on generalized div-curl lemmas given below will make use of this.

Exercises

1. Let $\varphi_j(\xi)^2 = \psi_j(\xi)$ be the partition of unity (5.37). Using the Littlewood–Paley estimates, show that, for $p \in (1,\infty),\ s \in \mathbb{R}$,

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} \approx \Big\| \Big\{ \sum_{k=0}^{\infty} 4^{ks} |\varphi_j(D)u|^2 \Big\}^{1/2} \Big\|_{L^p(\mathbb{R}^n)}. \tag{6.21}$$

(*Hint*: From (5.37), we have the left side of (6.21)

$$\approx \Big\| \Big\{ \sum_{k=0}^{\infty} \big|\Lambda^s \varphi_j(D)u\big|^2 \Big\}^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^n)}. \tag{6.22}$$

Now apply Exercise 2 of § 5.)

Exercises 2–4 lead up to a demonstration that if

$$\Psi_k(\xi) = \sum_{\ell \le k} \varphi_\ell(\xi)^2, \tag{6.23}$$

then, for $s > 0,\ p \in (1,\infty)$,

$$\Big\| \sum_{k=0}^{\infty} \Psi_k(D) f_k \Big\|_{H^{s,p}} \le C_{sp} \Big\| \Big\{ \sum_{k=0}^{\infty} 4^{ks} |f_k|^2 \Big\}^{1/2} \Big\|_{L^p}. \tag{6.24}$$

2. Show that the left side of (6.24) is

$$\approx \Big\| \Big\{ \sum_{\ell=0}^{\infty} \Big| \psi_\ell(D) \sum_{k=\ell}^{\infty} u_k \Big|^2 \Big\}^{1/2} \Big\|_{L^p} \approx \Big\| \Big\{ \sum_{\ell=0}^{\infty} 4^{\ell s} \Big| \psi_\ell(D) \sum_{k=\ell}^{\infty} f_k \Big|^2 \Big\}^{1/2} \Big\|_{L^p},$$

where $f_k = \Lambda^{-s} u_k$. (*Hint*: Use arguments similar to those needed for Exercise 1.)

3. Taking $w_k = 2^{ks} f_k$, argue that (6.24) follows given continuity of

$$\Gamma(D) : L^p(\mathbb{R}^n, \ell^2) \longrightarrow L^p(\mathbb{R}^n, \ell^2), \tag{6.25}$$

where

$$\begin{aligned} \Gamma_{k\ell}(\xi) &= \psi_k(\xi) 2^{-(\ell-k)s}, && \text{for } \ell \ge k, \\ &\qquad 0, && \text{for } \ell < k. \end{aligned} \tag{6.26}$$

4. Demonstrate the continuity (6.25), for $p \in (1,\infty),\ s > 0$.
(*Hint*: To apply Proposition 5.7, you need

$$\|D_\xi^\alpha \Gamma(\xi)\|_{\mathcal{L}(\ell^2)} \le C_s \langle \xi \rangle^{-|\alpha|}, \quad s > 0.$$

Obtain this by establishing

$$\sum_k |D_\xi^\alpha \Gamma_{k\ell}(\xi)| \le C\langle\xi\rangle^{-|\alpha|}, \quad s \ge 0,$$

and

$$\sum_\ell |D_\xi^\alpha \Gamma_{k\ell}(\xi)| \le C_s\langle\xi\rangle^{-|\alpha|}, \quad s > 0.)$$

5. If $u \in H^{n/p,p}(\mathbb{R}^n)$, $p \in (1,\infty)$, show that, for $q \in [p,\infty)$,

$$\|u\|_{L^q(\mathbb{R}^n)} \le C_n q^{(p-1)/p} \|u\|_{H^{n/p,p}(\mathbb{R}^n)}.$$

Deduce that, for some constant $\gamma = \gamma(u) > 0$,

$$\int_{\mathbb{R}^n} \Big(e^{\gamma|u(x)|^{p/(p-1)}} - 1\Big)\, dx < \infty, \tag{6.27}$$

thus extending Trudinger's estimate (4.10). See [Str].

The purpose of the next exercise is to extend the Gagliardo–Nirenberg estimates (3.10) to nonintegral cases, namely

$$\|u\|_{H^{\lambda,s/p}} \le C_1\|u\|_{L^{s/(p-\lambda)}} + C_2\|u\|_{H^{\lambda+\mu,s/(p+\mu)}}, \tag{6.28}$$

given real p, s, λ, and μ satisfying

$$1 < p < \infty,\ 0 < \mu < s - p,\ \text{ and }\ \lambda \in (0,p). \tag{6.29}$$

6. Establish the interpolation result

$$\big[L^{s/(p-\lambda)}(\mathbb{R}^n), H^{\lambda+\mu,s/(p+\mu)}(\mathbb{R}^n)\big]_\theta \subset H^{\lambda,s/p}(\mathbb{R}^n), \quad \theta = \frac{\lambda}{\lambda+\mu}, \tag{6.30}$$

under the hypotheses (6.29). Show that this implies (6.28).
(*Hint*: If $f = u(\theta)$ belongs to the left side of (6.30), with $u(z)$ holomorphic, $u(iy)$ and $u(1+iy)$ appropriately bounded, consider $v(z) = \Lambda^{-(\lambda+\mu)z}u(z)$. Use the interpolation result

$$\big[L^{s/(p-\lambda)}, L^{s/(p+\mu)}\big]_\theta = L^{s/p}, \quad \theta = \frac{\lambda}{\lambda+\mu}.)$$

Can you treat the $p = \lambda$ case, where $L^{s/(p-\lambda)} = L^\infty$?

7. Extend (6.30) to Sobolev inclusions for $[H^{s,p}, H^{\sigma,q}]_\theta$.

Exercises on generalized div-curl lemmas

Let M be a compact, oriented Riemannian manifold, and assume that, for $j = 1,\dots,k$, $\nu \in \mathbb{Z}^+$, $\sigma_{j\nu}$ are ℓ_j-forms on M, such that

$$\sigma_{j\nu} \longrightarrow \sigma_j \ \text{ weakly in } \ L^{p_j}(M), \quad \text{as } \nu \to \infty, \tag{6.31}$$

and

$$\{d\sigma_{j\nu} : \nu \ge 0\} \ \text{ compact in } \ H^{-1,p_j}(M). \tag{6.32}$$

Assume that

$$p_j \in (1,\infty), \quad \frac{1}{p_1} + \cdots + \frac{1}{p_k} \le 1. \tag{6.33}$$

The goal is to deduce that

$$\sigma_{1\nu} \wedge \cdots \wedge \sigma_{k\nu} \longrightarrow \sigma_1 \wedge \cdots \wedge \sigma_k \quad \text{in } \mathcal{D}'(M), \tag{6.34}$$

as $\nu \to \infty$. An exercise set in § 8 of Chap. 5 deals with the case $k = 2,\ p_1 = p_2 = 2$, which includes the div-curl lemma of F. Murat [Mur]. As in that exercise set, we follow [RRT].

1. Show that you can write $\sigma_{j\nu} = d\alpha_{j\nu} + \beta_{j\nu}$, where $\alpha_{j\nu} \to \alpha_j$ weakly in $H^{1,p_j}(M)$ and $\{\beta_{j\nu}\}$ is compact in $L^{p_j}(M)$. (*Hint*: Use the Hodge decomposition $\sigma = d\delta G\sigma + \delta d G\sigma + P\sigma$. Set $\alpha_{j\nu} = \delta G\sigma_{j\nu}$.)
2. Show that, for $j \le k$,
$$d\alpha_{1\nu} \wedge \cdots \wedge d\alpha_{j\nu} \longrightarrow d\alpha_1 \wedge \cdots \wedge d\alpha_j$$
in $\mathcal{D}'(M)$. If ${p_1}^{-1} + \cdots + {p_j}^{-1} = {q_j}^{-1} < 1$, show that this convergence holds weakly in $L^{q_j}(M)$.
(*Hint*: Use induction on j, via
$$\int d\alpha_{1\nu} \wedge \cdots \wedge d\alpha_{j+1,\nu} \wedge \varphi = \pm \int d\alpha_{1\nu} \wedge \cdots \wedge d\alpha_{j\nu} \wedge \alpha_{j+1,\nu} \wedge d\varphi.)$$
3. Now prove (6.34). (*Hint*: Expand $(d\alpha_{1\nu} + \beta_{1\nu}) \wedge \cdots \wedge (d\alpha_{k\nu} + \beta_{k\nu})$. For a term
$$\pm\big(d\alpha_{\ell_1\nu} \wedge \cdots \wedge d\alpha_{\ell_i\nu}\big) \wedge \big(\beta_{\ell_{i+1}\nu} \wedge \cdots \wedge \beta_{\ell_k\nu}\big),$$
establish and exploit weak L^q-convergence of the first factor (if $i < k$) plus strong L^r convergence of the second factor, with $q^{-1} + r^{-1} \le 1$.)
4. Localize the result (6.31)–(6.33) $\Rightarrow$ (6.34), replacing M by an open set $\Omega \subset \mathbb{R}^n$.
(*Hint*: Apply a cutoff $\chi \in C_0^\infty(\Omega)$.)
5. (The div-curl lemma.) Let $\dim M = 3$, and let X_ν and Y_ν be two sequences of vector fields such that
$$X_\nu \to X \text{ weakly in } L^{p_1}, \quad Y_\nu \to Y \text{ weakly in } L^{p_2},$$
$$\operatorname{div} X_\nu \text{ compact in } H^{-1,p_1}, \quad \operatorname{curl} Y_\nu \text{ compact in } H^{-1,p_2},$$
where $1 < p_j < \infty,\ {p_1}^{-1} + {p_2}^{-1} \le 1$. Show that $X_\nu \cdot Y_\nu \to X \cdot Y$ in $\mathcal{D}'$. Formulate the analogue for $\dim M = 2$.
6. Let $F_\nu : \mathbb{R}^n \to \mathbb{R}^n$ be a sequence of maps. Assume
$$F_\nu \to F \text{ weakly in } H^{1,n}(\mathbb{R}^n). \tag{6.35}$$
Show that
$$\det DF_\nu \to \det DF \ \text{ in } \mathcal{D}'(\mathbb{R}^n). \tag{6.36}$$
(*Hint*: Set $\sigma_{j\nu} = d\alpha_{j\nu} = F_\nu^* dx_j$.)
More generally, if $2 \le k \le n$ and
$$F_\nu \to F \ \text{ weakly in } H^{1,k}(\mathbb{R}^n), \tag{6.37}$$

then

$$\Lambda^k DF_\nu \to \Lambda^k DF \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{6.38}$$

and hence

$$\operatorname{Tr} \Lambda^k DF_\nu \to \operatorname{Tr} \Lambda^k DF \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{6.39}$$

7. L^p-spectral theory of the Laplace operator

We will apply material developed in §§ 5 and 6 to study spectral properties of the Laplace operator Δ on L^p-spaces. We first consider Δ on $L^p(M)$, where M is a compact Riemannian manifold, without boundary. For any $\lambda > 0$, $(\lambda - \Delta)^{-1}$ is bijective on $\mathcal{D}'(M)$, and results of § 6 imply $(\lambda - \Delta)^{-1}: L^p(M) \to H^{2,p}(M)$, provided $1 < p < \infty$. Thus if we define the unbounded operator Δ_p on $L^p(M)$ to be Δ acting on $H^{2,p}(M)$, it follows that Δ_p is a closed operator with nonempty resolvent set, and compact resolvent, hence a discrete spectrum, with finite-dimensional generalized eigenspaces. Elliptic regularity implies that each of these generalized eigenspaces consists of functions in $C^\infty(M)$, and then these functions are easily seen to be actual eigenfunctions. Thus, in such a case, the L^p-spectrum of Δ coincides with its L^2-spectrum.

It is desirable to mention properties of Δ_p, related to spectral properties. In particular, the heat semigroup $e^{t\Delta}$ defines a strongly continuous semigroup $H_p(t)$ on $L^p(M)$, for each $p \in [1, \infty)$. For $p \in [2, \infty)$, this can be seen by applying the L^2-theory, the maximum principle (for data in L^∞), and interpolating, to get $H_p(t) : L^p(M) \to L^p(M)$, for $p \in [2, \infty]$. Strong continuity for $p < \infty$ follows from denseness of $C^\infty(M)$ in $L^p(M)$. Then the action of $H_p(t)$ as a semigroup on $L^p(M)$ for $p \in (1, 2)$ follows by duality. One can also take the adjoint of the action of $e^{t\Delta}$ on $C(M)$ to get $e^{t\Delta}$ acting on $\mathfrak{M}(M)$, the space of finite Borel measures on M, and $e^{t\Delta}$ then preserves $L^1(M)$, the closure of $C^\infty(M)$ in $\mathfrak{M}(M)$.

Alternatively, the strongly continuous action of the heat semigroup on $L^p(M)$ for $p \in [1, \infty)$ can be perceived directly from the parametrix for $e^{t\Delta}$ constructed in Chap. 7, § 13.

Let $\mathcal{K}$ be a closed cone in the right half-plane of $\mathbb{C}$, with vertex at 0. Assume $\mathcal{K}$ is symmetric about the positive real axis and has angle $\alpha \in (0, \pi)$. If $P(z) : X \to X$ is a family of bounded operators on a Banach space X, for $z \in \mathcal{K}$, we say it is a holomorphic semigroup if it satisfies $P(z_1)P(z_2) = P(z_1 + z_2)$ for $z_j \in \mathcal{K}$, is strongly continuous in $z \in \mathcal{K}$, and is holomorphic in the interior, $z \in \overset{\circ}{\mathcal{K}}$. The strong continuity implies that $\|P(z)\|$ is locally uniformly bounded on $\mathcal{K}$.

Clearly, $e^{t\Delta}$ gives a holomorphic semigroup on $L^2(M)$. Also, $e^{z\Delta} f$ is defined in $\mathcal{D}'(M)$ whenever $f \in \mathcal{D}'(M)$ and Re $z \geq 0$, and $e^{z\Delta} f \in C^\infty(M)$ when Re $z > 0$. Also $u(z, x) = e^{z\Delta} f(x)$ is holomorphic in z in $\{\text{Re } z > 0\}$. This establishes all but one "small" point in the following.

Proposition 7.1. *$e^{z\Delta}$ defines a holomorphic semigroup $H_p(z)$ on $L^p(M)$, for each $p \in [1, \infty)$.*

Proof. Here, $\mathcal{K}$ can be any cone of the sort described above. It remains to establish strong continuity, $H_p(z)f \to f$ in $L^p(M)$ as $z \to 0$ in $\mathcal{K}$, for any $f \in L^p(M)$. Since $C^\infty(M)$ is dense in $L^p(M)$, it suffices to prove that $\{H_p(z) : z \in \mathcal{K}, |z| \leq 1\}$ has uniformly bounded operator norm on $L^p(M)$. This can be done by checking that the parametrix construction for $e^{t\Delta}$ extends from $t \in \mathbb{R}^+$ to $z \in \mathcal{K}$, yielding integral operators whose norms on $L^p(M)$ are readily bounded. The reader can check this.

Since the heat semigroup on $L^p(\Omega)$ for a compact manifold with boundary has a parametrix of a form more complicated than it does on $L^p(M)$, this "small" point gets bigger when we extend Proposition 7.1 to the case of compact manifolds with boundary.

Here is a useful property of holomorphic semigroups.

Proposition 7.2. *Let $P(z)$ be a holomorphic semigroup on a Banach space X, with generator A. Then*

$$t > 0,\ f \in X \Longrightarrow P(t)f \in \mathcal{D}(A) \tag{7.1}$$

and

$$\|AP(t)f\|_X \leq \frac{C}{t}\|f\|_X, \quad \textit{for}\ 0 < t \leq 1. \tag{7.2}$$

Proof. For some $a > 0$, there is a circle $\gamma(t)$, centered at t, of radius $a|t|$, such that $\gamma(t) \in \mathcal{K}$, for all $t \in (0, \infty)$. Thus

$$AP(t)f = P'(t)f = -\frac{1}{2\pi i}\int\limits_{\gamma(t)} (t-\zeta)^{-2} P(\zeta)f\, d\zeta. \tag{7.3}$$

Since $\|P(\zeta)f\| \leq C_2\|f\|$ for $\zeta \in \mathcal{K},\ |\zeta| \leq 1 + a$, we have (7.2).

In particular, we have that, for $p \in (1, \infty),\ 0 < t \leq 1$,

$$f \in L^p(M) \Longrightarrow \|e^{t\Delta} f\|_{H^{2,p}(M)} \leq \frac{C}{t}\|f\|_{L^p(M)}, \tag{7.4}$$

where $C = C_p$. This result could also be verified using the parametrix for $e^{t\Delta}$. Note that applying interpolation to (7.4) yields

$$\|e^{t\Delta} f\|_{H^{s,p}(M)} \leq C t^{-s/2}\, \|f\|_{L^p(M)}, \qquad \text{for } 0 \leq s \leq 2,\ 0 < t \leq 1, \tag{7.5}$$

when $p \in (1, \infty),\ C = C_p$. We will find it very useful to extend such an estimate to the case of $e^{t\Delta}$ acting on $L^p(\Omega)$ when Ω has a boundary.

We now look at Δ on a compact Riemannian manifold with (smooth) boundary $\overline{\Omega}$, with Dirichlet boundary condition. Assume $\overline{\Omega}$ is connected and $\partial\Omega \neq \emptyset$. We know that, for $\lambda \geq 0$,

$$R_\lambda = (\lambda - \Delta)^{-1} : L^2(\Omega) \to L^2(\Omega), \tag{7.6}$$

with range $H^2(\Omega) \cap H_0^1(\Omega)$. We can analyze $R_\lambda f$ for $f \in L^\infty(\Omega)$ by noting that R_λ is positivity preserving:

$$\lambda \geq 0,\ g \geq 0 \text{ on } \Omega \Longrightarrow R_\lambda g \geq 0 \text{ on } \Omega, \tag{7.7}$$

a result that follows from the positivity property of $e^{t\Delta}$ and the resolvent formula. From this and regularity estimates on $R_\lambda 1$, it easily follows that, for $\lambda \geq 0$,

$$R_\lambda : C(\overline{\Omega}) \to C(\overline{\Omega}) \quad \text{and} \quad R_\lambda : L^\infty(\Omega) \to L^\infty(\Omega). \tag{7.8}$$

Taking the adjoint of R_λ acting on $C(\overline{\Omega})$, we have R_λ acting on $\mathfrak{M}(\overline{\Omega})$, the space of finite Borel measures on $\overline{\Omega}$. Since the closure of $L^2(\Omega)$ in $\mathfrak{M}(\overline{\Omega})$ is $L^1(\Omega)$, we have

$$R_\lambda : L^1(\Omega) \to L^1(\Omega). \tag{7.9}$$

Interpolation yields

$$R_\lambda : L^p(\Omega) \longrightarrow L^p(\Omega), \quad 1 \leq p \leq \infty. \tag{7.10}$$

We next want to prove that

$$R_\lambda : L^p(\Omega) \longrightarrow H^{2,p}(\Omega), \quad p \in (1, \infty), \tag{7.11}$$

when $\lambda \geq 0$. To do this, it is convenient to assume that $\overline{\Omega} \subset M$, where M is a compact Riemannian manifold without boundary, diffeomorphic to the double of $\overline{\Omega}$. Let $R : M \to M$ be an involution that fixes $\partial\Omega$ and that, near $\partial\Omega$, is the reflection of each geodesic normal to $\partial\Omega$ about the point of intersection of the geodesic with $\partial\Omega$. Then extend f to be 0 on $M \setminus \Omega$, defining $\widetilde{f}$, and define v by

$$(\Delta - \lambda)v = \widetilde{f} \quad \text{on } M, \tag{7.12}$$

so $v \in H^{2,p}(M)$. Set $u = R_\lambda f$. Take

$$u_1(x) = v(x) - v\big(R(x)\big), \quad x \in \Omega. \tag{7.13}$$

With $v^r(x) = v\big(R(x)\big)$, we have $(L - \lambda)v^r(x) = \widetilde{f}\big(R(x)\big)$, where L is the Laplace operator for R^*g, the metric on M pulled back via R. Thus $L = \Delta + L^b$, where L^b is a differential operator of order 2, whose principal symbol vanishes on $\partial\Omega$. Thus $u_1 \in H^{2,p}(\Omega)$, $u_1 = 0$ on $\partial\Omega$, and $w_1 = u - u_1$ satisfies

$$(\Delta - \lambda)w_1 = r_1 \text{ on } \Omega, \quad w_1\big|_{\partial\Omega} = 0, \tag{7.14}$$

with

$$r_1 = (\Delta - \lambda)v^r\big|_\Omega = -L^b v^r\big|_\Omega. \tag{7.15}$$

It follows from (5.49) that

$$L^b v^r\big|_\Omega \in H^{1,p}(\Omega) \subset L^{p_2}(\Omega), \tag{7.16}$$

for some $p_2 > p$. If $p_2 < \infty$, repeat the construction above, applying it to (7.14), to obtain

$$w_1 = u_2 + w_2, \quad u_2 \in H^{2,p_2}(\Omega), \quad u_2\big|_{\partial\Omega} = 0, \tag{7.17}$$

and

$$(\Delta - \lambda)w_2 = r_2 \text{ on } \Omega, \quad w_2\big|_{\partial\Omega} = 0, \quad r_2 \in H^{1,p_2}(\Omega) \subset L^{p_3}(\Omega). \tag{7.18}$$

Continue, obtaining

$$u = u_1 + \cdots + u_k + w_k, \quad u_j \in H^{2,p_j}(\Omega), \quad u_j\big|_{\partial\Omega} = 0, \tag{7.19}$$

such that

$$(\Delta - \lambda)w_j = r_j \text{ on } \Omega, \quad w_j\big|_{\partial\Omega} = 0, \quad r_j \in H^{1,p_j}(\Omega) \subset L^{p_{j+1}}(\Omega). \tag{7.20}$$

We continue until $p_k > n = \dim \Omega$. At this point, we use a couple of results that will be established in the next section. Given $s \in (0,1)$, let $C^s(\overline{\Omega})$ denote the space of Hölder-continuous functions on $\overline{\Omega}$, with Hölder exponent s. We have

$$r_k \in H^{1,p_k}(\Omega) \subset C^s(\overline{\Omega}), \tag{7.21}$$

for some $s \in (0,1)$, appealing to Proposition 8.5 for the last inclusion in (7.21). Then the estimates in Theorem 8.9 imply

$$w_k \in C^{2+s}(\overline{\Omega}) \subset H^{2,p}(\Omega). \tag{7.22}$$

This proves (7.11).

Arguments parallel to those used for M show that the heat semigroup $e^{t\Delta}$, defined a priori on $L^2(\Omega)$, yields also a well-defined, strongly continuous semigroup

$H_p(t)$ on $L^p(\Omega)$, for each $p \in [1, \infty)$. If Δ_p denotes the generator of the heat semigroup on $L^p(\Omega)$, with Dirichlet boundary condition, then (7.11) implies

$$\mathcal{D}(\Delta_p) \subset H^{2,p}(\Omega), \quad p \in (1, \infty). \tag{7.23}$$

We see that Δ_p has compact resolvent. Furthermore, arguments such as used above for M show that the spectrum of Δ_p coincides with the L^2-spectrum of Δ.

We now extend Proposition 7.1.

Proposition 7.3. *For $p \in (1, \infty)$, $e^{z\Delta}$ defines a holomorphic semigroup on $L^p(\Omega)$, on any symmetric cone $\mathcal{K}$ about $\mathbb{R}^+$ of angle $< \pi$.*

Proof. As in the proof of Proposition 7.1, the point we need to establish is the local uniform boundedness of the $L^p(\Omega)$-operator norm of $e^{z\Delta}$, for $z \in \mathcal{K}$. In other words, we need estimates for the solution u to

$$\frac{\partial u}{\partial t} = \Delta u \text{ on } \mathcal{K} \times \Omega, \quad u(0) = f, \quad u\big|_{\mathcal{K} \times \partial\Omega} = 0, \tag{7.24}$$

of the form

$$\|u(t)\|_{L^p(\Omega)} \le C \|f\|_{L^p(\Omega)}, \quad t \in \mathcal{K}, \ \operatorname{Re} t \le 1. \tag{7.25}$$

By duality, it suffices to do this for $p \in (1, 2]$. The case $p = 2$ is obvious, so for the rest of the proof we will assume $p \in (1, 2)$. We will also assume $n = \dim \Omega > 1$, since the reflection principle works easily when $n = 1$.

To begin, define v by

$$\frac{\partial v}{\partial t} = \Delta v \text{ on } \mathcal{K} \times M, \quad v(0) = \widetilde{f} \in L^p(M), \tag{7.26}$$

where $\widetilde{f}$ is f on Ω, zero on $M \setminus \Omega$. Making use of Proposition 7.2, which we know applies to $e^{t\Delta}$ on $L^p(M)$, we have

$$\|v(t)\|_{H^{1,p}(M)} \le C|t|^{-1/2} \|f\|_{L^p(\Omega)}. \tag{7.27}$$

Now, if $R : M \to M$ is the involution on M used above, for $x \in \Omega$ we set

$$u_1(t, x) = v(t, x) - v\big(t, R(x)\big); \quad u_1 \in C\big(\mathcal{K}, L^p(\Omega)\big). \tag{7.28}$$

We have

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + g \text{ on } \mathcal{K} \times \Omega, \quad u_1(0) = f, \quad u_1\big|_{\mathcal{K} \times \partial\Omega} = 0, \tag{7.29}$$

and, by an argument parallel to (7.16), we derive from (7.27) an estimate

$$\|g(t)\|_{L^p(\Omega)} \le C\,|t|^{-1/2}\,\|f\|_{L^p(\Omega)}. \tag{7.30}$$

In this case, we replace appeal to (5.49) by the parametrix construction for $e^{t\Delta}$ on $\mathcal{D}'(M)$ made in Chap. 7, § 13.

We regard u_1 as a first approximation to u, but we seek a more accurate approximation rather than rely on an estimate at this point of the error. So now we define v_2 by

$$\frac{\partial v_2}{\partial t} = \Delta v_2 - \widetilde{g} \text{ on } \mathcal{K} \times M, \quad v_2(0) = 0, \tag{7.31}$$

where $\widetilde{g}$ is g on $\mathcal{K} \times \Omega$ and zero on $\mathcal{K} \times (M \setminus \Omega)$. We have

$$v_2(t) = -\int_0^t e^{(t-s)\Delta}\widetilde{g}(s)\,ds, \tag{7.32}$$

and the estimate $\|\widetilde{g}(s)\|_{L^p(M)} \le C\,|s|^{-1/2}$ from (7.30), together with the operator norm estimate of $e^{(t-s)\Delta}$ on $L^p(M)$, from Proposition 7.2, yields

$$v_2 \in C\big(\mathcal{K}, H^{1,p}(M)\big). \tag{7.33}$$

Now, for $x \in \Omega$, set

$$u_2(t,x) = v_2(t,x) - v_2\big(t, R(x)\big); \quad u_2 \in C\big(\mathcal{K}, H^{1,p}(\Omega)\big). \tag{7.34}$$

Thus

$$\frac{\partial u_2}{\partial t} = \Delta u_2 - g + g_2 \text{ on } \mathcal{K} \times \Omega, \quad u_2(0) = 0, \quad u_2\big|_{\mathcal{K}\times\partial\Omega} = 0, \tag{7.35}$$

and we have, parallel to but better than (7.30),

$$\|g_2(t)\|_{L^p(\Omega)} \le C\,\|f\|_{L^p(\Omega)}. \tag{7.36}$$

Next, solve

$$\frac{\partial v_3}{\partial t} = \Delta v_3 - \widetilde{g}_2 \text{ on } \mathcal{K} \times M, \quad v_3(0) = 0, \tag{7.37}$$

where $\widetilde{g}_2$ is g_2 on $\mathcal{K}\times\Omega$ and zero on $\mathcal{K}\times(M\setminus\Omega)$. The argument involving (7.32) and (7.33) this time yields the better estimate

$$v_3 \in C\big(\mathcal{K}, H^{2-\varepsilon,p}(M)\big), \quad \forall\, \varepsilon > 0, \tag{7.38}$$

hence, by the Sobolev imbedding result of Proposition 6.4, with $s = 1 - \varepsilon$,

$$v_3 \in C\big(\mathcal{K}, H^{1,p_3}(M)\big), \quad p_3 = \frac{np}{n-(1-\varepsilon)p} > p, \tag{7.39}$$

provided $p < n$. Now we set

$$u_3(t,x) = v_3(t,x) - v_3\big(t, R(x)\big); \quad u_3 \in C\big(\mathcal{K}, H^{1,p_3}(\Omega)\big), \tag{7.40}$$

and we get

$$\frac{\partial u_3}{\partial t} = \Delta u_3 - g_2 + g_3 \text{ on } \mathcal{K} \times \Omega, \quad u_3(0) = 0, \quad u_3\big|_{\mathcal{K}\times\partial\Omega} = 0, \tag{7.41}$$

with the following improvement on (7.36):

$$\|g_3(t)\|_{L^{p_3}(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{7.42}$$

Continuing in this fashion, we get

$$u_j \in C\big(\mathcal{K}, H^{2-\varepsilon,p_{j-1}}(\Omega)\big) \subset C\big(\mathcal{K}, H^{1,p_j}(\Omega)\big), \tag{7.43}$$

with $p = p_2 < p_3 < \cdots \nearrow$. Given $p \in (1,2)$, some p_k is ≥ 2. Then $u_k \in C\big(\mathcal{K}, H^1(\Omega)\big)$ satisfies

$$\frac{\partial u_k}{\partial t} = \Delta u_k - g_{k-1} + g_k \text{ on } \mathcal{K} \times \Omega, \quad u_k(0) = 0, \quad u_k\big|_{\mathcal{K}\times\partial\Omega} = 0, \tag{7.44}$$

with

$$g_k \in C\big(\mathcal{K}, L^2(\Omega)\big). \tag{7.45}$$

Now we solve for w the equation

$$\frac{\partial w}{\partial t} = \Delta w - g_k \text{ on } \mathcal{K} \setminus \Omega, \quad w(0) = 0, \quad w\big|_{\mathcal{K}\times\partial\Omega} = 0. \tag{7.46}$$

The easy L^2-estimates yield

$$w \in C\big(\mathcal{K}, H^{2-\varepsilon}(\Omega)\big), \tag{7.47}$$

and the solution to (7.24) is

$$u = u_1 + \cdots + u_k + w. \tag{7.48}$$

This proves the desired estimate (7.25), for $p \in (1,2)$, which is enough to prove Proposition 7.3.

We mention that an interpolation argument yields that $e^{z\Delta}$ is a holomorphic semigroup on $L^p(\Omega)$ on a cone $\mathcal{K}$ that is symmetric about $\mathbb{R}^+$ and has angle $\pi\big(1-|2/p-1|\big)$. (See [RS], Vol. 2, p. 255.) This result is valid even if Ω has nasty boundary, as well as in other settings. On the other hand, ingredients of the argument used above will also be useful for other results, presented below.

Note that once we have the holomorphy of $e^{t\Delta}$ on $L^p(\Omega)$, for all $p\in(1,\infty)$, we can apply Proposition 7.2. In particular, suppose we carry out the construction of the u_k above, not stopping as soon as $p_k\ge 2$, but letting p_k become arbitrarily large. Then (7.44) is replaced by $g_k\in C\big(\mathcal{K},L^{p_k}(\Omega)\big)$, and we can now apply Proposition 7.2 to improve (7.47) to

$$w\in C\big(\mathcal{K},H^{2-\varepsilon,p_k}(\Omega)\big), \tag{7.49}$$

making use of (7.2), (7.11), and interpolation to estimate the norm of $e^{t\Delta}: L^p(\Omega)\to H^{2-\varepsilon,p}(\Omega)$.

We now consider the construction (7.24)–(7.44) when $u(0)=f\in L^\infty(\Omega)$. We will restrict attention to $t\in\mathbb{R}^+$. A direct inspection of the parametrix for the heat kernel, constructed in Chap. 7, § 13, shows that $e^{t\Delta}: L^\infty(M)\to C^1(M)$, with norm $\le Ct^{-1/2}$, for $t\in(0,1]$, so v in (7.26) satisfies the estimate $\|v(t)\|_{C^1(M)}\le Ct^{-1/2}\|f\|_{L^\infty(\Omega)}$, and $\|u_1(t)\|_{C^1(\overline{\Omega})}$ satisfies a similar estimate. Thus g in (7.29) satisfies the estimate (7.30), with $p=\infty$, and consequently v_2 in (7.32) satisfies $\|v_2(t)\|_{C^1(M)}\le C$. Hence $\|u_2(t)\|_{C^1(\overline{\Omega})}\le C$, and g_2 in (7.35) satisfies (7.36) with $p=\infty$. Thus $u=u_1+u_2+w$, where w satisfies

$$\frac{\partial w}{\partial t}=\Delta w-g_2 \text{ on } \mathbb{R}^+\times\Omega,\quad w(0)=0,\quad w\big|_{\mathbb{R}^+\times\partial\Omega}=0. \tag{7.50}$$

By the holomorphy of $e^{t\Delta}$ on $L^p(\Omega)$ for $p\in(1,\infty)$, we have

$$w\in C\big([0,\infty),H^{2-\varepsilon,p}(\Omega)\big), \tag{7.51}$$

for any $\varepsilon>0$ and arbitrarily large $p<\infty$, hence $w\in C\big(\mathbb{R}^+,C^{2-\delta}(\overline{\Omega})\big)$, for any $\delta>0$. We deduce that

$$\|e^{t\Delta}f\|_{C^1(\overline{\Omega})}\le Ct^{-1/2}\,\|f\|_{L^\infty(\Omega)},\quad 0<t\le 1. \tag{7.52}$$

The estimate (7.52), together with the following result, will be useful for the study of semilinear parabolic equations on domains with boundary, in § 3 of Chap. 15.

Proposition 7.4. *If $\overline{\Omega}$ is a compact Riemannian manifold with boundary, on which the Dirichlet condition is placed, then $e^{t\Delta}$ defines a strongly continuous semigroup on the Banach space*

$$C^1_b(\overline{\Omega})=\big\{f\in C^1(\overline{\Omega}): f|_{\partial\Omega}=0\big\}. \tag{7.53}$$

Proof. It is easy to verify that, for $N > 1 + (\dim M)/4$,

$$\mathcal{D}(\Delta^N) \subset C_b^1(\overline{\Omega}), \quad \text{densely}.$$

Since $e^{t\Delta}$ is a strongly continuous semigroup on $\mathcal{D}(\Delta^N)$, it suffices to show that for each $f \in C_b^1(\overline{\Omega})$, $\{e^{t\Delta} f : 0 \le t \le 1\}$ is uniformly bounded in $\mathrm{Lip}(\overline{\Omega})$. To see this, we analyze solutions to

$$\frac{\partial u}{\partial t} = \Delta u, \text{ for } x \in \Omega, \quad u(0,x) = f(x), \quad u(t,x) = 0, \text{ for } x \in \partial\Omega,$$

when

$$f \in C^1(\overline{\Omega}), \quad f\big|_{\partial\Omega} = 0. \tag{7.54}$$

We will to some extent follow the proof of Proposition 7.3, and also use that result. In this case, for $\widetilde{f}$ equal to f on $\overline{\Omega}$ and to zero on $M \setminus \overline{\Omega}$, we have $\widetilde{f} \in \mathrm{Lip}(M)$. Thus, for v defined by

$$\frac{\partial v}{\partial t} = \Delta v \text{ on } \mathbb{R}^+ \times M, \quad v(0) = \widetilde{f},$$

we have

$$v \in \mathcal{C}\big(\mathbb{R}^+, \mathrm{Lip}(M)\big), \tag{7.55}$$

where the "$\mathcal{C}$" stands for "weak" continuity in t, (i.e., $v(t)$ is bounded in $\mathrm{Lip}(M)$ and continuous in t, with values in $H^{1,p}(M)$, for each $p < \infty$). Hence

$$u_1(t,x) = v(t,x) - v\big(t, R(x)\big)\big|_{\mathbb{R}^+ \times \overline{\Omega}}$$

satisfies

$$u_1 \in \mathcal{C}\big(\mathbb{R}^+, \mathrm{Lip}(\overline{\Omega})\big). \tag{7.56}$$

We have

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + g, \quad u_1(0) = f, \quad u_1\big|_{\mathbb{R}^+ \times \partial\Omega} = 0,$$

where

$$g = L^b v^r\big|_{\mathbb{R}^+ \times \Omega}.$$

Here, as in (7.15), L^b is a second-order differential operator whose principal symbol vanishes on $\partial\Omega$, and $v^r(x) = v\big(R(x)\big)$. Consequently, again an analogue of (5.49) gives

$$g \in \mathcal{C}\big(\mathbb{R}^+, L^\infty(\Omega)\big). \tag{7.57}$$

Now, we have $u = u_1 + w$, where w satisfies

$$\frac{\partial w}{\partial t} = \Delta w - g, \quad w(0) = 0, \quad w\big|_{\mathbb{R}^+ \times \partial\Omega} = 0, \tag{7.58}$$

and, by (7.57), $g \in C\big(\mathbb{R}^+, L^p(\Omega)\big)$, for all $p < \infty$. This implies

$$w \in C\big(\mathbb{R}^+, H^{2-\varepsilon,p}(\Omega)\big), \quad \forall\, p < \infty,\ \varepsilon > 0, \tag{7.59}$$

since $e^{t\Delta}$ is a holomorphic semigroup on $L^p(\Omega)$. This proves Proposition 7.4.

Exercises

1. Extend results of this section to the Neumann boundary condition.

   In Exercises 2 and 3, let Ω be an open subset, with smooth boundary, of a compact Riemannian manifold M. Assume there is an isometry $\tau : M \to M$ that is an involution, fixing $\partial\Omega$, so M is the isometric double of $\overline{\Omega}$.
2. Suppose X_j are smooth vector fields on $\overline{\Omega}$, $f_j \in L^p(\Omega)$ for some $p \in [2, \infty)$, and u is the unique solution in $H_0^{1,2}(\Omega)$ to

   $$\Delta u = \sum X_j f_j.$$

   Show that $u \in H^{1,p}(\Omega)$. (*Hint*: Reduce to the case where each X_j is a smooth vector field on M, such that $\tau_\# X_j = \pm X_j$. Extend f_j to $f_j \in L^p(M)$, so that $\tau^* f_j = \mp f_j$. Thus $\sum X_j f_j \in H^{-1,p}(M)$ is odd under τ.)
3. Extend the result of Exercise 2 to the case $f_j \in L^p(\Omega)$ when $1 < p < 2$, appropriately weakening the a priori hypothesis on u.
4. Try to extend the results of Exercises 2 and 3 to general, compact, smooth $\overline{\Omega}$, not necessarily having an isometric double.
5. Show that (7.5) can be improved to

   $$R_\lambda : L^\infty(\Omega) \longrightarrow C(\overline{\Omega}),$$

   for $\lambda \geq 0$. (*Hint*: Use (7.11). Show that, in fact, for $\lambda \geq 0$,

   $$R_\lambda : L^\infty(\Omega) \longrightarrow C^r(\overline{\Omega}), \quad \forall\, r < 2.)$$

   A sharper result will be contained in (8.54)–(8.55).

8. Hölder spaces and Zygmund spaces

If $0 < s < 1$, we define the space $C^s(\mathbb{R}^n)$ of Hölder-continuous functions on $\mathbb{R}^n$ to consist of bounded functions u such that

$$|u(x+y) - u(x)| \leq C|y|^s. \tag{8.1}$$

For $k = 0, 1, 2, \ldots$, we take $C^k(\mathbb{R}^n)$ to consist of bounded, continuous functions u such that $D^\beta u$ is bounded and continuous, for $|\beta| \le k$. If $s = k + r$, $0 < r < 1$, we define $C^s(\mathbb{R}^n)$ to consist of functions $u \in C^k(\mathbb{R}^n)$ such that, for $|\beta| = k$, $D^\beta u$ belongs to $C^r(\mathbb{R}^n)$.

For nonintegral s, the Hölder spaces $C^s(\mathbb{R}^n)$ have a characterization similar to that for L^p and more generally $H^{s,p}$, in (5.46) and (6.23), via the Littlewood–Paley partition of unity used in (5.37),

$$1 = \sum_{j=0}^{\infty} \varphi_j(\xi)^2,$$

with φ_j supported on $\langle\xi\rangle \sim 2^j$, and $\varphi_j(\xi) = \varphi_1(2^{1-j}\xi)$ for $j \ge 1$. Let $\psi_j(\xi) = \varphi_j(\xi)^2$.

Proposition 8.1. *If $u \in C^s(\mathbb{R}^n)$, then*

$$\sup_k \; 2^{ks} \|\psi_k(D)u\|_{L^\infty} < \infty. \tag{8.2}$$

Proof. To see this, first note that it is obvious for $s = 0$. For $s = \ell \in \mathbb{Z}^+$, it then follows from the elementary estimate

$$\begin{aligned} C_1 2^{k\ell} \|\psi_k(D)u(x)\|_{L^\infty} &\le \sum_{|\alpha| \le \ell} \|\psi_k(D) D^\alpha u(x)\|_{L^\infty} \\ &\le C_2 2^{k\ell} \|\psi_k(D)u(x)\|_{L^\infty}. \end{aligned} \tag{8.3}$$

Thus it suffices to establish that $u \in C^s$ implies (8.2) for $0 < s < 1$. Since $\hat{\psi}_1(x)$ has zero integral, we have, for $k \ge 1$,

$$\begin{aligned} |\psi_k(D)u(x)| &= \left| \int \hat{\psi}_k(y)\big[u(x-y) - u(x)\big]\, dy \right| \\ &\le C \int |\hat{\psi}_k(y)| \cdot |y|^s \, dy, \end{aligned} \tag{8.4}$$

which is readily bounded by $C \cdot 2^{-ks}$.

This result has a partial converse.

Proposition 8.2. *If s is not an integer, finiteness in (8.2) implies $u \in C^s(\mathbb{R}^n)$.*

Proof. It suffices to demonstrate this for $0 < s < 1$. With $\Psi_k(\xi) = \sum_{j \le k} \psi_j(\xi)$, if $|y| \sim 2^{-k}$, write

$$\begin{aligned} u(x+y) - u(x) &= \int_0^1 y \cdot \nabla \Psi_k(D) u(x + ty)\, dt \\ &\quad + \big(I - \Psi_k(D)\big)\big(u(x+y) - u(x)\big) \end{aligned} \tag{8.5}$$

and use (8.2) and (8.3) to dominate the L^∞-norm of both terms on the right by $C \cdot 2^{-sk}$, since $\|\nabla \Psi_k(D)u\|_{L^\infty} \le C \cdot 2^{(1-s)k}$.

This converse breaks down if $s \in \mathbb{Z}^+$. We define the *Zygmund space* $C^s_*(\mathbb{R}^n)$ to consist of u such that (8.2) is finite, using that to define the C^s_*-norm, namely,

$$\|u\|_{C^s_*} = \sup_k \, 2^{ks} \|\psi_k(D)u\|_{L^\infty}. \tag{8.6}$$

Thus

$$C^s = C^s_* \text{ if } s \in \mathbb{R}^+ \setminus \mathbb{Z}^+, \quad C^k \subset C^k_*, \ k \in \mathbb{Z}^+. \tag{8.7}$$

The class $C^s_*(\mathbb{R}^n)$ can be defined for any $s \in \mathbb{R}$, as the set of elements $u \in \mathcal{S}'(\mathbb{R}^n)$ such that (8.6) is finite.

The following complements previous boundedness results for Fourier multipliers $P(D)$ on $L^p(\mathbb{R}^n)$ and on $H^{s,p}(\mathbb{R}^n)$.

Proposition 8.3. *If $P(\xi) \in S^m_1(\mathbb{R}^n)$, then, for all $s \in \mathbb{R}$,*

$$P(D) : C^s_* \longrightarrow C^{s-m}_*. \tag{8.8}$$

Proof. Consider first the case $m = 0$. Pick $\tilde{\psi}_j(\xi) \in C^\infty_0(\mathbb{R}^n)$ such that $\tilde{\psi}_j(\xi) = 1$ on supp ψ_j and $\tilde{\psi}_j(\xi) = \tilde{\psi}_1(2^{1-j}\xi)$, for $j \ge 2$. It follows readily from the analysis of the Schwartz kernel of $P(D)$ made in § 2 of Chap. 7, particularly in the proof of Proposition 2.2 there, that

$$P(\xi) \in S^0_1(\mathbb{R}^n) \Longrightarrow \sup_j \|\tilde{\psi}_j(\xi)P(\xi)\|_{\mathcal{F}L^1} < \infty, \tag{8.9}$$

where $\|Q\|_{\mathcal{F}L^1} = \|\widehat{Q}\|_{L^1}$. Also, it is clear that

$$\|\psi_k(D)P(D)u\|_{L^\infty} \le C\|\tilde{\psi}_k P\|_{\mathcal{F}L^1} \cdot \|\psi_k(D)u\|_{L^\infty}, \tag{8.10}$$

which implies (8.8) for $m = 0$. The extension to general $m \in \mathbb{R}$ is straightforward.

In particular, with $\Lambda = (1-\Delta)^{1/2}$,

$$\Lambda^m : C^s_* \longrightarrow C^{s-m}_* \quad \text{is an isomorphism.} \tag{8.11}$$

Note that in light of (8.9) and (8.10), we have

$$\|P(D)u\|_{C^s_*} \le C \sup_{\xi \in \mathbb{R}^n, |\alpha| \le [n/2]+1} \|P^{(\alpha)}(\xi)\langle \xi \rangle^{|\alpha|}\|_{L^\infty} \cdot \|u\|_{C^s_*}. \tag{8.12}$$

In particular, for $y \in \mathbb{R}$,

$$\|\Lambda^{iy} u\|_{C^s_*} \le C\langle y\rangle^{n/2+1} \|u\|_{C^s_*}. \tag{8.13}$$

Compare with (5.47).

The Sobolev imbedding theorem, Proposition 6.3, can be sharpened and extended to the following:

Proposition 8.4. *For all $s \in \mathbb{R},\ p \in (1, \infty)$,*

$$H^{s,p}(\mathbb{R}^n) \subset C^r_*(\mathbb{R}^n), \quad r = s - \frac{n}{p}. \tag{8.14}$$

Proof. In light of (8.11), it suffices to consider the case $s = n/p$. Let $L_m(\xi) \in S^m_1(\mathbb{R}^n)$ be nowhere vanishing and satisfy $L_m(\xi) = |\xi|^m$, for $|\xi| \ge 1/100$. It suffices to show that, for $p \in (1, \infty)$,

$$\|\psi_k(D) L_{-n/p}(D) u\|_{L^\infty} \le C\|u\|_{L^p(\mathbb{R}^n)}, \tag{8.15}$$

with C independent of k. We can restrict attention to $k \ge 2$. Then $A_k(\xi) = \psi_k(\xi) L_{-n/p}(\xi)$ satisfies

$$A_{k+1}(\xi) = 2^{-nk/p}\, A_1(2^{-k}\xi).$$

Hence $\widehat{A}_{k+1}(x) \in \mathcal{S}(\mathbb{R}^n)$ and

$$\|\widehat{A}_{k+1}\|_{L^{p'}(\mathbb{R}^n)} = C, \ \ \text{independent of } k \ge 2. \tag{8.16}$$

Thus the left side of (8.15) is dominated by $\|\widehat{A}_k\|_{L^{p'}} \cdot \|u\|_{L^p}$, which in turn is dominated by the right side of (8.15). This completes the proof.

It is useful to extend Proposition 8.3 to the following.

Proposition 8.5. *If $p(x, \xi) \in S^m_{1,0}(\mathbb{R}^n)$, then, for $s \in \mathbb{R}$,*

$$p(x, D) : C^s_*(\mathbb{R}^n) \longrightarrow C^{s-m}_*(\mathbb{R}^n). \tag{8.17}$$

Proof. In light of (8.11), it suffices to consider the case $m = 0$. Also, it suffices to consider one fixed s, which we can take to be positive. First we prove (8.17) in the special case where $p(x, \xi)$ has compact support in x. Then we can write

$$p(x, D)u = \int e^{ix\cdot\eta}\, q_\eta(D) u \, d\eta, \tag{8.18}$$

with

$$q_\eta(\xi) = (2\pi)^{-n} \int e^{-ix\cdot\eta}\, p(x, \xi)\, dx. \tag{8.19}$$

Via the estimates used to prove Proposition 8.3, it follows that, for any given $s \in \mathbb{R}$, $q_\eta(D) \in \mathcal{L}\big(C^s_*(\mathbb{R}^n)\big)$ has an operator norm that is a rapidly decreasing function of η. It is easy to establish the estimate

$$\|e^{ix\cdot\eta}\, u\|_{C^s_*} \le C(s)\, \langle\eta\rangle^s\, \|u\|_{C^s_*} \quad (s > 0), \tag{8.20}$$

first for $s \notin \mathbb{Z}^+$, by using the characterization (8.1) of $C^s = C^s_*$, then for general $s > 0$ by interpolation. The desired operator bound on (8.18) follows easily.

To do the general case, one can use a partition of unity in the x-variables, of the form

$$1 = \sum_{j\in\mathbb{Z}^n} \varphi_j(x), \quad \varphi_j(x) = \varphi_0(x+j), \ \ \varphi_0 \in C_0^\infty(\mathbb{R}^n),$$

and exploit the estimates on $p_j(x,D)u = \varphi_j(x)p(x,D)u$ obtained by the argument above, in concert with the rapid decrease of the Schwartz kernel of the operator $p(x,D)$ away from the diagonal. Details are left to the reader.

In § 9 we will establish a result that is somewhat stronger than Proposition 8.5, but this relatively simple result is already useful for Hölder estimates on solutions to linear, elliptic PDE.

It is useful to note that we can define Zygmund spaces $C^s_*(\mathbb{T}^n)$ on the torus just as in (8.6), but using Fourier series. We again have (8.7) and Propositions 8.3–8.5.

The issue of how Zygmund spaces form a complex interpolation scale is more subtle than the analogous situation for L^p-Sobolev spaces, treated in § 6. A different type of complex interpolation functor, $[X,Y]^b_\theta$, defined in Appendix A at the end of this chapter, does a better job than $[X,Y]_\theta$. We have the following result established in Appendix A.

Proposition 8.6. *For $r, s \in \mathbb{R}$, $\theta \in (0,1)$,*

$$[C^r_*(\mathbb{T}^n), C^s_*(\mathbb{T}^n)]^b_\theta = C^{\theta s+(1-\theta)r}_*(\mathbb{T}^n). \tag{8.21}$$

It is straightforward to extend the notions of Hölder and Zygmund spaces to spaces $C^s(M)$ and $C^s_*(M)$ when M is a compact manifold without boundary. Furthermore, the analogue of (8.14) is readily established, and we have

$$P : C^s_*(M) \longrightarrow C^{s-m}_*(M) \quad \text{if } P \in OPS^m_{1,0}(M). \tag{8.22}$$

If $\overline{\Omega}$ is a compact manifold with boundary, there is an obvious notion of $C^s(\overline{\Omega})$, for $s \ge 0$. We will define $C^s_*(\overline{\Omega})$ below, for $s \ge 0$. For now we look further at $C^s(\overline{\Omega})$. The following simple observation is useful. Give $\overline{\Omega}$ a Riemannian metric and let $\delta(x) = \ \mathrm{dist}(x, \partial\Omega)$.

Proposition 8.7. *Let $r \in (0,1)$. Assume $f \in C^1(\Omega)$ satisfies*

$$|\nabla f(x)| \le C\,\delta(x)^{r-1}, \quad x \in \Omega. \tag{8.23}$$

Then f extends continuously to $\overline{\Omega}$, as an element of $C^r(\overline{\Omega})$.

Proof. There is no loss of generality in assuming that Ω is the unit ball in $\mathbb{R}^n$. When estimating $f(x_2) - f(x_1)$, we may as well assume that x_1 and x_2 are a distance $\le 1/4$ from $\partial\Omega$ and $|x_1 - x_2| \le 1/4$. Write

$$f(x_2) - f(x_1) = \int_\gamma df(x),$$

where γ is a path from x_1 to x_2 of the following sort. Let y_j lie on the ray segment from 0 to x_j, a distance $d = |x_1 - x_2|$ from x_j. Then γ goes from x_1 to y_1 on a line, from y_1 to y_2 on a line, and from y_2 to x_2 on a line, as illustrated in Fig. 8.1. Then

$$|f(x_j) - f(y_j)| \le C \int_{1-d}^{1} (1-\rho)^{r-1}\,d\rho = C \int_0^d \tau^{r-1}\,d\tau \le C'd^r, \tag{8.24}$$

while

$$|f(y_1) - f(y_2)| \le C|y_1 - y_2|d^{r-1} \le C'd^r, \tag{8.25}$$

so

$$|f(x_2) - f(x_1)| \le C|x_1 - x_2|^r, \tag{8.26}$$

as asserted.

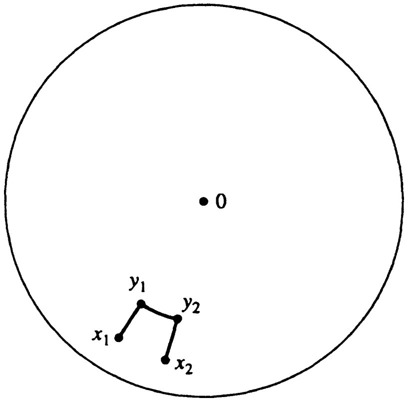

FIGURE 8.1 Path from x_1 to x_2

Now consider $\overline{\Omega}$ of the form $\overline{\Omega} = [0,1] \times M$, where M is a compact Riemannian manifold without boundary. We want to consider the action on $f \in C^r(M)$ of a family of operators of Poisson integral type, such as were studied in Chap. 7, § 12, to construct parametrices for regular elliptic boundary problems. We recall from (12.35) of Chap. 7 the class $OP\mathcal{P}^{-j}$ consisting of families $A(y)$ of pseudodifferential operators on M, parameterized by $y \in [0,1]$:

$$(8.27) \qquad A(y) \in OP\mathcal{P}^{-j} \Longleftrightarrow y^k D_y^\ell A(y) \text{ bounded in } OPS^{-j-k+\ell}_{1,0}(M).$$

Furthermore, if $L \in OPS^1(M)$ is a positive, self-adjoint, elliptic operator, then operators of the form $A(y)e^{-yL}$, with $A(y) \in OP\mathcal{P}^{-j}$, belong to $OP\mathcal{P}_e^{-j}$. In addition (see (12.50)), any $A(y) \in OP\mathcal{P}_e^{-j}$ can be written in the form $e^{-yL}B(y)$ for some such elliptic L and some $B(y) \in OP\mathcal{P}^{-j}$. The following result is useful for Hölder estimates on solutions to elliptic boundary problems.

Proposition 8.8. *If $A(y) \in OP\mathcal{P}_e^{-j}$ and $f \in C^r_*(M)$, then*

$$(8.28) \qquad u(y,x) = A(y)f(x) \Longrightarrow u \in C^{j+r}(I \times M),$$

provided $j + r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$.

Note that we allow $r < 0$ if $j > 0$.

Proof. First consider the case $j = 0,\ 0 < r < 1$, and write

$$(8.29) \qquad A(y)f = e^{-\kappa y \Lambda}\, B(y)f, \quad B(y) \in OP\mathcal{P}^0.$$

We can assume without loss of generality that $\Lambda = (1-\Delta)^{1/2}$, and we can replace M by $\mathbb{R}^n$. In such a case, we will show that

$$(8.30) \qquad |\nabla_{y,x} u(y,x)| \le C y^{r-1}\, \|u\|_{C^r}$$

if $0 < r < 1$, which by Proposition 8.7 will yield $u \in C^r(I \times M)$. Now if we set $\partial_j = \partial/\partial x_j$ for $1 \le j \le n,\ \partial_0 = \partial/\partial y$, then we can write

$$(8.31) \qquad y\partial_j u(y,x) = y\Lambda e^{-\kappa y \Lambda} B_j(y)f, \quad B_j(y) \in OP\mathcal{P}^0.$$

Now, given $f \in C^r(M),\ 0 < r < 1$, we have $B_j(y)f$ bounded in $C^r(M)$, for $y \in [0,1]$. Then the estimate (8.30) follows from

$$(8.32) \qquad \|\varphi(y\Lambda)g\|_{L^\infty} \le C y^r \|g\|_{C^r_*},$$

for $0 < r < 1$, where $\varphi(\lambda) = \lambda e^{-\kappa\lambda}$, which vanishes at $\lambda = 0$ and is rapidly decreasing as $\lambda \to +\infty$. In turn, this follows easily from the characterization (8.6) of the C^r_*-norm.

If $f \in C^{k+r}(M),\ k \in \mathbb{Z}^+,\ 0 < r < 1$, and $j = 0$ then given $|\alpha| \leq k$,

$$D^\alpha_{y,x} u = e^{-\kappa y \Lambda} B_\alpha(y) \Lambda^k f, \quad B_\alpha(y) \in OP\mathcal{P}^0, \tag{8.33}$$

so the analysis of (8.29), with f replaced by $\Lambda^k f$, applies to yield $D^\alpha_{y,x} u \in C^r(I \times M)$, for $|\alpha| \leq k$.

Similarly, the extension from $j = 0$ to general $j \in \mathbb{Z}^+$ is straightforward, so Proposition 8.8 is proved.

As we have said above, Proposition 8.8 is important because it yields Hölder estimates on solutions to elliptic boundary problems, as defined in Chap. 5, § 11. The principal consequence is the following:

Theorem 8.9. *Let $(P, B_j, 1 \leq j \leq \ell)$ be a regular elliptic boundary problem. Suppose P has order m and each B_j has order m_j. If u solves*

$$Pu = 0 \ on\ \Omega, \quad B_j u = g_j \ on\ \partial\Omega, \tag{8.34}$$

then, for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$,

$$g_j \in C^{r-m_j}_*(\partial\Omega) \Longrightarrow u \in C^r(\overline{\Omega}). \tag{8.35}$$

Proof. Of course, $u \in C^\infty(\Omega)$. On a collar neighborhood of $\partial\Omega$, diffeomorphic to $[0,1] \times \partial\Omega$, we can write, modulo $C^\infty\big([0,1] \times \partial\Omega\big)$,

$$u = \sum Q_j(y) g_j, \quad Q_j(y) \in OP\mathcal{P}^{-m_j}_e, \tag{8.36}$$

by Theorem 12.6 of Chap. 7, so the implication (8.35) follows directly from (8.28).

We next want to define Zygmund spaces on domains with boundary. Let Ω be an open set with smooth boundary (and closure $\overline{\Omega}$) in a compact manifold M. We want to consider Zygmund spaces $C^r_*(\overline{\Omega}),\ r > 0$. The approach we will take is to define $C^r_*(\overline{\Omega})$ by interpolation:

$$C^r_*(\overline{\Omega}) = \big[C^{s_1}(\overline{\Omega}), C^{s_2}(\overline{\Omega})\big]^b_\theta, \tag{8.37}$$

where $0 < s_1 < r < s_2,\ 0 < \theta < 1,\ r = (1-\theta)s_1 + \theta s_2$ (and $s_j \notin \mathbb{Z}$). As in (8.21), we are using the complex interpolation functor defined in Appendix A. We need to show that this is independent of choices of such s_j. Using an argument parallel to one in § 6, for any $N \in \mathbb{Z}^+$, we have an extension operator

$$E : C^s(\overline{\Omega}) \longrightarrow C^s(M), \quad s \in (0, N) \setminus \mathbb{Z}, \tag{8.38}$$

providing a right inverse for the surjective restriction operator

$$\rho : C^s(M) \longrightarrow C^s(\overline{\Omega}). \tag{8.39}$$

From Proposition 8.6, we can deduce that whenever $r > 0$ and s_j and θ are as above, $C^r_*(M) = \big[C^{s_1}(M), C^{s_2}(M)\big]^b_\theta$. Thus, by interpolation, we have, for $r > 0$,

$$E : C^r_*(\overline{\Omega}) \longrightarrow C^r_*(M), \quad \rho : C^r_*(M) \longrightarrow C^r_*(\overline{\Omega}), \tag{8.40}$$

and $\rho E = I$ on $C^r_*(\overline{\Omega})$. Hence

$$C^r_*(\overline{\Omega}) \approx C^r_*(M)\big/\{u \in C^r_*(M) : u\big|_\Omega = 0\}. \tag{8.41}$$

This characterization is manifestly independent of the choices made in (8.37). Note that the right side of (8.41) is meaningful even for $r \le 0$.

By Propositions 8.1 and 8.2, we know that $C^r_*(M) = C^r(M)$, for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, so

$$C^r_*(\overline{\Omega}) = C^r(\overline{\Omega}), \quad \text{for } r \in \mathbb{R}^+ \setminus \mathbb{Z}^+. \tag{8.42}$$

Using the spaces $C^r_*(\overline{\Omega})$, we can fill in the gaps (at $r \in \mathbb{Z}^+$) in the estimates of Theorem 8.9.

Proposition 8.10. *If $(P, B_j, 1 \le j \le \ell)$ is a regular elliptic boundary problem as in Theorem 8.9 and u solves (8.34), then, for all $r \in (0, \infty)$,*

$$g_j \in C^{r-m_j}_*(\partial\Omega) \Longrightarrow u \in C^r_*(\overline{\Omega}). \tag{8.43}$$

Proof. For $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, this is equivalent to (8.35). Since the solution u is given, mod $C^\infty(\overline{\Omega})$, by the operator (8.36), the rest follows by interpolation.

In a sense, the C^0_*-norm is only a tad weaker than the C^0-norm. The following is a quantitative version of this statement, which will prove very useful for the study of nonlinear evolution equations, particularly in Chap. 17.

Proposition 8.11. *If $s > n/2 + \delta$, then there is $C < \infty$ such that, for all $\varepsilon \in (0, 1]$,*

$$\|u\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Big(\log \frac{1}{\varepsilon}\Big) \|u\|_{C^0_*}. \tag{8.44}$$

Proof. By (8.6), $\|u\|_{C^0_*} = \sup \|\psi_k(D)u\|_{L^\infty}$. Now, with $\Psi_j = \sum_{\ell \le j} \psi_\ell$, make the decomposition $u = \Psi_j(D)u + \big(1 - \Psi_j(D)\big)u$; let $\varepsilon = 2^{-j}$. Clearly,

$$\|\Psi_j(D)u\|_{L^\infty} \le j\,\|u\|_{C^0_*}. \tag{8.45}$$

Meanwhile, using the Sobolev imbedding theorem, since $n/2 < s - \delta$,

$$\begin{aligned}\|(1-\Psi_j(D))u\|_{L^\infty} &\le C\|(1-\Psi_j(D))u\|_{H^{s-\delta}} \\ &\le C\, 2^{-j\delta}\|(1-\Psi_j(D))u\|_{H^s},\end{aligned} \tag{8.46}$$

the last identity holding since $\{2^{j\delta}\langle\xi\rangle^{-\delta}\big(1-\Psi_j(\xi)\big) : j \in \mathbb{Z}^+\}$ is uniformly bounded. This proves (8.44).

Suppose the norms satisfy $\|u\|_{C^0_*} \le C\|u\|_{H^s}$. If we substitute $\varepsilon^\delta = C^{-1}\|u\|_{C^0_*}/\|u\|_{H^s}$ into (8.44), we obtain the estimate (for a new $C = C(\delta)$)

$$\|u\|_{L^\infty} \le C\|u\|_{C^0_*}\left[1+\log\left(\frac{\|u\|_{H^s}}{\|u\|_{C^0_*}}\right)\right]. \tag{8.47}$$

We note that a number of variants of (8.44) and (8.47) hold. For some of them, it is useful to strengthen the last observation in the proof above to

$$\big\{2^{j\delta}\langle\xi\rangle^{-\delta}\big(1-\Psi_j(\xi)\big) : j \in \mathbb{Z}^+\big\} \text{ is bounded in } S^0_1(\mathbb{R}^n). \tag{8.48}$$

An argument parallel to the proof of Proposition 8.11 gives estimates

$$\|u\|_{C^k(M)} \le C\varepsilon^\delta\|u\|_{H^s(M)} + C\Big(\log\frac{1}{\varepsilon}\Big)\|u\|_{C^k_*(M)}, \tag{8.49}$$

given $k \in \mathbb{Z}^+,\ s > n/2 + k + \delta$, and consequently

$$\|u\|_{C^k(M)} \le C\|u\|_{C^k_*(M)}\left[1+\log\left(\frac{\|u\|_{H^s}}{\|u\|_{C^k_*}}\right)\right] \tag{8.50}$$

when M is a compact manifold without boundary.

We can also establish such an estimate for the $C^k(\overline{\Omega})$-norm when $\overline{\Omega}$ is a compact manifold with boundary. If $\overline{\Omega} \subset M$ as above, this follows easily from (8.50), via:

Lemma 8.12. *For any $r \in (0, N)$,*

$$\|u\|_{C^r_*(\overline{\Omega})} \approx \|Eu\|_{C^r_*(M)}. \tag{8.51}$$

Proof. If $Eu_j \to v$ in $C^r_*(M)$, then $\rho Eu_j \to \rho v$ in $C^r_*(\overline{\Omega})$, that is, $u_j \to \rho v$ in $C^r_*(\overline{\Omega})$, since $\rho Eu_j = u_j$. Thus $v = E\rho v$, in this case. This proves the lemma, which is also equivalent to the statement that E in (8.40) has closed range.

We also have such a result for Sobolev spaces:

$$\|u\|_{H^{r,p}(\Omega)} \approx \|Eu\|_{H^{r,p}(M)}, \quad 1 < p < \infty. \tag{8.52}$$

Thus (8.50) yields

$$\|u\|_{C^k(\overline{\Omega})} \le C\,\|u\|_{C^k_*(\overline{\Omega})}\left[1+\log\left(\frac{\|u\|_{H^s(\Omega)}}{\|u\|_{C^k_*(\overline{\Omega})}}\right)\right], \tag{8.53}$$

provided $s > n/2 + k$.

Exercises

1. Extend the estimates of Theorem 8.9 and Proposition 8.10 to solutions of

$$Pu = f \text{ on } \Omega, \quad B_j u = g_j \text{ on } \partial\Omega. \tag{8.54}$$

Show that, for $r \in (\mu, \infty),\ \mu = \max(m_j)$,

$$f \in C^{r-m}_*(\overline{\Omega}),\ g_j \in C^{r-m_j}_*(\partial\Omega) \Longrightarrow u \in C^r_*(\overline{\Omega}). \tag{8.55}$$

Note that we allow $r - m < 0$, in which case $C^{r-m}_*(\overline{\Omega})$ is defined by the right side of (8.41) (with r replaced by $r - m$).

2. Establish the following result, similar to (8.44):

$$\|u\|_{L^\infty} \le C\varepsilon^\delta\|u\|_{H^{s,p}} + C\Bigl(\log\ \frac{1}{\varepsilon}\Bigr)^{1-1/q}\|u\|_{H^{n/q,q}}, \tag{8.56}$$

where $s > n/p + \delta,\ q \in [2,\infty)$, and a similar estimate for $q \in (1,2]$, using $\bigl(\log\ 1/\varepsilon\bigr)^{1/q}$. (See [BrG] and [BrW].)

3. From (8.15) it follows that $H^{1,p}(\mathbb{R}^n) \subset C^r(\mathbb{R}^n)$ if $p > n,\ r = 1 - n/p$. Demonstrate the following more precise result:

$$|u(x) - u(y)| \le C|x-y|^{1-n/p}\ \|\nabla u\|_{L^p(B_{xy})}, \quad p > n, \tag{8.57}$$

where $B_{xy} = B_{|x-y|}(x) \cap B_{|x-y|}(y)$.
(*Hint*: Apply scaling to (2.16) to obtain

$$|v(re_1) - v(0)| \le C r^{p-n} \int\limits_{B_r(0)} |\nabla v(x)|^p\ dx.$$

To pass from $B_{|x-y|}(x)$ to B_{xy} in (8.57), note what the support of φ is in Exercise 5 of § 2.) There is a stronger estimate, known as *Morrey's inequality*. See Chap. 14 for more on this.

9. Pseudodifferential operators with nonregular symbols

We establish here some results on Hölder and Sobolev space continuity for pseudodifferential operators $p(x,D)$ with symbols $p(x,\xi)$ which are somewhat more ill behaved than those for which we had L^2-Sobolev estimates in Chap. 7 or L^p-Sobolev estimates and Hölder estimates in §§ 5 and 8 of this chapter. These results will be very useful in the analysis of nonlinear, elliptic PDE in Chap. 14 and will also be used in Chaps. 15 and 16.

Let $r \in (0, \infty)$. We say $p(x, \xi) \in C^r_* S^m_{1,\delta}(\mathbb{R}^n)$ provided

$$|D^\alpha_\xi p(x, \xi)| \le C_\alpha \langle \xi \rangle^{m-|\alpha|} \tag{9.1}$$

and

$$\|D^\alpha_\xi p(\cdot, \xi)\|_{C^r_*(\mathbb{R}^n)} \le C_\alpha \langle \xi \rangle^{m-|\alpha|+\delta r}. \tag{9.2}$$

Here $\delta \in [0, 1]$. The following rather strong result is due to G. Bourdaud [Bou], following work of E. Stein [S2].

Theorem 9.1. *If $r > 0$ and $p \in (1, \infty)$, then, for $p(x, \xi) \in C^r_* S^m_{1,1}$,*

$$p(x, D) : H^{s+m,p} \longrightarrow H^{s,p}, \tag{9.3}$$

provided $0 < s < r$. Furthermore, under these hypotheses,

$$p(x, D) : C^{s+m}_* \longrightarrow C^s_*. \tag{9.4}$$

Before giving the proof of this result, we record some implications. Note that any $p(x, \xi) \in S^m_{1,1}$ satisfies the hypotheses for all $r > 0$. Since operators in $OPS^m_{1,\delta}$ possess good multiplicative properties for $\delta \in [0, 1)$, we have the following:

Corollary 9.2. *If $p(x, \xi) \in S^m_{1,\delta}$, $0 \le \delta < 1$, we have the mapping properties (9.3) and (9.4) for all $s \in \mathbb{R}$.*

It is known that elements of $OPS^0_{1,1}$ need not be bounded on L^p, even for $p = 2$, but by duality and interpolation we have the following:

Corollary 9.3. *If $p(x, D)$ and $p(x, D)^*$ belong to $OPS^m_{1,1}$, then (9.3) holds for all $s \in \mathbb{R}$.*

We prepare to prove Theorem 9.1. It suffices to treat the case $m = 0$. Following [Bou] and also [Ma2], we make use of the following results from Littlewood–Paley theory. These results follow from (6.23) and (6.25), respectively.

Lemma 9.4. *Let $f_k \in \mathcal{S}'(\mathbb{R}^n)$ be such that, for some $A > 0$,*

$$\text{supp } \hat{f}_k \subset \{\xi : A \cdot 2^{k-1} \le |\xi| \le A \cdot 2^{k+1}\}, \quad k \ge 1. \tag{9.5}$$

Say $\hat{f}_0$ has compact support. Then, for $p \in (1, \infty)$, $s \in \mathbb{R}$, we have

$$\Big\| \sum_{k=0}^\infty f_k \Big\|_{H^{s,p}} \le C \Big\| \Big\{ \sum_{k=0}^\infty 4^{ks} |f_k|^2 \Big\}^{1/2} \Big\|_{L^p}. \tag{9.6}$$

If $f_k = \varphi_k(D)f$ *with* φ_k *supported in the shell defined in (9.5) and bounded in* $S^0_{1,0}$*, then the converse of the estimate (9.6) also holds.*

Lemma 9.5. *Let* $f_k \in \mathcal{S}'(\mathbb{R}^n)$ *be such that*

$$\text{supp } \hat{f}_k \subset \{\xi : |\xi| \le A \cdot 2^{k+1}\}, \quad k \ge 0. \tag{9.7}$$

Then, for $p \in (1,\infty),\ s > 0$*, we have*

$$\Big\| \sum_{k=0}^{\infty} f_k \Big\|_{H^{s,p}} \le C \Big\| \Big\{ \sum_{k=0}^{\infty} 4^{ks} |f_k|^2 \Big\}^{1/2} \Big\|_{L^p}. \tag{9.8}$$

The next ingredient is a symbol decomposition. We begin with the Littlewood–Paley partition of unity (5.37),

$$1 = \sum \varphi_j(\xi)^2 = \sum \psi_j(\xi), \tag{9.9}$$

and with

$$p(x,\xi) = \sum_{j=0}^{\infty} p(x,\xi)\psi_j(\xi) = \sum_{j=0}^{\infty} p_j(x,\xi). \tag{9.10}$$

Now, let us take a basis of $L^2\big(|\xi_j| < \pi\big)$ of the form

$$e_\alpha(\xi) = e^{i\alpha\cdot\xi},$$

and write (for $j \ge 1$)

$$p_j(x,\xi) = \sum_{\alpha} p_{j\alpha}(x) e_\alpha(2^{-j}\xi)\psi_j^{\#}(\xi), \tag{9.11}$$

where $\psi_1^{\#}(\xi)$ has support on $1/2 < |\xi| < 2$ and is 1 on supp ψ_1, $\psi_j^{\#}(\xi) = \psi_1^{\#}(2^{-j+1}\xi)$, with an analogous decomposition for $p_0(\xi)$. Inserting these decompositions into (9.10) and summing over j, we obtain $p(x,\xi)$ as a sum of a rapidly decreasing sequence of elementary symbols.

By definition, an elementary symbol in $C^r_* S^0_{1,\delta}$ is of the form

$$q(x,\xi) = \sum_{k=0}^{\infty} Q_k(x)\varphi_k(\xi), \tag{9.12}$$

where φ_k is supported on $\langle\xi\rangle \sim 2^k$ and bounded in S^0_1—in fact, $\varphi_k(\xi) = \varphi_1(2^{-k+1}\xi)$, for $k \ge 2$—and $Q_k(x)$ satisfies

$$|Q_k(x)| \le C, \ \|Q_k\|_{C^r_*} \le C \cdot 2^{kr\delta}. \tag{9.13}$$

For the purpose of proving Theorem 9.1, we take $\delta = 1$. It suffices to estimate the $H^{r,p}$-operator norm of $q(x,D)$ when $q(x,\xi)$ is such an elementary symbol.

Set $Q_{kj}(x) = \psi_j(D)Q_k(x)$, with $\{\psi_j\}$ the partition of unity described in (9.9). Set

$$\begin{aligned} q(x,\xi) &= \sum_k \left\{ \sum_{j=0}^{k-4} Q_{kj}(x) + \sum_{j=k-3}^{k+3} Q_{kj}(x) + \sum_{j=k+4}^{\infty} Q_{kj}(x) \right\} \varphi_k(\xi) \\ &= q_1(x,\xi) + q_2(x,\xi) + q_3(x,\xi). \end{aligned} \tag{9.14}$$

We will perform separate estimates of these three pieces. Set $f_k = \varphi_k(D) f$.

First we estimate $q_1(x,D)f$. By Lemma 9.4, since $\langle \xi \rangle \sim 2^j$ on the spectrum of Q_{kj},

$$\begin{aligned} \|q_1(x,D)f\|_{H^{s,p}} &\le C \left\| \left\{ \sum_{k=4}^{\infty} 4^{ks} \Big| \sum_{j=0}^{k-4} Q_{kj} f_k \Big|^2 \right\}^{1/2} \right\|_{L^p} \\ &\le C \left\| \left\{ \sum_{k=4}^{\infty} 4^{ks} \|Q_k\|^2_{L^\infty} |f_k|^2 \right\}^{1/2} \right\|_{L^p} \\ &\le C \|f\|_{H^{s,p}}, \end{aligned} \tag{9.15}$$

for all $s \in \mathbb{R}$.

To estimate $q_2(x,D)f$, note that $\|Q_{kj}\|_{L^\infty} \le C \cdot 2^{-jr+kr}$. Then Lemma 9.5 implies

$$\|q_2(x,D)f\|_{H^{s,p}} \le C \left\| \left\{ \sum_{k=0}^{\infty} 4^{ks} |f_k|^2 \right\}^{1/2} \right\|_{L^p} \le C \|f\|_{H^{s,p}}, \tag{9.16}$$

for $s > 0$.

To estimate $q_3(x,D)f$, we apply Lemma 9.4 to $h_j = \sum_{k=0}^{j-4} Q_{kj} f_k$, to obtain

$$\begin{aligned} \|q_3(x,D)f\|_{H^{s,p}} &\le C \left\| \left\{ \sum_{j=4}^{\infty} 4^{js} \Big| \sum_{k=0}^{j-4} Q_{kj} f_k \Big|^2 \right\}^{1/2} \right\|_{L^p} \\ &\le C \left\| \left\{ \sum_{j=4}^{\infty} 4^{j(s-r)} \left(\sum_{k=0}^{j-4} 2^{kr} |f_k| \right)^2 \right\}^{1/2} \right\|_{L^p}. \end{aligned} \tag{9.17}$$

Now, if we set $g_j = \sum_{k=0}^{j-4} 2^{(k-j)r} |f_k|$ and then set $G_j = 2^{js} g_j$ and $F_j = 2^{js} |f_j|$, we see that

$$G_j = \sum_{k=0}^{j-4} 2^{(k-j)(r-s)} F_k.$$

As long as $r > s$, Young's inequality (see Exercise 1 at the end of this section) yields $\|(G_j)\|_{\ell^2} \leq C \|(F_j)\|_{\ell^2}$, so the last line in (9.17) is bounded by

$$C \Big\| \Big\{ \sum_{j=0}^{\infty} 4^{js} |f_j|^2 \Big\}^{1/2} \Big\|_{L^p} \leq C \|f\|_{H^{s,p}}.$$

This proves (9.3).

The proof of (9.4) is similar. We replace (9.6) by

$$\|f\|_{C^r_*} \sim \sup_{k \geq 0} \, 2^{kr} \|\psi_k(D) f\|_{L^\infty}, \quad r > 0. \tag{9.18}$$

We also need an analogue of Lemma 9.5:

Lemma 9.6. *If $f_k \in \mathcal{S}'(\mathbb{R}^n)$ and supp $\hat{f}_k \subset \{\xi : |\xi| \leq A \cdot 2^{k+1}\}$, then, for $r > 0$,*

$$\Big\| \sum_{k=0}^{\infty} f_k \Big\|_{C^r_*} \leq C \sup_{k \geq 0} \, 2^{kr} \|f_k\|_{L^\infty}. \tag{9.19}$$

Proof. For some finite N, we have $\psi_j(D) \sum_{k \geq 0} f_k = \psi_j(D) \sum_{k \geq j-N} f_k$. Suppose $\sup_k \, 2^{kr} \|f_k\|_{L^\infty} = S$. Then

$$\Big\| \psi_j(D) \sum_{k \geq 0} f_k \Big\|_{L^\infty} \leq CS \sum_{k \geq j-N} 2^{-kr} \leq C' S 2^{-jr}.$$

This proves (9.19).

Now, to prove (9.4), as before it suffices to consider elementary symbols, of the form (9.12)–(9.13), and we use again the decomposition $q(x,\xi) = q_1 + q_2 + q_3$ of (9.14). Thus it remains to obtain analogues of the estimates (9.15)–(9.17).

Parallel to (9.15), using the fact that $\sum_{j=0}^{k-4} Q_{kj}(x) f_k$ has spectrum in the shell $\langle \xi \rangle \sim 2^k$, and $\|Q_k\|_{L^\infty} \leq C$, we obtain

$$\begin{aligned} \|q_1(x,D) f\|_{C^s_*} &\leq C \sup_{k \geq 0} \, 2^{ks} \Big\| \sum_{j=0}^{k-4} Q_{kj} f_k \Big\|_{L^\infty} \\ &\leq C \sup_{k \geq 0} \, 2^{ks} \|f_k\|_{L^\infty} \\ &\leq C \|f\|_{C^s_*}, \end{aligned} \tag{9.20}$$

for all $s \in \mathbb{R}$. Parallel to (9.16), using $\|Q_{kj}\|_{L^\infty} \le C \cdot 2^{-jr+kr}$ and Lemma 9.6, we have

$$\begin{aligned} \|q_2(x,D)f\|_{C_*^s} &\le \Big\| \sum_{k=0}^{\infty} g_k \Big\|_{C_*^s} \\ &\le C \sup_{k\ge 0}\ 2^{ks}\|g_k\|_{L^\infty} \\ &\le C \sup_{k\ge 0}\ 2^{ks}\|f_k\|_{L^\infty} \le C\|f\|_{C_*^s}, \end{aligned} \tag{9.21}$$

for all $s > 0$, where the sum of seven terms

$$g_k = \sum_{j=k-3}^{k+3} Q_{kj}(x) f_k$$

has spectrum contained in $|\xi| \le C \cdot 2^k$, and $\|g_k\|_{L^\infty} \le C\|f_k\|_{L^\infty}$.

Finally, parallel to (9.17), since $\sum_{k=0}^{j-4} Q_{kj} f_k$ has spectrum in the shell $\langle\xi\rangle \sim 2^j$, we have

$$\begin{aligned} \|q_3(x,D)f\|_{C_*^s} &\le C \sup_{j\ge 0}\ 2^{js}\Big\| \sum_{k=0}^{j-4} Q_{kj} f_k \Big\|_{L^\infty} \\ &\le C \sup_{j\ge 0}\ 2^{j(s-r)} \sum_{k=0}^{j-4} 2^{kr}\|f_k\|_{L^\infty}. \end{aligned} \tag{9.22}$$

If we bound this last sum by

$$\Big[\sum_{k=0}^{j-4} 2^{k(r-s)}\Big] \sup_k\ 2^{ks}\|f_k\|_{L^\infty}, \tag{9.23}$$

then

$$\|q_3(x,D)f\|_{C_*^s} \le C\Big[\sup_{j\ge 0}\ 2^{j(s-r)} \sum_{k=0}^{j-4} 2^{k(r-s)}\Big]\|f\|_{C_*^s}, \tag{9.24}$$

and the factor in brackets is finite as long as $s < r$. The proof of Theorem 9.1 is complete.

Things barely blow up in (9.24) when $s = r$. We will establish the following result here. A sharper result (for $p(x,\xi) \in C_*^r S_{1,\delta}^m$ with $\delta < 1$) is given in (9.43).

Proposition 9.7. *If* $p(x,\xi) \in C^r_* S^m_{1,1}$, *then*

$$p(x,D) : C^{m+r+\varepsilon}_* \longrightarrow C^r_*, \ \ \textit{for all } \varepsilon > 0. \tag{9.25}$$

Proof. It suffices to treat the case $m = 0$. We follow the proof of (9.4). The estimates (9.20) and (9.21) continue to work; (9.22) yields

$$\begin{aligned} \|q_3(x,D)f\|_{C^r_*} &\leq C \sup_{j\geq 0} \sum_{k=0}^{j-4} 2^{kr} \|f_k\|_{L^\infty} \\ &= C \sum_{k=0}^{\infty} 2^{kr} \|f_k\|_{L^\infty} \\ &\leq C \sum_{k=0}^{\infty} 2^{kr} \cdot 2^{-kr-k\varepsilon} \|f\|_{C^{r+\varepsilon}_*}, \end{aligned} \tag{9.26}$$

which proves (9.25).

The way symbols in $C^r_* S^m_{1,\delta}$ most frequently arise is the following. One has in hand a symbol $p(x,\xi) \in C^r_* S^m_{1,0}$, such as the symbol of a *differential* operator, with Hölder-continuous coefficients. One is then motivated to decompose $p(x,\xi)$ as a sum

$$p(x,\xi) = p^\#(x,\xi) + p^b(x,\xi), \tag{9.27}$$

where $p^\#(x,\xi) \in S^m_{1,\delta}$, for some $\delta \in (0,1)$, and there is a good operator calculus for $p^\#(x,D)$, while $p^b(x,\xi) \in C^r_* S^\mu_{1,\delta}$ (for some $\mu < m$) is treated as a remainder term, to be estimated. We will refer to this construction as *symbol smoothing*.

The symbol decomposition (9.27) is constructed as follows. Use the partition of unity $\psi_j(\xi)$ of (9.9). Given $p(x,\xi) \in C^r_* S^m_{1,0}$, choose $\delta \in (0,1]$ and set

$$p^\#(x,\xi) = \sum_{j=0}^{\infty} J_{\varepsilon_j} p(x,\xi)\, \psi_j(\xi), \tag{9.28}$$

where J_ε is a smoothing operator on functions of x, namely

$$J_\varepsilon f(x) = \phi(\varepsilon D) f(x), \tag{9.29}$$

with $\phi \in C^\infty_0(\mathbb{R}^n)$, $\phi(\xi) = 1$ for $|\xi| \leq 1$ (e.g., $\phi = \psi_0$), and we take

$$\varepsilon_j = 2^{-j\delta}. \tag{9.30}$$

We then define $p^b(x,\xi)$ to be $p(x,\xi) - p^\#(x,\xi)$, yielding (9.27).

To analyze these terms, we use the following simple result.

Lemma 9.8. *For* $\varepsilon \in (0, 1]$,

$$\|D_x^\beta J_\varepsilon f\|_{C_*^s} \le C_\beta\, \varepsilon^{-|\beta|} \|f\|_{C_*^s} \tag{9.31}$$

and

$$\|f - J_\varepsilon f\|_{C_*^{s-t}} \le C\varepsilon^t \|f\|_{C_*^s}, \quad \textit{for } t \ge 0. \tag{9.32}$$

Furthermore, if $s > 0$,

$$\|f - J_\varepsilon f\|_{L^\infty} \le C_s \varepsilon^s \|f\|_{C_*^s}. \tag{9.33}$$

Proof. The estimate (9.31) follows from the fact that, for each $\beta \ge 0$,

$$\varepsilon^{|\beta|} D_x^\beta \phi(\varepsilon D) \text{ is bounded in } OPS^0_{1,0},$$

and the estimate (9.32) follows from the fact that, with $\Lambda = (1-\Delta)^{1/2}$,

$$\Lambda^t : C_*^s \longrightarrow C_*^{s-t} \text{ isomorphically,}$$

plus the fact that

$$\varepsilon^{-t}\Lambda^{-t}(1-\phi(\varepsilon D)) \text{ is bounded in } OPS^0_{1,0},$$

for $0 < \varepsilon \le 1$. As for (9.33), if $\varepsilon \sim 2^{-j}$, we have

$$\|\big(1-\phi(\varepsilon D)\big) f\|_{L^\infty} \le \sum_{\ell \ge j} \|\psi_\ell(D) f\|_{L^\infty} \le C \sum_{\ell \ge j} 2^{-\ell s} \|f\|_{C_*^s},$$

and since $\sum_{\ell \ge j} 2^{-\ell s} \le C_s 2^{-js}$ for $s > 0$, (9.33) follows.

Using this, we easily derive the following conclusion:

Proposition 9.9. *If* $p(x, \xi) \in C_*^r S^m_{1,0}$, *then, in the decomposition (9.27),*

$$p^\#(x, \xi) \in S^m_{1,\delta} \tag{9.34}$$

and

$$p^b(x, \xi) \in C_*^r S^{m-r\delta}_{1,\delta}. \tag{9.35}$$

Proof. The estimate (9.31) yields

$$\|D_x^\beta D_\xi^\alpha p^\#(\cdot, \xi)\|_{C_*^r} \le C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|+\delta|\beta|}, \tag{9.36}$$

which implies (9.34).

That $p^b(x,\xi)$ satisfies an estimate of the form (9.2), with m replaced by $m-r\delta$, follows from (9.32), with $t=0$. That it satisfies (9.1), with m replaced by $m-r\delta$, is a consequence of the estimate (9.33).

It will also be useful to smooth out a symbol $p(x,\xi)\in C^r_*S^m_{1,\delta}$, for $\delta\in(0,1)$. Pick $\gamma\in(\delta,1)$, and apply (9.28), with $\varepsilon_j=2^{-j(\gamma-\delta)}$, obtaining $p^\#(x,\xi)$ and hence a decomposition of the form (9.27). In this case, we obtain

$$p(x,\xi)\in C^r_*S^m_{1,\delta}\Longrightarrow p^\#(x,\xi)\in S^m_{1,\gamma},\quad p^b(x,\xi)\in C^r_*S^{m-(\gamma-\delta)r}_{1,\gamma}. \tag{9.37}$$

We use the symbol decomposition (9.27) to establish the following variant of Theorem 9.1, which will be most useful in Chap. 14.

Proposition 9.10. *If $\delta\in[0,1)$ and $p(x,\xi)\in C^r_*S^m_{1,\delta}$, then*

$$\begin{aligned} p(x,D)&:H^{s+m,p}\longrightarrow H^{s,p},\\ p(x,D)&:C^{s+m}_*\longrightarrow C^s_*, \end{aligned} \tag{9.38}$$

provided $p\in(1,\infty)$ and

$$-(1-\delta)r<s<r. \tag{9.39}$$

Proof. The result follows directly from Theorem 9.1 if $0<s<r$, so it remains to consider $s\in\big(-(1-\delta)r,0\big]$. Use the decomposition (9.27), $p=p^\#+p^b$, with (9.37) holding. Thus $p^\#(x,D)$ has the mapping property (9.38) for all $s\in\mathbb{R}$. Applying Theorem 9.1 to $p^b(x,D)$ yields mapping properties such as

$$p^b(x,D):H^{\sigma+m-(\gamma-\delta)r,p}\longrightarrow H^{\sigma,p},\quad \sigma>0,$$

or, setting $s=\sigma-(\gamma-\delta)r$,

$$p^b(x,D):H^{s+m,p}\longrightarrow H^{s+(\gamma-\delta)r,p}\subset H^{s,p},\quad s>-(\gamma-\delta)r,$$

and similar results on C^{s+m}_*. Then letting $\gamma\nearrow 1$ completes the proof of (9.38).

Recall that, for $r\in(0,\infty)$, we have defined $p(x,\xi)$ to belong to the space $C^r_*S^m_{1,\delta}(\mathbb{R}^n)$ provided the estimates (9.1) and (9.2) hold. If $r\in[0,\infty)$, we will say that $p(x,\xi)\in C^rS^m_{1,\delta}(\mathbb{R}^n)$ provided that (9.1)–(9.2) hold and, additionally,

$$\|D^\alpha_\xi p(\cdot,\xi)\|_{C^j(\mathbb{R}^n)}\le C_\alpha\,\langle\xi\rangle^{m-|\alpha|+j\delta},\quad 0\le j\le r,\quad j\in\mathbb{Z}. \tag{9.40}$$

In particular, we make a semantic distinction between $C^r_*S^m_{1,\delta}$ and $C^rS^m_{1,\delta}$ even when $r\notin\mathbb{Z}^+$, in which cases C^r_* and C^r coincide. The differences between the two symbol classes are minor, especially when $r\notin\mathbb{Z}$, but natural examples of symbols often do have this additional property, and we sometimes use the symbol classes just defined to record this fact.

Exercises

1. Young's inequality implies

$$\|f * g\|_{\ell^q} \le \|f\|_{\ell^1}\ \|g\|_{\ell^q},$$

where $f = (f_j),\ g = (g_j)$, and $(f * g)_j = \sum_k f_{j-k} g_k$. Show how this applies (with $q = 2$) to the estimate of (9.17).

2. Supplement Lemma 9.8 with the estimates

$$\begin{aligned} \|D_x^\beta J_\varepsilon f\|_{L^\infty} \le C\|f\|_{C^s}, &\qquad |\beta| \le s, \\ C\varepsilon^{-(|\beta|-s)}\ \|f\|_{C_*^s}, &\qquad |\beta| > s, \end{aligned} \tag{9.41}$$

given $s > 0$.

3. Show that if $p(x,\xi) \in C_*^r S_{1,0}^m$ has the decomposition (9.27), then

$$\begin{aligned} D_x^\beta p^\#(x,\xi) \in\ S_{1,\delta}^m, &\qquad \text{for } |\beta| < r, \\ S_{1,\delta}^{m+\delta(|\beta|-r)}, &\qquad \text{for } |\beta| > r. \end{aligned} \tag{9.42}$$

4. Strengthen part of Proposition 9.10 to obtain, for $\delta \in [0,1),\ r > 0$,

$$p(x,\xi) \in C_*^r S_{1,\delta}^m \Longrightarrow p(x,D) : C_*^{s+m} \longrightarrow C_*^s,\ \text{for } -(1-\delta)r < s \le r. \tag{9.43}$$

(*Hint*: Apply Proposition 9.7 to $p^b(x,D)$, arising in (9.37).)

5. Given $s \in \mathbb{R},\ 1 \le p,q \le \infty$, we say $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the *Triebel space* $F_{p,q}^s(\mathbb{R}^n)$ provided

$$\|f\|_{F_{p,q}^s} = \left\|\{2^{js}\psi_j(D)f\}\right\|_{L^p(\mathbb{R}^n,\ell^q)} < \infty, \tag{9.44}$$

where $\{\psi_j\}$ is the partition of unity (9.9). Note that $F_{p,2}^s = H^{s,p}$ if $1 < p < \infty$, by Lemma 9.4. Also, we say that $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the *Besov space* $B_{p,q}^s(\mathbb{R}^n)$ provided

$$\|f\|_{B_{p,q}^s} = \left\|\{2^{js}\psi_j(D)f\}\right\|_{\ell^q(L^p(\mathbb{R}^n))} < \infty. \tag{9.45}$$

Note that $B_{\infty,\infty}^s = C_*^s$. Also, $B_{2,2}^s = H^s$, since $\ell^2(L^2(\mathbb{R}^n)) = L^2(\mathbb{R}^n,\ell^2)$. Extend Theorem 9.1 to results of the form

$$p(x,D) : F_{p,q}^{s+m} \to F_{p,q}^s, \quad p(x,D) : B_{p,q}^{s+m} \to B_{p,q}^s.$$

(See [Ma1].)

6. We define the symbol class $C_*^r S_{cl}^m$ to consist of $p(x,\xi) \in C_*^r S_{1,0}^m$ such that

$$p(x,\xi) \sim \sum_{j\ge 0} p_j(x,\xi) \tag{9.46}$$

where $p_j(x,\xi) \in C_*^r S_{1,0}^m$ is homogeneous of degree $m-j$ in ξ, for $|\xi| \ge 1$, and (9.46) means that the difference between the left side and the sum over $0 \le j < N$ belongs to $C_*^r S_{1,0}^{m-N}$. If $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, we also denote the symbol class by $C^r S_{cl}^m$. Show that estimates of the form (9.3) and (9.4) have simpler proofs in this case, derived from expansions of the form

$$p_j(x,\xi) = \sum_\nu p_{j\nu}(x)|\xi|^{m-j}\omega_\nu\big(|\xi|^{-1}\xi\big), \tag{9.47}$$

for $|\xi| \geq 1$, where $\{\omega_\nu\}$ is an orthonormal basis of $L^2(S^{n-1})$ consisting of eigenfunctions of the Laplace operator.

10. Paradifferential operators

Here we develop the paradifferential operator calculus, introduced by J.-M. Bony in [Bon]. We begin with Y. Meyer's ingenious formula for $F(u)$ as $M(x,D)u+R$ where F is smooth in its argument(s), u belongs to a Hölder or Sobolev space, $M(x,D)$ is a pseudodifferential operator of type $(1,1)$, and R is smooth. From there, one applies symbol smoothing to $M(x,\xi)$ and makes use of results established in § 9.

Following [Mey], we discuss the connection between $F(u)$, for smooth nonlinear F, and the action on u of certain pseudodifferential operators of type $(1,1)$. Let $\psi_j(\xi) = \varphi_j(\xi)^2$ be the Littlewood–Paley partition of unity (5.37), and set $\Psi_k(\xi) = \sum_{j\le k}\psi_j(\xi)$. Given u (e.g., in $C^r(\mathbb{R}^n)$), set

$$u_k = \Psi_k(D)u, \tag{10.1}$$

and write

$$F(u) = F(u_0) + [F(u_1) - F(u_0)] + \cdots + [F(u_{k+1}) - F(u_k)] + \cdots . \tag{10.2}$$

Then write

$$\begin{aligned} F(u_{k+1}) - F(u_k) &= F\big(u_k + \psi_{k+1}(D)u\big) - F(u_k) \\ &= m_k(x)\psi_{k+1}(D)u, \end{aligned} \tag{10.3}$$

where

$$m_k(x) = \int_0^1 F'\big(\Psi_k(D)u + t\psi_{k+1}(D)u\big)\,dt. \tag{10.4}$$

Consequently, we have

$$\begin{aligned} F(u) &= F(u_0) + \sum_{k=0}^\infty m_k(x)\psi_{k+1}(D)u \\ &= M(x,D)u + F(u_0), \end{aligned} \tag{10.5}$$

where

$$M(x,\xi) = \sum_{k=0}^{\infty} m_k(x)\psi_{k+1}(\xi) = M_F(u;x,\xi). \tag{10.6}$$

We claim

$$M(x,\xi) \in S^0_{1,1}, \tag{10.7}$$

provided u is continuous. To estimate $M(x,\xi)$, note first that by (10.4)

$$\|m_k\|_{L^\infty} \le \sup |F'(\lambda)|. \tag{10.8}$$

To estimate higher derivatives, we use the elementary estimate

$$\|D^\ell g(h)\|_{L^\infty} \le C \sum_{\ell_1+\cdots+\ell_\nu \le \ell} \|g'\|_{C^{\nu-1}} \|D^{\ell_1} h\|_{L^\infty} \cdots \|D^{\ell_\nu} h\|_{L^\infty} \tag{10.9}$$

to obtain

$$\|D^\ell_x m_k\|_{L^\infty} \le C_\ell \|F''\|_{C^{\ell-1}} \langle \|u\|_{L^\infty}\rangle^{\ell-1} \cdot 2^{k\ell}, \tag{10.10}$$

granted the following estimates, which hold for all $u \in L^\infty$:

$$\|\Psi_k(D)u + t\psi_{k+1}(D)u\|_{L^\infty} \le C\|u\|_{L^\infty} \tag{10.11}$$

and

$$\|D^\ell[\Psi_k(D)u + t\psi_{k+1}(D)u]\|_{L^\infty} \le C_\ell 2^{k\ell}\|u\|_{L^\infty} \tag{10.12}$$

for $t \in [0,1]$. Consequently, (10.6) yields

$$|D^\alpha_\xi M(x,\xi)| \le C_\alpha \sup_\lambda |F'(\lambda)| \langle\xi\rangle^{-|\alpha|} \tag{10.13}$$

and, for $|\beta| \ge 1$,

$$|D^\beta_x D^\alpha_\xi M(x,\xi)| \le C_{\alpha\beta} \|F''\|_{C^{|\beta|-1}} \langle \|u\|_{L^\infty}\rangle^{|\beta|-1} \langle\xi\rangle^{|\beta|-|\alpha|}. \tag{10.14}$$

We give a formal statement of the result just established.

Proposition 10.1. *If F is C^∞ and $u \in C^r$ with $r \ge 0$, then*

$$F(u) = M_F(u;x,D)u + R(u), \tag{10.15}$$

where

$$R(u) = F(\psi_0(D)u) \in C^\infty$$

and

$$M_F(u; x, \xi) = M(x, \xi) \in S^0_{1,1}. \tag{10.16}$$

Following [Bon] and [Mey], we call $M_F(u; x, D)$ a *paradifferential operator*.

Applying Theorem 9.1, we have

$$\|M(x, D)f\|_{H^{s,p}} \le K\|f\|_{H^{s,p}}, \tag{10.17}$$

for $p \in (1, \infty)$, $s > 0$, with

$$K = K_N(F, u) = C\|F'\|_{C^N}[1 + \|u\|^N_{L^\infty}], \tag{10.18}$$

provided $0 < s < N$, and similarly

$$\|M(x, D)f\|_{C^s_*} \le K\|f\|_{C^s_*}. \tag{10.19}$$

Using $f = u$, we have the following important Moser-type estimates, extending Proposition 3.9:

Proposition 10.2. *If F is smooth with $\|F'\|_{C^N(\mathbb{R})} < \infty$, and $0 < s < N$, then*

$$\|F(u)\|_{H^{s,p}} \le K_N(F, u)\|u\|_{H^{s,p}} + \|R(u)\|_{H^{s,p}} \tag{10.20}$$

and

$$\|F(u)\|_{C^s_*} \le K_N(F, u)\|u\|_{C^s_*} + \|R(u)\|_{C^s_*}, \tag{10.21}$$

given $1 < p < \infty$, with $K_N(F, u)$ as in (10.18).

This expression for $K_N(F, u)$ involves the L^∞-norm of u, and one can use $\|F'\|_{C^N(I)}$, where I contains the range of u. Note that if $F(u) = u^2$, then $F'(u) = 2u$, and higher powers of $\|u\|_{L^\infty}$ do not arise; hence we obtain the estimate

$$\|u^2\|_{H^{s,p}} \le C_s\|u\|_{L^\infty} \cdot \|u\|_{H^{s,p}}, \quad s > 0, \tag{10.22}$$

and a similar estimate on $\|u^2\|_{C^s_*}$.

It will be useful to have further estimates on the symbol $M(x, \xi) = M_F(u; x, \xi)$ when $u \in C^r$ with $r > 0$. The estimate (10.12) extends to

$$\left\|D^\ell\left[\Psi_k(D)f + t\psi_{k+1}(D)f\right]\right\|_{L^\infty} \le \begin{array}{ll} C_\ell\|f\|_{C^r}, & \ell \le r, \\ C_\ell 2^{k(\ell-r)}\|f\|_{C^r}, & \ell > r, \end{array} \tag{10.23}$$

so we have, when $u \in C^r$,

$$|D_x^\beta D_\xi^\alpha M(x,\xi)| \le \begin{array}{ll} K_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}, & |\beta| \le r, \\ K_{\alpha\beta}\langle\xi\rangle^{-|\alpha|+|\beta|-r}, & |\beta| > r, \end{array} \tag{10.24}$$

with

$$K_{\alpha\beta} = K_{\alpha\beta}(F,u) = C_{\alpha\beta}\|F'\|_{C^{|\beta|}}[1 + \|u\|_{C^r}^{|\beta|}]. \tag{10.25}$$

Also, since $\Psi_k(D) + t\psi_{k+1}(D)$ is uniformly bounded on C^r, for $t \in [0,1]$ and $k \ge 0$, we have

$$\|D_\xi^\alpha M(\cdot,\xi)\|_{C^r} \le K_{\alpha r}\langle\xi\rangle^{-|\alpha|}, \tag{10.26}$$

where $K_{\alpha r}$ is as in (10.25), with $|\beta| = [r]+1$. This last estimate shows that

$$u \in C^r \Longrightarrow M_F(u;x,\xi) \in C^r S^0_{1,0}. \tag{10.27}$$

This is useful additional information; for example, (10.17) and (10.19) hold for $s > -r$, and of course we can apply the symbol smoothing of § 9.

It will be useful to have terminology expressing the structure of the symbols we produce. Given $r \ge 0$, we say

$$\begin{aligned} p(x,\xi) \in \mathcal{A}^r S^m_{1,\delta} \Longleftrightarrow \|D_\xi^\alpha p(\cdot,\xi)\|_{C^r} &\le C_\alpha\langle\xi\rangle^{m-|\alpha|} \quad \text{and} \\ |D_x^\beta D_\xi^\alpha p(x,\xi)| &\le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|+\delta(|\beta|-r)}, \quad |\beta| > r. \end{aligned} \tag{10.28}$$

Thus (10.24)–(10.26) yield

$$M(x,\xi) \in \mathcal{A}^r S^0_{1,1} \tag{10.29}$$

for the $M(x,\xi)$ of Proposition 10.1. If $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, the class $\mathcal{A}^r S^m_{1,1}$ coincides with the symbol class denoted by $\mathcal{A}^m_r$ by Meyer [Mey]. Clearly, $\mathcal{A}^0 S^m_{1,\delta} = S^m_{1,\delta}$, and

$$\mathcal{A}^r S^m_{1,\delta} \subset C^r S^m_{1,0} \cap S^m_{1,\delta}.$$

Also, from the definition we see that

$$p(x,\xi) \in \mathcal{A}^r S^m_{1,\delta} \Longrightarrow D_x^\beta p(x,\xi) \in \begin{array}{ll} S^m_{1,\delta}, & \text{for } |\beta| \le r, \\ S^{m+\delta(|\beta|-r)}_{1,\delta}, & \text{for } |\beta| \ge r. \end{array} \tag{10.30}$$

It is also natural to consider a slightly smaller symbol class:

$$p(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta} \Longleftrightarrow \|D_\xi^\alpha p(\cdot,\xi)\|_{C^{r+s}} \le C_{\alpha s}\langle\xi\rangle^{m-|\alpha|+\delta s}, \quad s \ge 0. \tag{10.31}$$

Considering the cases $s = 0$ and $s = |\beta| - r$, we see that

$$\mathcal{A}_0^r S^m_{1,\delta} \subset \mathcal{A}^r S^m_{1,\delta}.$$

We also say

$$p(x,\xi) \in {}^r S^m_{1,\delta} \iff \text{the right side of (10.30) holds,} \tag{10.32}$$

so

$$\mathcal{A}^r S^m_{1,\delta} \subset {}^r S^m_{1,\delta}.$$

The following result refines (10.29).

Proposition 10.3. *For the symbol $M(x,\xi) = M_F(u; x,\xi)$ of Proposition 10.1, we have*

$$M(x,\xi) \in \mathcal{A}_0^r S^0_{1,1}, \tag{10.33}$$

provided $u \in C^r,\ r \geq 0$.

Proof. For this, we need

$$\|m_k\|_{C^{r+s}} \leq C \cdot 2^{ks}. \tag{10.34}$$

Now, extending (10.9), we have

$$\|g(h)\|_{C^{r+s}} \leq C\|g\|_{C^N}[1 + \|h\|^N_{L^\infty}](\|h\|_{C^{r+s}} + 1), \tag{10.35}$$

with $N = [r+s]+1$, as a consequence of (10.21) when $r+s$ is not an integer, and by (10.9) when it is. This gives, via (10.4),

$$\|m_k\|_{C^{r+s}} \leq C(\|u\|_{L^\infty}) \sup_{t\in I} \|(\Psi_k + t\psi_{k+1})u\|_{C^{r+s}}, \tag{10.36}$$

where $I = [0,1]$. However,

$$\|(\Psi_k + t\psi_{k+1})u\|_{C^{r+s}} \leq C \cdot 2^{ks}\|u\|_{C^r}. \tag{10.37}$$

For $r+s \in \mathbb{Z}^+$, this follows from (9.41); for $r+s \notin \mathbb{Z}^+$, it follows as in the proof of Lemma 9.8, since

$$2^{-ks}\Lambda^s(\Psi_k + t\psi_{k+1}) \ \text{ is bounded in } OPS^0_{1,0}. \tag{10.38}$$

This establishes (10.34), and hence (10.33) is proved.

Returning to symbol smoothing, if we use the method of § 9 to write

$$M(x,\xi) = M^{\#}(x,\xi) + M^{b}(x,\xi), \tag{10.39}$$

then (10.27) implies

$$M^{\#}(x,\xi) \in S^{m}_{1,\delta}, \quad M^{b}(x,\xi) \in C^{r} S^{m-r\delta}_{1,\delta}. \tag{10.40}$$

We now refine these results; for $M^{\#}$ we have a general result:

Proposition 10.4. *For the symbol decomposition defined by the formulas (9.27)–(9.30),*

$$p(x,\xi) \in C^{r} S^{m}_{1,0} \Longrightarrow p^{\#}(x,\xi) \in \mathcal{A}^{r}_{0} S^{m}_{1,\delta}. \tag{10.41}$$

Proof. This is a simple modification of (9.42) which essentially says that $p^{\#}(x,\xi) \in \mathcal{A}^{r} S^{m}_{1,\delta}$; we simply supplement (9.41) with

$$\|J_{\varepsilon} f\|_{C^{r+s}_{*}} \le C\, \varepsilon^{-s} \|f\|_{C^{r}_{*}},\ s \ge 0, \tag{10.42}$$

which is basically the same as (10.37).

To treat $M^{b}(x,\xi)$, we have, for $\delta \le \gamma$,

$$p(x,\xi) \in \mathcal{A}^{r}_{0} S^{m}_{1,\gamma} \Longrightarrow p^{b}(x,\xi) \in C^{r} S^{m-\delta r}_{1,\delta} \cap \mathcal{A}^{r}_{0} S^{m}_{1,\gamma} \subset S^{m-\delta r}_{1,\gamma}, \tag{10.43}$$

where containment in $C^{r} S^{m-\delta r}_{1,\delta}$ follows from (9.35). To see the last inclusion, note that for $p^{b}(x,\xi)$ to belong to the intersection above implies

$$\begin{aligned} \|D^{\alpha}_{\xi} p^{b}(\cdot,\xi)\|_{C^{s}} \le\ & C \langle\xi\rangle^{m-|\alpha|-\delta r+\delta s}, && \text{for } 0 \le s \le r, \\ & C \langle\xi\rangle^{m-|\alpha|+(s-r)\gamma}, && \text{for } s \ge r. \end{aligned} \tag{10.44}$$

In particular, these estimates imply $p^{b}(x,\xi) \in S^{m-r\delta}_{1,\gamma}$. This proves the following:

Proposition 10.5. *For the symbol $M(x,\xi) = M_{F}(u;x,\xi)$ with decomposition (10.39),*

$$u \in C^{r} \Longrightarrow M^{b}(x,\xi) \in S^{-r\delta}_{1,1}. \tag{10.45}$$

Results discussed above extend easily to the case of a function F of several variables, say $u = (u_1,\dots,u_L)$. Directly extending (10.2)–(10.6), we have

$$F(u) = \sum_{j=1}^{L} M_j(x,D)u_j + F(\Psi_0(D)u), \tag{10.46}$$

with

$$M_j(x,\xi) = \sum_k m_k^j(x)\psi_{k+1}(\xi), \tag{10.47}$$

where

$$m_k^j(x) = \int_0^1 (\partial_j F)\big(\Psi_k(D)u + t\psi_{k+1}(D)u\big)\, dt. \tag{10.48}$$

Clearly, the results established above apply to the $M_j(x,\xi)$ here; for example,

$$u \in C^r \Longrightarrow M_j(x,\xi) \in \mathcal{A}_0^r S_{1,1}^m. \tag{10.49}$$

In the particular case $F(u,v) = uv$, we obtain

$$uv = A(u;x,D)v + A(v;x,D)u + \Psi_0(D)u \cdot \Psi_0(D)v, \tag{10.50}$$

where

$$A(u;x,\xi) = \sum_{k=1}^{\infty} \Big[\Psi_k(D)u + \frac{1}{2}\psi_{k+1}(D)u\Big]\psi_{k+1}(\xi). \tag{10.51}$$

Since this symbol belongs to $S_{1,1}^0$ for $u \in L^\infty$, we obtain the following extension of (10.22), which generalizes the Moser estimate (3.21):

Corollary 10.6. *For $s > 0,\ 1 < p < \infty$, we have*

$$\|uv\|_{H^{s,p}} \le C\big[\|u\|_{L^\infty}\|v\|_{H^{s,p}} + \|u\|_{H^{s,p}}\|v\|_{L^\infty}\big]. \tag{10.52}$$

We now analyze a nonlinear differential operator in terms of a paradifferential operator. If F is smooth in its arguments, in analogy with (10.46)–(10.48) we have

$$F(x, D^m u) = \sum_{|\alpha| \le m} M_\alpha(x,D)D^\alpha u + F(x, D^m \Psi_0(D)u), \tag{10.53}$$

where $F(x, D^m\Psi_0(D)u) \in C^\infty$ and

$$M_\alpha(x,\xi) = \sum_k m_k^\alpha(x)\psi_{k+1}(\xi), \tag{10.54}$$

with

$$m_k^\alpha(x) = \int_0^1 (\partial F/\partial \zeta_\alpha)\big(\Psi_k(D)D^m u + t\psi_{k+1}(D)D^m u\big)\, dt. \tag{10.55}$$

As in Propositions 10.1 and 10.3, we have, for $r \geq 0$,

$$u \in C^{m+r} \Longrightarrow M_\alpha(x, \xi) \in \mathcal{A}_0^r S_{1,1}^0 \subset S_{1,1}^0 \cap C^r S_{1,0}^0. \tag{10.56}$$

In other words, if we set

$$M(u; x, D) = \sum_{|\alpha| \leq m} M_\alpha(x, D) D^\alpha, \tag{10.57}$$

we obtain

Proposition 10.7. *If* $u \in C^{m+r}$, $r \geq 0$, *then*

$$F(x, D^m u) = M(u; x, D)u + R, \tag{10.58}$$

with $R \in C^\infty$ *and*

$$M(u; x, \xi) \in \mathcal{A}_0^r S_{1,1}^m \subset S_{1,1}^m \cap C^r S_{1,0}^m. \tag{10.59}$$

As in Propositions 10.4 and 10.5, in this case symbol smoothing yields

$$M(u; x, \xi) = M^\#(x, \xi) + M^b(x, \xi), \tag{10.60}$$

with

$$M^\#(x, \xi) \in \mathcal{A}_0^r S_{1,\delta}^m, \quad M^b(x, \xi) \in S_{1,1}^{m-r\delta}. \tag{10.61}$$

A specific choice for symbol smoothing which leads to paradifferential operators of [Bon] and [Mey] is the following operation on $M(x, \xi)$:

$$M^\#(x, \xi) = \sum_k \Psi_{k-5} M(x, \xi)\, \psi_k(\xi), \tag{10.62}$$

where, as in (9.28), Ψ_{k-5} acts on $M(x, \xi)$ as a function of x. We use $\Psi_{k-5} = \Psi_{k-5}(D)$, with $\Psi_\ell(\xi) = \sum_{j \leq \ell} \psi_j(\xi)$. We have

$$M(x, \xi) \in L^\infty S_{1,0}^m \Longrightarrow M^\#(x, \xi) \in \mathcal{B}_\rho S_{1,1}^m, \tag{10.63}$$

with $\rho = 1/16$, where we define $\mathcal{B}_\rho S_{1,1}^m$ for $\rho < 1$ to be

$$\mathcal{B}_\rho S_{1,1}^m = \{b(x, \xi) \in S_{1,1}^m : \hat{b}(\eta, \xi) \ \text{supported in } |\eta| \leq \rho|\xi|\}, \tag{10.64}$$

and where $\hat{b}(\eta, \xi) = \int b(x, \xi) e^{-i\eta \cdot x}\, dx$. Set $\mathcal{B}S_{1,1}^m = \cup_{\rho<1} \mathcal{B}_\rho S_{1,1}^m$.

Most of the applications of the material of this section made in the following chapters of this book will involve symbol smoothing, (10.60)–(10.61), with $\delta < 1$. However, we will establish some basic results on operator calculus for symbols of the form (10.64).

We will analyze products $a(x,D)b(x,D) = p(x,D)$ when we are given $a(x,\xi) \in S^{\mu}_{1,1}(\mathbb{R}^n)$ and $b(x,\xi) \in \mathcal{B}S^m_{1,1}(\mathbb{R}^n)$. We are particularly interested in estimating the remainder $r_\nu(x,\xi)$, arising in

$$a(x,D)b(x,D) = p_\nu(x,D) + r_\nu(x,D), \tag{10.65}$$

where

$$p_\nu(x,\xi) = \sum_{|\alpha|\le\nu} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi)\cdot\partial_x^\alpha b(x,\xi). \tag{10.66}$$

Proposition 10.8 below is a variant of results of [Bon] and [Mey], established in [AT].

To begin the analysis, we have the formula

$$r_\nu(x,\xi) = \frac{1}{(2\pi)^n}\int \Big[a(x,\xi+\eta) - \sum_{|\alpha|\le\nu}\frac{\eta^\alpha}{\alpha!}\,\partial_\xi^\alpha a(x,\xi)\Big] e^{ix\cdot\eta}\hat{b}(\eta,\xi)\,d\eta. \tag{10.67}$$

Write

$$r_\nu(x,\xi) = \sum_{j\ge 0} r_{\nu j}(x,\xi), \tag{10.68}$$

with

$$\begin{aligned} r_{\nu j}(x,\xi) &= \int \widehat{A}_{\nu j}(x,\xi,\eta)\widehat{B}_j(x,\xi,\eta)\,d\eta \\ &= \int A_{\nu j}(x,\xi,y)B_j(x,\xi,-y)\,dy, \end{aligned} \tag{10.69}$$

where the terms in these integrands are defined as follows. Pick $\vartheta > 1$, and take a Littlewood–Paley partition of unity $\{\varphi_j^2 : j \ge 0\}$, such that $\varphi_0(\eta)$ is supported in $|\eta| \le 1$, while for $j \ge 1$, $\varphi_j(\eta)$ is supported in $\vartheta^{j-1} \le |\eta| \le \vartheta^{j+1}$. Then we set

$$\begin{aligned} \widehat{A}_{\nu j}(x,\xi,\eta) &= \frac{1}{(2\pi)^n}\Big[a(x,\xi+\eta) - \sum_{|\alpha|\le\nu}\frac{\eta^\alpha}{\alpha!}\partial_\xi^\alpha a(x,\xi)\Big]\varphi_j(\eta), \\ \widehat{B}_j(x,\xi,\eta) &= \hat{b}(\eta,\xi)\varphi_j(\eta)e^{ix\cdot\eta}. \end{aligned} \tag{10.70}$$

Note that

$$B_j(x,\xi,y) = \varphi_j(D_y)b(x+y,\xi). \tag{10.71}$$

Thus

$$\|B_j(x,\xi,\cdot)\|_{L^\infty} \le C\vartheta^{-rj}\|b(\cdot,\xi)\|_{C^r_*}. \tag{10.72}$$

Also,

$$\operatorname{supp}\hat{b}(\eta,\xi) \subset \{|\eta| < \rho|\xi|\} \Longrightarrow B_j(x,\xi,y) = 0, \quad \text{for } \vartheta^{j-1} \ge \rho|\xi|. \tag{10.73}$$

We next estimate the L^1-norm of $A_{\nu j}(x,\xi,\cdot)$. Now, by a standard proof of Sobolev's imbedding theorem, given $K > n/2$, we have

$$\|A_{\nu j}(x,\xi,\cdot)\|_{L^1} \le C\|\Gamma_j \widehat{A}_{\nu j}(x,\xi,\cdot)\|_{H^K}, \tag{10.74}$$

where $\Gamma_j f(\eta) = f(\vartheta^j\eta)$, so $\Gamma_j\widehat{A}_{\nu j}$ is supported in $|\eta| \le \vartheta$. Let us use the integral formula for the remainder term in the power-series expansion to write

$$\begin{aligned}&\widehat{A}_{\nu j}(x,\xi,\vartheta^j\eta) = \\ &\frac{\varphi_j(\vartheta^j\eta)}{(2\pi)^n}\sum_{|\alpha|=\nu+1}\frac{\nu+1}{\alpha!}\left(\int_0^1(1-s)^{\nu+1}\partial_\xi^\alpha a(x,\xi+s\vartheta^j\eta)\,ds\right)\vartheta^{j|\alpha|}\eta^\alpha.\end{aligned} \tag{10.75}$$

Since $|\eta| \le \vartheta$ on the support of $\Gamma_j\widehat{A}_{\nu j}$, if also $\vartheta^{j-1} < \rho|\xi|$, then $|\vartheta^j\eta| < \rho\vartheta^2|\xi|$. Now, given $\rho \in (0,1)$, choose $\vartheta > 1$ such that $\rho\vartheta^3 < 1$. This implies $\langle\xi\rangle \sim \langle\xi + s\vartheta^j\eta\rangle$, for all $s \in [0,1]$. We deduce that the hypothesis

$$|\partial_\xi^\alpha a(x,\xi)| \le C_\alpha\langle\xi\rangle^{\mu_2-|\alpha|}, \quad \text{for } |\alpha| \ge \nu+1, \tag{10.76}$$

implies

$$\|A_{\nu j}(x,\xi,\cdot)\|_{L^1} \le C_\nu\vartheta^{j(\nu+1)}\langle\xi\rangle^{\mu_2-\nu-1}, \quad \text{for } \vartheta^{j-1} < \rho|\xi|. \tag{10.77}$$

Now, when (10.72) and (10.77) hold, we have

$$|r_{\nu j}(x,\xi)| \le C_\nu\vartheta^{j(\nu+1-r)}\langle\xi\rangle^{\mu_2-\nu-1}\|b(\cdot,\xi)\|_{C^r_*}, \tag{10.78}$$

and if (10.73) also applies, we have

$$|r_\nu(x,\xi)| \le C_\nu\langle\xi\rangle^{\mu_2-r}\|b(\cdot,\xi)\|_{C^r_*} \quad \text{if } \nu+1 > r, \tag{10.79}$$

since

$$\sum_{\vartheta^{j-1}<\rho|\xi|}\vartheta^{j(\nu+1-r)} \le C|\xi|^{\nu+1-r}$$

in such a case.

To estimate derivatives of $r_\nu(x,\xi)$, we can write

$$D_x^\beta D_\xi^\gamma r_{\nu j}(x,\xi) =$$

(10.80)

$$\sum_{\beta_1+\beta_2=\beta}\ \sum_{\gamma_1+\gamma_2=\gamma} \binom{\beta}{\beta_1}\binom{\gamma}{\gamma_1} \int D_x^{\beta_1} D_\xi^{\gamma_1} A_{\nu j}(x,\xi,y) \cdot D_x^{\beta_2} D_\xi^{\gamma_2} B_j(x,\xi,-y)\,dy.$$

Now $D_x^{\beta_1} D_\xi^{\gamma_1} A_{\nu j}(x,\xi,y)$ is produced just like $A_{\nu j}(x,\xi,y)$, with the symbol $a(x,\xi)$ replaced by $D_x^{\beta_1} D_\xi^{\gamma_1} a(x,\xi)$, and $D_x^{\beta_2} D_\xi^{\gamma_2} B_j(x,\xi,-y)$ is produced just like $B_j(x,\xi,-y)$, with $b(x,\xi)$ replaced by $D_x^{\beta_2} D_\xi^{\gamma_2} b(x,\xi)$. Thus, if we strengthen the hypothesis (10.76) to

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{\mu_2-|\alpha|+|\beta|}, \text{ for } |\alpha| \ge \nu+1, \tag{10.81}$$

we have

$$\|D_x^{\beta_1} D_\xi^{\gamma_1} A_{\nu j}(x,\xi,\cdot)\|_{L^1} \le C_\nu \vartheta^{j(\nu+1)} \langle\xi\rangle^{\mu_2-|\gamma_1|+|\beta_1|-\nu-1}, \tag{10.82}$$

for $\vartheta^{j-1} < \rho|\xi|$. Furthermore, extending (10.72), we have

$$\|D_x^{\beta_2} D_\xi^{\gamma_2} B_j(x,\xi,\cdot)\|_{L^\infty} \le C\vartheta^{(|\beta_2|-r)j} \|D_\xi^{\gamma_2} b(\cdot,\xi)\|_{C_*^r}. \tag{10.83}$$

Now

$$\sum_{\vartheta^{j-1}<\rho|\xi|} \vartheta^{j(\nu+1+|\beta_2|-r)} \le C|\xi|^{\nu+1+|\beta_2|-r} \tag{10.84}$$

if $\nu+1>r$, so as long as (10.73) applies, (10.82) and (10.83) yield

$$|D_x^\beta D_\xi^\gamma r_\nu(x,\xi)| \le C \sum_{\gamma_1+\gamma_2=\gamma} \langle\xi\rangle^{\mu_2+|\beta|-|\gamma_1|-r} \|D_\xi^{\gamma_2} b(\cdot,\xi)\|_{C_*^r} \tag{10.85}$$

if $\nu+1>r$. These estimates lead to the following result:

Proposition 10.8. *Assume*

$$a(x,\xi) \in S^\mu_{1,1}, \quad b(x,\xi) \in \mathcal{B}S^m_{1,1}. \tag{10.86}$$

Then

$$a(x,D)b(x,D) = p(x,D) \in OPS^{\mu+m}_{1,1}. \tag{10.87}$$

Assume furthermore that

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{\mu_2-|\alpha|+|\beta|}, \quad \textit{for } |\alpha| \ge \nu+1, \tag{10.88}$$

with $\mu_2 \le \mu$, *and that*

$$\|D_\xi^\alpha b(\cdot,\xi)\|_{C_*^r} \le C_\alpha \langle \xi \rangle^{m_2-|\alpha|}. \tag{10.89}$$

Then, if $\nu + 1 > r$, *we have (10.65)–(10.66), with*

$$r_\nu(x, D) \in OPS_{1,1}^{\mu_2+m_2-r}. \tag{10.90}$$

The following is a commonly encountered special case of Proposition 10.8.

Corollary 10.9. *In Proposition 10.8, replace the hypothesis (10.89) by*

$$D_x^\beta b(x,\xi) \in S_{1,1}^{m_2}, \ \ \textit{for} \ \ |\beta| = K, \tag{10.91}$$

where $K \in \{1, 2, 3, \dots\}$ *is given. Then we have (10.65)–(10.66), with*

$$r_\nu(x,\xi) \in OPS_{1,1}^{\mu_2+m_2-K} \quad \textit{if} \ \ \nu \ge K. \tag{10.92}$$

Proof. The hypothesis (10.91) implies (10.89), with $r = K$.

We can also deduce from Proposition 10.8 that $a(x, D)b(x, D)$ has a complete asymptotic expansion if $b(x,\xi)$ is a symbol of type $(1,\delta)$ with $\delta < 1$.

Corollary 10.10. *If* $0 \le \delta < 1$ *and*

$$a(x,\xi) \in S_{1,1}^{\mu}, \quad b(x,\xi) \in S_{1,\delta}^{m}, \tag{10.93}$$

then $a(x, D)b(x, D) \in OPS_{1,1}^{\mu+m}$, *and we have (10.65)–(10.66), with*

$$r_\nu(x, D) \in OPS_{1,1}^{\mu+m-\nu(1-\delta)}. \tag{10.94}$$

Proof. Altering $b(x,\xi)$ by an element of $S_{1,0}^{-\infty}$, one can arrange that the condition (10.73) on supp $\hat{b}(\eta,\xi)$ hold. Then, apply Corollary 10.9, with $m_2 = m + K\delta$, so $m_2 - K = m - K(1-\delta)$, and take $K = \nu$.

Note that, under the hypotheses of Corollary 10.10,

$$\sum_{|\alpha|=\nu} \frac{1}{\alpha!}\, \partial_\xi^\alpha a(x,\xi) \cdot \partial_x^\alpha b(x,\xi) \in S_{1,1}^{\mu+m-\nu(1-\delta)}, \tag{10.95}$$

so we actually have

$$r_{\nu-1}(x, D) \in OPS_{1,1}^{\mu+m-\nu(1-\delta)}. \tag{10.96}$$

The family $\cup_m OP\mathcal{B}S_{1,1}^m$ does not form an algebra, but the following result is a useful substitute:

Proposition 10.11. *If* $p_j(x,\xi) \in \mathcal{B}_{\rho_j} S^{m_j}_{1,1}$ *and* $\rho = \rho_1 + \rho_2 + \rho_1\rho_2 < 1$, *then*

$$\begin{aligned} p_1(x,\xi)p_2(x,\xi) &\in \mathcal{B}_\rho S^{m_1+m_2}_{1,1}, \\ p_1(x,D)p_2(x,D) &\in OP\mathcal{B}_\rho S^{m_1+m_2}_{1,1}. \end{aligned} \tag{10.97}$$

Proof. The result for the symbol product is obvious; in fact, one can replace ρ by $\rho_1 + \rho_2$. As for $A(x,D) = p_1(x,D)p_2(x,D)$, we already have from Proposition 10.8 that $A(x,\xi) \in S^{m_1+m_2}_{1,1}$; we merely need to check the support of $\widehat{A}(\eta,\xi)$. We can do this using the formula

$$\widehat{A}(\eta,\xi) = \int \hat{p}_1(\eta-\zeta,\xi+\zeta)\hat{p}_2(\zeta,\xi)\, d\zeta. \tag{10.98}$$

Note that given (η,ξ), if there exists $\zeta \in \mathbb{R}^n$ such that $\hat{p}_1(\eta-\zeta,\xi+\zeta) \neq 0$ and $\hat{p}_2(\zeta,\xi) \neq 0$, then

$$|\eta-\zeta| \le \rho_1|\xi+\zeta|, \quad |\zeta| \le \rho_2|\xi|,$$

so

$$|\eta| \le \rho_1|\xi+\zeta| + |\zeta| \le \rho_1|\xi| + \rho_1|\zeta| + \rho_2|\xi| \le (\rho_1+\rho_2+\rho_1\rho_2)|\xi|.$$

This completes the proof.

Exercises

1. Prove the commutator property:

$$[OPS^\mu, OP\mathcal{A}^r_0 S^m_{1,\delta}] \subset OPS^{m+\mu-r}_{1,\delta}, \quad 0 \le r < 1,\ 0 \le \delta < 1. \tag{10.99}$$

2. Prove that, for $0 \le \delta < 1$,

$$P \in OP\mathcal{A}^r_0 S^m_{1,\delta} \Longrightarrow P^* \in OP\mathcal{A}^r_0 S^m_{1,\delta}. \tag{10.100}$$

(*Hint*: Use $P(x,D)^* = P^*(x,D)$, with $P^*(x,\xi) \sim \sum D^\alpha_x D^\alpha_\xi \overline{p(x,\xi)}$. Show that

$$p(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta} \Longrightarrow D^\alpha_x D^\alpha_\xi p(x,\xi) \in \mathcal{A}^r_0 S^{m-(1-\delta)|\alpha|}_{1,\delta}.)$$

3. Show that

$$\sum_{|\alpha|\le m} a_\alpha(x, D^{m-1}u) D^\alpha u = M(u;x,D)u + R, \tag{10.101}$$

where $R \in C^\infty$ and, for $0 < r < 1$,

$$u \in C^{m-1+r} \Longrightarrow M(u;x,\xi) \in \mathcal{A}^r_0 S^m_{1,1} + S^{m-r}_{1,1}. \tag{10.102}$$

Deduce that you can write

$$M(u;x,\xi)=M^{\#}(x,\xi)+M^{b}(x,\xi), \tag{10.103}$$

with

$$M^{\#}(x,\xi)\in\mathcal{A}_0^r S^m_{1,\delta},\quad M^b(x,\xi)\in S^{m-r\delta}_{1,1}. \tag{10.104}$$

Note that the hypothesis on u is weaker than in Proposition 10.7.

4. The estimate (10.9) follows from the formula

$$D^\alpha g(h)=\sum_{\alpha_1+\cdots+\alpha_\nu=\alpha} C(\alpha_1,\dots,\alpha_\nu)h^{(\alpha_1)}\cdots h^{(\alpha_\nu)}g^{(\nu)}(h),$$

which is a consequence of the chain rule. Show that the following Moser-type estimate holds:

$$\|D^\ell g(h)\|_{L^\infty}\le C\sum_{1\le\nu\le\ell}\|g'\|_{C^{\nu-1}}\|h\|^{\nu-1}_{L^\infty}\|D^\ell h\|_{L^\infty}. \tag{10.105}$$

5. The paraproduct of J.-M. Bony [Bon] is defined by applying symbol smoothing to the multiplication operator, $M_f u=fu$. One takes

$$T_f u=\sum_k \Psi_{k-5}(D)f\cdot\psi_k(D)u, \tag{10.106}$$

where, as in (10.62), $\Psi_\ell(\xi)=\sum_{j\le\ell}\psi_j(\xi)$. Show that, with $T_f=F(x,D)$,

$$f\in L^\infty(\mathbb{R}^n)\Longrightarrow F(x,\xi)\in S^0_{1,1}(\mathbb{R}^n). \tag{10.107}$$

Show that, for any $r\in\mathbb{R}$,

$$f\in C^r_*(\mathbb{R}^n)\Longrightarrow |D_x^\beta D_\xi^\alpha F(x,\xi)|\le C_{\alpha\beta}\|f\|_{C^r_*}\langle\xi\rangle^{-r-|\alpha|+|\beta|},\quad \text{for } |\alpha|\ge 1. \tag{10.108}$$

6. Using Propositions 10.8–10.11, show that if $p(x,\xi)\in\mathcal{B}_{1/2}S^m_{1,1}$, then

$$f\in C^0_*\Longrightarrow [T_f,p(x,D)]\in OP\mathcal{B}S^m_{1,1}. \tag{10.109}$$

Applications of this are given in [AT].

7. Show that $p(x,\xi)\in\mathcal{B}S^m_{1,1}$ implies $p(x,D)^*\in OPS^m_{1,1}$, and, if ρ is sufficiently small,

$$p(x,\xi)\in\mathcal{B}_\rho S^m_{1,1}\Longrightarrow p(x,D)^*\in OP\mathcal{B}S^m_{1,1}. \tag{10.110}$$

8. Investigate properties of operators with symbols in

$$\mathcal{B}^r S^m_{1,1}=\mathcal{B}S^m_{1,1}\cap\mathcal{A}_0^r S^m_{1,1}. \tag{10.111}$$

11. Young measures and fuzzy functions

Limits in the weak* topology of sequences $f_j \in L^p(\Omega)$ are often not well behaved under the pointwise application of nonlinear functions. For example,

$$\sin nx \to 0 \quad \text{weak}^* \text{ in } L^\infty\big([0,\pi]\big), \tag{11.1}$$

while

$$\sin^2 nx \to \frac{1}{2} \quad \text{weak}^* \text{ in } L^\infty\big([0,\pi]\big) \tag{11.2}$$

(see Fig. 11.1). A *fuzzy function* is endowed with an extra piece of structure, allowing for convergence under nonlinear mappings.

Assume Ω is an open set in $\mathbb{R}^n$. Given $1 \le p \le \infty$, we define an element of $Y^p(\Omega)$ to be a pair (f, λ), where $f \in L^p(\Omega)$ and λ is a positive Borel measure on $\Omega \times \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} = [-\infty, \infty]$), having the properties

$$y \in L^p\big(\Omega \times \overline{\mathbb{R}}, d\lambda(x,y)\big), \tag{11.3}$$

(so, in particular, $\Omega \times \{\pm\infty\}$ has measure zero),

$$\lambda(E \times \mathbb{R}) = \mathcal{L}^n(E), \tag{11.4}$$

for Borel sets $E \subset \Omega$, where $\mathcal{L}^n$ is Lebesgue measure on Ω, and

$$\iint\limits_{E\times\mathbb{R}} y \, d\lambda(x,y) = \int\limits_E f(x)\,dx, \tag{11.5}$$

for each Borel set $E \subset \Omega$. We can equivalently state (11.4) and (11.5) as

$$\iint \varphi(x)\,d\lambda(x,y) = \int \varphi(x)\,dx \tag{11.6}$$

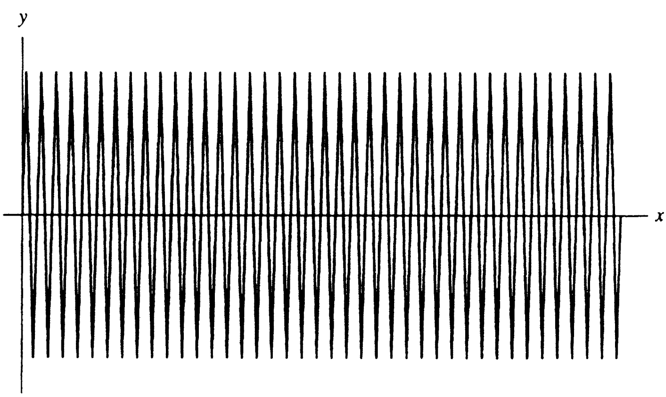

FIGURE 11.1 Approaching a Fuzzy Limit

and

$$\iint \varphi(x)y \, d\lambda(x, y) = \int \varphi(x) f(x) \, dx, \tag{11.7}$$

for $\varphi \in C_0(\Omega)$, that is, for continuous and compactly supported φ.

Note that (11.5) implies

$$\int_E |f(x)| \, dx \le \iint_{E\times\mathbb{R}} |y| \, d\lambda(x, y), \tag{11.8}$$

since we can write $E = E_1 \cup E_2$ with $f \ge 0$ on E_1 and $f < 0$ on E_2. If we partition E into tiny sets, on each of which f is nearly constant, we obtain

$$\int_E |f(x)|^p \, dx \le \iint_{E\times\mathbb{R}} |y|^p \, d\lambda(x, y). \tag{11.9}$$

We say that (f, λ) is a fuzzy function, and λ is a *Young measure*, representing f.

A special case of such λ is γ_f, defined by

$$\iint \psi(x, y) \, d\gamma_f(x, y) = \int \psi\big(x, f(x)\big) \, dx, \tag{11.10}$$

for $\psi \in C_0(\Omega \times \mathbb{R})$. We say (f, γ_f) is sharply defined.

Fuzzy functions arise as limits of sharply defined functions in the following sense. Suppose $f_j \in L^p(\Omega),\ 1 < p \le \infty$, and $(f, \lambda) \in Y^p(\Omega)$. We say

$$f_j \to (f, \lambda) \quad \text{in } \ Y^p(\Omega), \tag{11.11}$$

provided

$$f_j \to f \quad \text{weak}^* \text{ in } \ L^p(\Omega) \tag{11.12}$$

and

$$\gamma_{f_j} \to \lambda \quad \text{weak}^* \text{ in } \ \mathcal{M}(\Omega \times \overline{\mathbb{R}}), \tag{11.13}$$

and furthermore,

$$\|y\|_{L^p(\Omega\times\overline{\mathbb{R}}, d\gamma_{f_j})} \le C < \infty. \tag{11.14}$$

Actually, (11.12) is a consequence of (11.13) and (11.14), thanks to (11.9).

To take an example, if $\Omega = (0, \pi)$ and $f_n(x) = \sin\, nx$, as in (11.1), it is easily seen that

$$f_n \to (0, \lambda_0) \quad \text{in } \ Y^\infty(\Omega), \tag{11.15}$$

where

$$d\lambda_0(x, y) = \chi_{[-1,1]}(y)\, \frac{2\, dx\, dy}{\sqrt{1-y^2}}. \tag{11.16}$$

Also,

$$f_n^2 \to \left(\frac{1}{2}, \lambda_1\right) \quad \text{in } \ Y^\infty(\Omega), \tag{11.17}$$

where

$$d\lambda_1(x, y) = \chi_{[0,1]}(y)\, \frac{2\, dx\, dy}{\sqrt{y(y-1)}}. \tag{11.18}$$

The following result illustrates the use of $Y^p(\Omega)$ in controlling the behavior of nonlinear maps. We make rather restrictive hypotheses for this first result, to keep the argument short and reveal its basic simplicity.

Proposition 11.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be continuous. If $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, then*

$$\Phi(f_j) \to g \quad \textit{weak}^* \textit{ in } \ L^\infty(\Omega), \tag{11.19}$$

where $g \in L^\infty(\Omega)$ is specified by

$$\int g(x)\varphi(x)\, dx = \iint \Phi(y)\varphi(x)\, d\lambda(x, y), \quad \varphi \in C_0(\Omega). \tag{11.20}$$

Proof. We need to check the behavior of $\int \Phi(f_j)\varphi\, dx$. Since $\Phi(f_j)$ is bounded in $L^\infty(\Omega)$, it suffices to take φ in $C_0(\Omega)$, which is dense in $L^1(\Omega)$. Let I be a compact interval in $(-\infty, \infty)$, containing the range of each function $\Phi(f_j)$. Now, for any $\varphi \in C_0(\Omega)$,

$$\begin{aligned} \int_\Omega \Phi(f_j)\varphi\, dx &= \iint_{\Omega\times I} \varphi(x)\Phi(y)\, d\gamma_{f_j}(x, y) \\ &\to \iint_{\Omega\times I} \varphi(x)\Phi(y)\, d\lambda(x, y), \end{aligned} \tag{11.21}$$

since $\gamma_{f_j} \to \lambda$ weak* in $\mathcal{M}(\Omega \times I)$. This proves the proposition.

Under the hypotheses of Proposition 11.1, we see that, more precisely than (11.19),

$$\Phi(f_j) \to (g, \nu) \quad \text{in } Y^\infty(\Omega), \tag{11.22}$$

where g is given by (11.20) and ν is specified by

$$\iint \psi(x, y)\, d\nu(x, y) = \iint \psi\big(x, \Phi(y)\big)\, d\lambda(x, y), \quad \psi \in C_0(\Omega \times \overline{\mathbb{R}}). \tag{11.23}$$

Thus ν is the natural image of λ under the map $\widetilde{\Phi}(x, y) = \big(x, \Phi(y)\big)$ of $\Omega \times I \to \Omega \times \overline{\mathbb{R}}$. One often writes $\nu = \widetilde{\Phi}_* \lambda$. The extra information carried by (11.22) is that $\gamma_{\Phi(f_j)} \to \nu,$ weak* in $\mathcal{M}(\Omega \times \overline{\mathbb{R}})$, which follows from

$$\begin{aligned} \iint \psi(x, y)\, d\gamma_{\Phi(f_j)}(x, y) &= \iint \psi\big(x, \Phi(y)\big)\, d\gamma_{f_j}(x, y) \\ &\to \iint \psi\big(x, \Phi(y)\big)\, d\lambda(x, y). \end{aligned} \tag{11.24}$$

We can extend Proposition 11.1 and its refinement (11.22) to

$$f_j \to (f, \lambda) \text{ in } Y^p(\Omega) \Longrightarrow \Phi(f_j) \to (g, \nu) \text{ in } Y^q(\Omega), \tag{11.25}$$

with $1 < p, q < \infty$, where g and ν are given by the same formulas as above, provided that $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|\Phi(y)| \le C|y|^{p/q}. \tag{11.26}$$

We need this only for large $|y|$ if Ω has finite measure.

This result suggests defining the action of Φ on a fuzzy function (f, λ) by

$$\Phi(f, \lambda) = (g, \nu), \tag{11.27}$$

where g and ν are given by the formulas (11.20) and (11.23). Thus (11.22) can be restated as

$$f_j \to (f, \lambda) \text{ in } Y^\infty(\Omega) \Longrightarrow \Phi(f_j) \to \Phi(f, \lambda) \text{ in } Y^\infty(\Omega). \tag{11.28}$$

It is now natural to extend the notion of convergence $f_j \to (f, \lambda)$ in $Y^p(\Omega)$ to $(f_j, \lambda_j) \to (f, \lambda)$ in $Y^p(\Omega)$, provided all these objects belong to $Y^p(\Omega)$ and we have, parallel to (11.12)–(11.14),

$$f_j \to f \quad \text{weak}^* \text{ in } L^p(\Omega), \tag{11.29}$$

$$\lambda_j \to \lambda \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega \times \overline{\mathbb{R}}), \tag{11.30}$$

and

$$\|y\|_{L^p(\Omega\times\overline{\mathbb{R}},d\lambda_j)} \le C < \infty. \tag{11.31}$$

As before, (11.29) is actually a consequence of (11.30) and (11.31). Now (11.28) is easily extended to

$$(f_j,\lambda_j) \to (f,\lambda) \text{ in } Y^\infty(\Omega) \Longrightarrow \Phi(f_j,\lambda_j) \to \Phi(f,\lambda) \text{ in } Y^\infty(\Omega), \tag{11.32}$$

for continuous $\Phi : \mathbb{R} \to \mathbb{R}$. There is a similar extension of (11.25), granted the bound (11.26) on $\Phi(y)$.

We say that f_j (or more generally (f_j,λ_j)) converges *sharply* in $Y^p(\Omega)$, if it converges, in the sense defined above, to (f,λ) with $\lambda = \gamma_f$. It is of interest to specify conditions under which we can guarantee sharp convergence. We will establish some results in that direction a bit later.

When one has a fuzzy function (f,λ), it can be conceptually useful to pass from the measure λ on $\Omega \times \overline{\mathbb{R}}$ to a family of probability measures λ_x on $\overline{\mathbb{R}}$, defined for a.e. $x \in \Omega$. We discuss how this can be done. From (11.4) we have

$$\Big| \iint\limits_{E\times\overline{\mathbb{R}}} \psi(y)\, d\lambda(x,y) \Big| \le \sup |\psi|\, \mathcal{L}^n(E), \tag{11.33}$$

and hence

$$\Big| \iint\limits_{\Omega\times\overline{\mathbb{R}}} \varphi(x)\psi(y)\, d\lambda(x,y) \Big| \le \sup |\psi| \cdot \|\varphi\|_{L^1(\Omega)}. \tag{11.34}$$

It follows that there is a linear transformation

$$T : C(\overline{\mathbb{R}}) \longrightarrow L^\infty(\Omega), \quad \|T\psi\|_{L^\infty(\Omega)} \le \sup |\psi|, \tag{11.35}$$

such that

$$\iint\limits_{\Omega\times\overline{\mathbb{R}}} \varphi(x)\psi(y)\, d\lambda(x,y) = \int\limits_{\Omega} \varphi(x) T\psi(x)\, dx. \tag{11.36}$$

Using the separability of $C(\overline{\mathbb{R}})$, we can deduce that there is a set $S \subset \Omega$, of Lebesgue measure zero, such that, for *all* $\psi \in C(\overline{\mathbb{R}})$, $T\psi(x)$ is defined *pointwise*, for $x \in \Omega \setminus S$. Note that T is positivity preserving and $T(1) = 1$. Thus for each $x \in \Omega \setminus S$, there is a probability measure λ_x on $\overline{\mathbb{R}}$ such that

$$T\psi(x) = \int\limits_{\overline{\mathbb{R}}} \psi(y)\, d\lambda_x(y). \tag{11.37}$$

Hence

$$\iint\limits_{\Omega\times\overline{\mathbb{R}}} \varphi(x)\psi(y)\, d\lambda(x,y) = \int\limits_{\Omega}\left(\int\limits_{\overline{\mathbb{R}}} \varphi(x)\psi(y)\, d\lambda_x(y)\right) dx. \tag{11.38}$$

From this it follows that

$$\iint\limits_{\Omega\times\overline{\mathbb{R}}} \psi(x,y)\, d\lambda(x,y) = \int\limits_{\Omega}\left(\int\limits_{\overline{\mathbb{R}}} \psi(x,y)\, d\lambda_x(y)\right) dx, \tag{11.39}$$

for any Borel-measurable function ψ that is either positive or integrable with respect to $d\lambda$. Thus we can reformulate Proposition 11.1:

Corollary 11.2. *If $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous and $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, then*

$$\Phi(f_j) \to g \quad \text{weak}^* \text{ in } L^\infty(\Omega), \tag{11.40}$$

where

$$g(x) = \int\limits_{\mathbb{R}} \Phi(y)\, d\lambda_x(y), \quad \text{a.e. } x \in \Omega. \tag{11.41}$$

One key feature of the notion of convergence of a sequence of fuzzy functions is that, while it is preserved under nonlinear maps, we also retain the sort of compactness property that weak* convergence has.

Proposition 11.3. *Let $(f_j, \lambda_j) \in Y^\infty(\Omega)$, and assume $\|f_j\|_{L^\infty(\Omega)} \le M$. Then there exist $(f, \lambda) \in Y^\infty(\Omega)$ and a subsequence $(f_{j_\nu}, \lambda_{j_\nu})$ such that*

$$(f_{j_\nu}, \lambda_{j_\nu}) \longrightarrow (f, \lambda). \tag{11.42}$$

Proof. The well-known weak* compactness (and metrizability) of $\{g \in L^\infty(\Omega) : \|g\|_{L^\infty} \le M\}$ implies that one can pass to a subsequence (which we continue to denote by (f_j, λ_j)) such that $f_j \to f$ weak* in $L^\infty(\Omega)$.

Each measure λ_j is supported on $\Omega \times I$, $I = [-M, M]$. Now we exploit the weak* compactness and metrizability of $\{\mu \in \mathcal{M}(K \times I) : \|\mu\| \le \mathcal{L}^n(K)\}$, for each compact $K \subset \Omega$, together with a standard diagonal argument, to obtain a further subsequence such that $\lambda_{j_\nu} \to \lambda$ weak* in $\mathcal{M}(\Omega \times I)$. The identities (11.6) and (11.7) are preserved under passage to such a limit, so the proposition is proved.

So far we have dealt with real-valued fuzzy functions, but we can as easily consider fuzzy functions with values in a finite-dimensional, normed vector space V. We define $Y^p(\Omega, V)$ to consist of pairs (f, λ), where $f \in L^p(\Omega, V)$ is a

V-valued L^p function and λ is a positive Borel measure on $\Omega \times \overline{V}$ ($\overline{V} = V$ plus the sphere S_∞ at infinity), having the properties

$$|y| \in L^p\big(\Omega \times \overline{V}, d\lambda(x,y)\big), \tag{11.43}$$

so in particular $\Omega \times S_\infty$ has measure zero,

$$\lambda(E \times V) = \mathcal{L}^n(E), \tag{11.44}$$

for Borel sets $E \subset \Omega$, and

$$\iint\limits_{E\times V} y \, d\lambda(x,y) = \int\limits_E f(x)\, dx \in V, \tag{11.45}$$

for each Borel set $E \subset \Omega$.

All of the preceding results of this section extend painlessly to this case. Instead of considering $\Phi : \mathbb{R} \to \mathbb{R}$, we take $\Phi : V_1 \to V_2$, where V_j are two normed finite-dimensional vector spaces. This time, a Young measure λ "disintegrates" into a family λ_x of probability measures on $\overline{V}$.

There is a natural map

$$\& : Y^\infty(\Omega, V_1) \times Y^\infty(\Omega, V_2) \longrightarrow Y^\infty(\Omega, V_1 \oplus V_2) \tag{11.46}$$

defined by

$$(f_1, \lambda_1)\&(f_2, \lambda_2) = (f_1 \oplus f_2, \nu), \tag{11.47}$$

where, for a.e. $x \in \Omega$, Borel $F_j \subset V_j$,

$$\nu_x(F_1 \times F_2) = \lambda_{1x}(F_1)\lambda_{2x}(F_2). \tag{11.48}$$

Using this, we can define an "addition" on elements of $Y^\infty(\Omega, V)$:

$$(f_1, \lambda_1) + (f_2, \lambda_2) = S\big((f_1, \lambda_1)\&(f_2, \lambda_2)\big), \tag{11.49}$$

where $S : V \oplus V \to V$ is given by $S(v, w) = v + w$, and we extend S to a map $S : Y^\infty(\Omega, V \oplus V) \to Y^\infty(\Omega, V)$ by the same process as used in (11.27).

Of course, multiplication by a scalar $a \in \mathbb{R}$, $M_a : V \to V$, induces a map M_a on $Y^\infty(\Omega, V)$, so we have what one might call a "fuzzy linear structure" on $Y^\infty(\Omega, V)$. It is not truly a linear structure since certain basic requirements on vector space operations do not hold here. For example (in the case

$V = \mathbb{R}$), $(f,\lambda) \in Y^\infty(\Omega)$ has a natural "negative," namely $(-f,\check{\lambda})$, where $\check{\lambda}(E) = \lambda(-E)$. However, $(f,\lambda) + (-f,\check{\lambda}) \neq (0,\gamma_0)$ unless (f,λ) is sharply defined. Similarly, $(f,\lambda) + (f,\lambda) \neq 2(f,\lambda)$ unless (f,λ) is sharply defined, so the distributive law fails.

We now derive some conditions under which, for a given sequence $u_j \to (u,\lambda)$ in $Y^\infty(\Omega)$ and a given nonlinear function F, we also have $F(u_j) \to F(u)$ weak* in $L^\infty(\Omega)$, which is the same here as $F(u) = \overline{F}$. The following result is of the nature that weak* convergence of the dot product of the $\mathbb{R}^2$-valued functions $\big(u_j, F(u_j)\big)$ with a certain family of $\mathbb{R}^2$-valued functions $V(u_j)$ to $(u,\overline{F})\cdot\overline{V}$ will imply $\overline{F} = F(u)$. The specific choice of $V(u_j)$ will perhaps look curious; we will explain below how this choice arises.

Proposition 11.4. *Suppose $u_j \to (u,\lambda)$ in $Y^\infty(\Omega)$, and let $F : \mathbb{R} \to \mathbb{R}$ be C^1. Suppose you know that*

$$u_j q(u_j) - F(u_j)\eta(u_j) \longrightarrow u\overline{q} - \overline{F}\,\overline{\eta} \quad \text{weak* in } L^\infty(\Omega), \tag{11.50}$$

for every convex function $\eta : \mathbb{R} \to \mathbb{R}$, with q given by

$$q(y) = \int_c^y \eta'(s)F'(s)\,ds, \tag{11.51}$$

and where

$$q(u,\lambda) = (\overline{q},\nu_1), \quad F(u,\lambda) = (\overline{F},\nu_2), \quad \eta(u,\lambda) = (\overline{\eta},\nu_3). \tag{11.52}$$

Then

$$F(u_j) \to F(u) \quad \text{weak* in } L^\infty(\Omega). \tag{11.53}$$

Proof. It suffices to prove that $\overline{F} = F(u)$ a.e. on Ω. Now, applying Corollary 11.2 to $\Phi(y) = yq(y) - F(y)\eta(y)$, we have the left side of (11.50) converging weak* in $L^\infty(\Omega)$ to

$$v(x) = \int \big[yq(y) - F(y)\eta(y)\big]\,d\lambda_x(y),$$

so the hypothesis (11.50) implies

$$v = u\overline{q} - \overline{F}\,\overline{\eta}, \quad \text{a.e. on } \Omega.$$

Rewrite this as

$$\int \Big\{\big(F(y) - \overline{F}(x)\big)\eta(y) - \big(u(x) - y\big)q(y)\Big\}\,d\lambda_x(y) = 0, \quad \text{a.e. } x \in \Omega. \tag{11.54}$$

Now we make the following special choices of functions η and q:

$$\eta_a(y) = |y - a|, \quad q_a(y) = \operatorname{sgn}(y - a)\,\big(F(y) - F(a)\big). \tag{11.55}$$

We use these in (11.54), with $a = u(x)$, obtaining, after some cancellation,

$$\big(F\big(u(x)\big) - \overline{F}(x)\big) \int |y - u(x)|\, d\lambda_x(y) = 0, \quad \text{a.e. } x \in \Omega. \tag{11.56}$$

Thus, for a.e. $x \in \Omega$, either $\overline{F}(x) = F\big(u(x)\big)$ or $\lambda_x = \delta_{u(x)}$, which also implies $\overline{F}(x) = F\big(u(x)\big)$. The proof is complete.

Why is one motivated to work with such functions $\eta(u)$ and $q(u)$? They arise in the study of solutions to some nonlinear PDE on $\Omega \subset \mathbb{R}^2$. Let us use coordinates (t, x) on Ω. As long as u is a Lipschitz-continuous, real-valued function on Ω, it follows from the chain rule that

$$u_t + F(u)_x = 0 \Longrightarrow \eta(u)_t + q(u)_x = 0, \tag{11.57}$$

provided $q'(y) = \eta'(y)F'(y)$, that is, q is given by (11.52). (For general $u \in L^\infty(\Omega)$, the implication (11.57) does not hold.) Our next goal is to establish the following:

Proposition 11.5. *Assume $u_j \in L^\infty(\Omega)$, of norm $\le M < \infty$. Assume also that*

$$\partial_t u_j + \partial_x F(u_j) \to 0 \quad \textit{in } \; H^{-1}_{loc}(\Omega) \tag{11.58}$$

and

$$\partial_t \eta(u_j) + \partial_x q(u_j) \quad \textit{precompact in } \; H^{-1}_{loc}(\Omega), \tag{11.59}$$

for each convex function $\eta : \mathbb{R} \to \mathbb{R}$, with q given by (11.51). If $u_j \to u$ weak* in $L^\infty(\Omega)$, then

$$\partial_t u + \partial_x F(u) = 0. \tag{11.60}$$

Proof. By Proposition 11.3, passing to a subsequence, we have $u_j \to (u, \lambda)$ in $Y^\infty(\Omega)$. Then, by Proposition 11.1, $F(u_j) \to \overline{F}$, $q(u_j) \to \overline{q}$, and $\eta(u_j) \to \overline{\eta}$ weak* in $L^\infty(\Omega)$. Consider the vector-valued functions

$$v_j = \big(u_j, F(u_j)\big), \quad w_j = \big(q(u_j), -\eta(u_j)\big). \tag{11.61}$$

Thus $v_j \to (u, \overline{F})$, $w_j \to (\overline{q}, -\overline{\eta})$ weak* in $L^\infty(\Omega)$. The hypotheses (11.58)–(11.59) are equivalent to

$$\operatorname{div} v_j, \;\; \operatorname{rot} w_j \;\; \text{precompact in } \; H^{-1}_{\text{loc}}(\Omega). \tag{11.62}$$

Also, the hypothesis on $\|u_j\|_{L^\infty}$ implies that v_j and w_j are bounded in $L^\infty(\Omega)$, and a fortiori in $L^2_{\mathrm{loc}}(\Omega)$. The div-curl lemma hence implies that

$$v_j \cdot w_j \to v \cdot w \ \text{ in } \ \mathcal{D}'(\Omega), \quad v = (u, \overline{F}), \ \ w = (\overline{q}, -\overline{\eta}). \tag{11.63}$$

In view of the L^∞-bounds, we hence have

$$u_j q(u_j) - F(u_j)\eta(u_j) \longrightarrow u\overline{q} - \overline{F}\overline{\eta} \quad \text{weak}^* \text{ in } \ L^\infty(\Omega). \tag{11.64}$$

Since this is the hypothesis (11.50) of Proposition 11.4, we deduce that

$$F(u_j) \longrightarrow F(u) \quad \text{weak}^* \text{ in } \ L^\infty(\Omega). \tag{11.65}$$

Hence $\partial_t u_j + \partial_x F(u_j) \to \partial_t u + \partial_x F(u)$ in $\mathcal{D}'(\Omega)$, so we have (11.60).

One of the most important cases leading to the situation dealt with in Proposition 11.5 is the following; for $\varepsilon \in (0, 1]$, consider the PDE

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_x^2 u_\varepsilon \ \text{ on } \ \Omega = (0, \infty) \times \mathbb{R}, \quad u_\varepsilon(0) = f. \tag{11.66}$$

Say $f \in L^\infty(\mathbb{R})$. The unique solvability of (11.66), for $t \in [0, \infty)$, for each $\varepsilon > 0$, will be established in Chap. 15, and results there imply

$$u_\varepsilon \in C^\infty(\Omega), \tag{11.67}$$

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty}, \tag{11.68}$$

and

$$\varepsilon \int_0^\infty \int_{-\infty}^\infty (\partial_x u_\varepsilon)^2 \, dx \, dt \le \frac{1}{2} \|f\|_{L^2}^2. \tag{11.69}$$

The last result implies that $\sqrt{\varepsilon}\partial_x u_\varepsilon$ is bounded in $L^2(\Omega)$. Hence $\varepsilon\partial_x^2 u_\varepsilon \to 0$ in $H^{-1}(\Omega)$, as $\varepsilon \to 0$. Thus, if u_{ε_j} is relabeled u_j, with $\varepsilon_j \to 0$, we have hypothesis (11.58) of Proposition 11.5. We next check hypothesis (11.59).

Using the chain rule and (11.66), we have

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_x^2 \eta(u_\varepsilon) - \varepsilon \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2, \tag{11.70}$$

at least when η is C^2 and q satisfies (11.52). Parallel to (11.69), we have

$$\varepsilon \int_0^T \int \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 \, dx \, dt = \int \eta\big(f(x)\big) \, dx - \int \eta\big(u_\varepsilon(T, x)\big) \, dx. \tag{11.71}$$

A simple approximation argument, taking smooth $\eta_\delta \to \eta$, shows that whenever η is nonnegative and convex, C^2 or not,

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \partial_x^2 \eta(u_\varepsilon) - R_\varepsilon, \tag{11.72}$$

with

$$R_\varepsilon \text{ bounded in } \mathcal{M}(\Omega). \tag{11.73}$$

Since $\partial_x \eta(u_\varepsilon) = \eta'(u_\varepsilon)\partial_x u_\varepsilon$, and any convex η is locally Lipschitz, we deduce from (11.68) and (11.69) that $\sqrt{\varepsilon}\partial_x \eta(u_\varepsilon)$ is bounded in $L^2(\Omega)$. Hence

$$\varepsilon \partial_x^2 \eta(u_\varepsilon) \to 0 \text{ in } H^{-1}(\Omega), \quad \text{as } \varepsilon \to 0. \tag{11.74}$$

We thus have certain bounds on the right side of (11.72), by (11.73) and (11.74). Meanwhile, the left side of (11.72) is certainly bounded in $H^{-1,p}_{\text{loc}}(\Omega)$, $\forall\, p < \infty$. This situation is treated by the following lemma of F. Murat.

Lemma 11.6. *Suppose $\mathcal{F}$ is bounded in $H^{-1,p}_{loc}(\Omega)$, for some $p > 2$, and $\mathcal{F} \subset \mathcal{G} + \mathcal{H}$, where $\mathcal{G}$ is precompact in $H^{-1}_{loc}(\Omega)$ and $\mathcal{H}$ is bounded in $\mathcal{M}_{loc}(\Omega)$. Then $\mathcal{F}$ is precompact in $H^{-1}_{loc}(\Omega)$.*

Proof. Multiplying by a cut-off $\chi \in C_0^\infty(\Omega)$, we reduce to the case where all $f \in \mathcal{F}$ are supported in some compact K, and the decomposition $f = g + h$, $g \in \mathcal{G}$, $h \in \mathcal{H}$ also has g, h supported in K. Putting K in a box and identifying opposite sides, we are reduced to establishing an analogue of the lemma when Ω is replaced by $\mathbb{T}^n$.

Now Sobolev imbedding theorems imply

$$\mathcal{M}(\mathbb{T}^n) \subset H^{-s,q}(\mathbb{T}^n), \quad s \in (0, n), \ \ q \in \Big(1, \frac{n}{n-s}\Big).$$

Via Rellich's compactness result (6.9), it follows that

$$\iota : \mathcal{M}(\mathbb{T}^n) \hookrightarrow H^{-1,q}(\mathbb{T}^n), \quad \text{compact } \forall\, q \in \Big(1, \frac{n}{n-1}\Big). \tag{11.75}$$

Hence $\mathcal{H}$ is precompact in $H^{-1,q}(\mathbb{T}^n)$, for any $q < n/(n-1)$, so we have

$$\mathcal{F} \text{ precompact in } H^{-1,q}(\mathbb{T}^n), \quad \text{bounded in } H^{-1,p}(\mathbb{T}^n), \ \ p > 2. \tag{11.76}$$

By a simple interpolation argument, (11.76) implies that $\mathcal{F}$ is precompact in $H^{-1}(\mathbb{T}^n)$, so the lemma is proved.

We deduce that if the family $\{u_\varepsilon : 0 < \varepsilon \le 1\}$ of solutions to (11.66) satisfies (11.67)–(11.69), then

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) \quad \text{precompact in } H^{-1}_{\mathrm{loc}}(\Omega), \tag{11.77}$$

which is hypothesis (11.59) of Proposition 11.5. Therefore, we have the following:

Proposition 11.7. *Given solutions u_ε, $0 < \varepsilon \le 1$ to (11.66), satisfying (11.67)–(11.69), a weak* limit u in $L^\infty(\Omega)$, as $\varepsilon = \varepsilon_j \to 0$, satisfies*

$$\partial_t u + \partial_x F(u) = 0. \tag{11.78}$$

The approach to the solvability of (11.78) used above is given in [Tar]. In Chap. 16, § 6, we will obtain global existence results containing that of Proposition 11.7, using different methods, involving uniform estimates of $\|\partial_x u_\varepsilon(t)\|_{L^1(\mathbb{R})}$. On the other hand, in § 9 of Chap. 16 we will make use of techniques involving fuzzy functions and the div-curl lemma to establish some global solvability results for certain 2×2 hyperbolic systems of conservation laws, following work of R. DiPerna [DiP].

The notion of fuzzy function suggests the following notion of a "fuzzy solution" to a PDE, of the form

$$\sum_j \frac{\partial}{\partial x_j} F_j(u) = 0. \tag{11.79}$$

Namely, $(u, \lambda) \in Y^\infty(\Omega)$ is a fuzzy solution to (11.79) if

$$\sum_j \frac{\partial}{\partial x_j} \overline{F}_j = 0 \ \text{ in } \ \mathcal{D}'(\Omega), \quad \overline{F}_j(x) = \int F_j(y)\, d\lambda_x(y). \tag{11.80}$$

This notion was introduced in [DiP], where (u, λ) is called a "measure-valued solution" to (11.79). Given $|F_j(y)| \le C\langle y\rangle^p$, we can also consider the concept of a fuzzy solution $(u, \lambda) \in Y^p(\Omega)$. Contrast the following simple result with Proposition 11.5:

Proposition 11.8. *Assume $(u_j, \lambda_j) \in Y^\infty(\Omega)$, $\|u_j\|_{L^\infty} \le M$, and $(u_j, \lambda_j) \to (u, \lambda)$ in $Y^\infty(\Omega)$. If*

$$\sum_k \partial_k F_k(u_j) \to 0 \quad \text{in } \ \mathcal{D}'(\Omega), \tag{11.81}$$

as $j \to \infty$, then u is a fuzzy solution to (11.79).

Proof. By Proposition 11.1, $F_k(u_j) \to \overline{F}_k$ weak* in $L^\infty(\Omega)$. The result follows immediately from this.

In [DiP] there are some results on when one can say that, when $(u, \lambda) \in Y^\infty(\Omega)$ is a fuzzy solution to (11.79), then $u \in L^\infty(\Omega)$ is a weak solution to (11.79), results that in particular lead to another proof of Proposition 11.7.

Exercises

1. If $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, we say the convergence is *sharp* provided $\lambda = \gamma_f$. Show that sharp convergence implies
$$f_j \to f \ \text{ in } \ L^2(\Omega_0),$$
for any $\Omega_0 \subset\subset \Omega$.
(*Hint*: Sharp convergence implies $|f_j|^2 \to |f|^2$ weak* in $L^\infty(\Omega)$. Thus $f_j \to f$ weakly in L^2 and also $\|f_j\|_{L^2(\Omega_0)} \to \|f\|_{L^2(\Omega_0)}$.)
2. Deduce that, given $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, the convergence is sharp if and only if, for some subsequence, $f_{j_\nu} \to f$ a.e. on Ω.
3. Given $(f, \lambda) \in Y^\infty(\Omega)$ and the associated family of probability measures $\lambda_x,\ x \in \Omega$, as in (11.37)–(11.39), show that $\lambda = \gamma_f$ if and only if, for a.e. $x \in \Omega,\ \lambda_x$ is a point mass.
4. Complete the interpolation argument cited in the proof of Lemma 11.6. Show that (with $X = \Lambda^{-1}(\mathcal{F})$) if $q < 2 < p$,
$$X \text{ precompact in } L^q(\mathbb{T}^n), \text{ bounded in } L^p(\mathbb{T}^n) \Longrightarrow X \text{ precompact in } L^2(\mathbb{T}^n).$$
(*Hint*: If $f_n \in X,\ f_n \to f$ in $L^q(\mathbb{T}^n)$, use
$$\|f_n - f\|_{L^2} \le \|f_n - f\|_{L^q}^\alpha \|f_n - f\|_{L^p}^{1-\alpha}.)$$
5. Extend various propositions of this section from $Y^\infty(\Omega)$ to $Y^p(\Omega),\ 1 < p \le \infty$.

12. Hardy spaces

The Hardy space $\mathfrak{H}^1(\mathbb{R}^n)$ is a subspace of $L^1(\mathbb{R}^n)$ defined as follows. Set

$$(\mathcal{G}f)(x) = \sup\bigl\{|\varphi_t * f(x)| : \varphi \in \mathcal{F},\ t > 0\bigr\}, \tag{12.1}$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$ and

$$\mathcal{F} = \bigl\{\varphi \in C_0^\infty(\mathbb{R}^n) : \varphi(x) = 0 \text{ for } |x| \ge 1,\ \|\nabla\varphi\|_{L^\infty} \le 1\bigr\}. \tag{12.2}$$

This is called the *grand maximal function* of f. Then we define

$$\mathfrak{H}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \mathcal{G}f \in L^1(\mathbb{R}^n)\}. \tag{12.3}$$

A related (but slightly larger) space is $\mathfrak{h}^1(\mathbb{R}^n)$, defined as follows. Set

$$(\mathcal{G}^b f)(x) = \sup\bigl\{|\varphi_t * f(x)| : \varphi \in \mathcal{F},\ 0 < t \le 1\bigr\}, \tag{12.4}$$

and define

$$\mathfrak{h}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \mathcal{G}^b f \in L^1(\mathbb{R}^n)\}. \tag{12.5}$$

An important tool in the study of Hardy spaces is another maximal function, the *Hardy-Littlewood maximal function*, defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{\mathrm{vol}(B_r)} \int_{B_r(x)} |f(y)|\, dy. \tag{12.6}$$

The basic estimate on this maximal function is the following weak type-(1,1) estimate:

Proposition 12.1. *There is a constant $C = C(n)$ such that, for any $\lambda > 0,\ f \in L^1(\mathbb{R}^n)$, we have the estimate*

$$meas\big(\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > \lambda\}\big) \le \frac{C}{\lambda}\|f\|_{L^1}. \tag{12.7}$$

Note that the estimate

$$\mathrm{meas}\big(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big) \le \frac{1}{\lambda}\|f\|_{L^1}$$

follows by integrating the inequality $|f| \ge \lambda\chi_{S_\lambda}$, where $S_\lambda = \{|f| > \lambda\}$.

To begin the proof of Proposition 12.1, let

$$F_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}. \tag{12.8}$$

We remark that, for any $f \in L^1(\mathbb{R}^n)$ and any $\lambda > 0,\ F_\lambda$ is open. Given $x \in F_\lambda$, pick $r = r_x$ such that $A_r|f|(x) > \lambda$, and let $B_x = B_{r_x}(x)$. Thus $\{B_x : x \in F_\lambda\}$ is a covering of F_λ by balls. We will be able to obtain the estimate (12.7) from the following "covering lemma," due to N. Wiener.

Lemma 12.2. *If $\mathcal{C} = \{B_\alpha : \alpha \in \mathfrak{A}\}$ is a collection of open balls in $\mathbb{R}^n$, with union U, and if $m_0 <$ meas(U), then there is a finite collection of disjoint balls $B_j \in \mathcal{C},\ 1 \le j \le K$, such that*

$$\sum meas(B_j) > 3^{-n} m_0. \tag{12.9}$$

We show how the lemma allows us to prove (12.7). In this case, let $\mathcal{C} = \{\overset{\circ}{B}_x : x \in F_\lambda\}$. Thus, if $m_0 <$ meas(F_λ), there exist disjoint balls $B_j = \overset{\circ}{B}_{r_j}(x_j)$ such that meas$(\cup B_j) > 3^{-n} m_0$. This implies

$$m_0 < 3^n \sum \mathrm{meas}(B_j) \le \frac{3^n}{\lambda} \sum \int_{B_j} |f(x)|\, dx \le \frac{3^n}{\lambda} \int |f(x)|\, dx, \tag{12.10}$$

for all $m_0 <$ meas(F_λ), which yields (12.7), with $C = 3^n$.

We now turn to the proof of Lemma 12.2. We can pick a compact $K \subset U$ such that $m(K) > m_0$. Then the covering $\mathcal{C}$ yields a finite covering of K, say $A_1, \dots, A_N$. Let B_1 be the ball A_j of the largest radius. Throw out all A_ℓ that meet B_1, and let B_2 be the remaining ball of largest radius. Continue until $\{A_1, \dots, A_N\}$ is exhausted. One gets disjoint balls $B_1, \dots, B_K$ in $\mathcal{C}$. Now each A_j meets some B_ℓ, having the property that the radius of B_ℓ is $\geq$ the radius of A_j. Thus, if $\widehat{B}_j$ is the ball concentric with B_j, with three times the radius, we have

$$\bigcup_{j=1}^{K} \widehat{B}_j \supset \bigcup_{\ell=1}^{N} A_\ell \supset K.$$

This yields (12.9).

Note that clearly

$$f \in L^\infty(\mathbb{R}^n) \Longrightarrow \|\mathcal{M}(f)\|_{L^\infty} \leq \|f\|_{L^\infty}. \tag{12.11}$$

Now the method of proof of the Marcinkiewicz interpolation theorem, Proposition 5.4, yields the following.

Corollary 12.3. *If* $1 < p < \infty$, *then*

$$\|\mathcal{M}(f)\|_{L^p} \leq C_p \|f\|_{L^p}. \tag{12.12}$$

Our first result on Hardy spaces is the following, relating $\mathfrak{h}^1(\mathbb{R}^n)$ to the smaller space $\mathfrak{H}^1(\mathbb{R}^n)$.

Proposition 12.4. *If* $u \in \mathfrak{h}^1(\mathbb{R}^n)$ *has compact support and* $\int u(x)\, dx = 0$, *then* $u \in \mathfrak{H}^1(\mathbb{R}^n)$.

Proof. It suffices to show that

$$v(x) = \sup\{|\varphi_t * u(x)| : \varphi \in \mathcal{F}, t \geq 1\} \tag{12.13}$$

belongs to $L^1(\mathbb{R}^n)$. Clearly, v is bounded. Also, if $\operatorname{supp} u \subset \{|x| \leq R\}$, then we can write $u = \sum \partial_j u_j,\ u_j \in L^1(B_R)$. Then

(12.14)
$$\varphi_t * u(x) = \sum_j t^{-1} \psi_{jt} * u_j(x), \quad \psi_{jt}(x) = t^{-n} \psi_j(t^{-1}x),\ \psi_j(x) = \partial_j \varphi(x).$$

If $|x| = R + 1 + \rho$, then $\psi_{jt} * u_j(x) = 0$ for $t < \rho$, so

$$v(x) \leq C\rho^{-1} \sum_j \mathcal{M}(u_j)(x). \tag{12.15}$$

The weak (1,1) bound (12.7) on $\mathcal{M}$ now readily yields an L^1-bound on $v(x)$.

One advantage of $\mathfrak{h}^1(\mathbb{R}^n)$ is its localizability. We have the following useful result:

Proposition 12.5. *If $r > 0$ and $g \in C^r(\mathbb{R}^n)$ has compact support, then*

$$u \in \mathfrak{h}^1(\mathbb{R}^n) \Longrightarrow gu \in \mathfrak{h}^1(\mathbb{R}^n). \tag{12.16}$$

Proof. If $g \in C^r$ and $0 < r \leq 1$, we have, for all $\varphi \in \mathcal{F}$,

$$\left|\varphi_t * (gu)(x) - g(x)\varphi_t * u(x)\right| \leq C t^{r-n} \int\limits_{B_t(x)} |u(y)|\, dy. \tag{12.17}$$

Hence it suffices to show that

$$v(x) = \sup_{0<t\leq 1} t^{r-n} \int\limits_{B_t(x)} |u(y)|\, dy \tag{12.18}$$

belongs to $L^1(\mathbb{R}^n)$. Since

$$v(x) \leq \int \frac{\chi(x-y)}{|x-y|^{n-r}} |u(y)|\, dy, \tag{12.19}$$

where $\chi(x)$ is the characteristic function of $\{|x| \leq 1\}$, this is clear.

Given $\Omega \subset \mathbb{R}^n$ open, $u \in L^1_{\text{loc}}(\Omega)$, we say

$$u \in \mathfrak{H}^1_{\text{loc}}(\Omega) \Longleftrightarrow gu \in \mathfrak{h}^1(\mathbb{R}^n), \quad \forall\, g \in C_0^\infty(\Omega). \tag{12.20}$$

This is equivalent to the statement that, for any compact $K \subset \Omega$, there is a $v \in \mathfrak{H}^1(\mathbb{R}^n)$ such that $u = v$ on a neighborhood of K. To see this, note that if $u \in \mathfrak{H}^1_{\text{loc}}(\Omega)$ and $g \in C_0^\infty(\Omega)$, $g = 1$ on a neighborhood of K, then $gu \in \mathfrak{h}^1(\mathbb{R}^n)$. Now take $v = gu + h$, where $h \in C_0^\infty(\mathbb{R}^n)$ has support disjoint from supp g, and $\int h(x)\, dx = -\int g(x)u(x)\, dx$. By Proposition 12.4, $v \in \mathfrak{H}^1(\mathbb{R}^n)$. The converse is established similarly.

Not every compactly supported element of $L^1(\mathbb{R}^n)$ belongs to $\mathfrak{h}^1(\mathbb{R}^n)$, but we do have the following.

Proposition 12.6. *If $p > 1$ and $u \in L^p(\mathbb{R}^n)$ has compact support, then $u \in \mathfrak{h}^1(\mathbb{R}^n)$.*

Proof. We have

$$(\mathcal{G}^b f)(x) \leq (\mathcal{G} f)(x) \leq C\mathcal{M} f(x). \tag{12.21}$$

Hence, given $p > 1,\ u \in L^p(\mathbb{R}^n) \Rightarrow \mathcal{G}^b u \in L^p(\mathbb{R}^n)$. Also, $\mathcal{G}^b u$ has support in $|x| \le R+1$ if supp $u \subset \{|x| \le R\}$, so $\mathcal{G}^b u \in L^1(\mathbb{R}^n)$.

The spaces $\mathfrak{H}^1(\mathbb{R}^n)$ and $\mathfrak{h}^1(\mathbb{R}^n)$ are Banach spaces, with norms

$$\|u\|_{\mathfrak{H}^1} = \|\mathcal{G}u\|_{L^1}, \quad \|u\|_{\mathfrak{h}^1} = \|\mathcal{G}^b u\|_{L^1}. \tag{12.22}$$

It is useful to have the following approximation result.

Proposition 12.7. *Fix $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\int \psi(x)\,dx = 1$. If $u \in \mathfrak{H}^1(\mathbb{R}^n)$, then*

$$\|\psi_\varepsilon * u - u\|_{\mathfrak{H}^1} \longrightarrow 0, \ \text{ as } \ \varepsilon \to 0. \tag{12.23}$$

Proof. One easily verifies from the definition that, for some $C < \infty$, $\mathcal{G}(\psi_\varepsilon * u)(x) \le C\mathcal{G}u(x)$, $\forall\, x, \forall\, \varepsilon \in (0,1]$. Hence, by the dominated convergence theorem, it suffices to show that

$$\mathcal{G}(\psi_\varepsilon * u - u)(x) \longrightarrow 0, \quad \text{a.e. } x, \ \text{ as } \ \varepsilon \to 0; \tag{12.24}$$

that is,

$$\sup_{t>0,\varphi\in\mathcal{F}} \left|(\varphi_t * \psi_\varepsilon * u - \varphi_t * u)(x)\right| \to 0, \quad \text{a.e. } x, \ \text{ as } \ \varepsilon \to 0.$$

To prove this, it suffices to show that

$$\lim_{\varepsilon,\delta\to 0}\ \sup_{0<t\le\delta}\ \sup_{\varphi\in\mathcal{F}} \left|(\varphi_t * \psi_\varepsilon * u - \varphi_t * u)(x)\right| = 0, \ \text{ a.e. } x, \tag{12.25}$$

and that, for each $\delta > 0$,

$$\lim_{\varepsilon\to 0}\ \sup_{t\ge\delta}\ \sup_{\varphi\in\mathcal{F}} \left|(\varphi_t * \psi_\varepsilon - \varphi_t) * u(x)\right| = 0. \tag{12.26}$$

In fact, (12.25) holds whenever x is a *Lebesgue point* for u (see the exercises for more on this), and (12.26) holds for all $x \in \mathbb{R}^n$, since $u \in L^1(\mathbb{R}^n)$ and, for all $\varphi \in \mathcal{F}$, we have $\|\varphi_t * \psi_\varepsilon - \varphi_t\|_{L^\infty} \le C\varepsilon t^{-n-1}$.

Corollary 12.8. *Let $T_y u(x) = u(x+y)$. Then, for $u \in \mathfrak{H}^1(\mathbb{R}^n)$,*

$$\|T_y u - u\|_{\mathfrak{H}^1} \longrightarrow 0, \ \text{ as } \ |y| \to 0. \tag{12.27}$$

Proof. Since $\|T\|_{\mathcal{L}(\mathfrak{H}^1)} = 1$ for all y, it suffices to show that (12.27) holds for u in a dense subspace of $\mathfrak{H}^1(\mathbb{R}^n)$. Thus it suffices to show that, for each $\varepsilon > 0$, $u \in \mathfrak{H}^1(\mathbb{R}^n)$,

$$\lim_{|y|\to 0}\ \|T_y(\psi_\varepsilon * u) - \psi_\varepsilon * u\|_{\mathfrak{H}^1} = 0. \tag{12.28}$$

But $T_y(\psi_\varepsilon * u) - \psi_\varepsilon * u = (\psi_{\varepsilon y} - \psi_\varepsilon) * u$, where

$$\psi_{\varepsilon y}(x) - \psi_\varepsilon(x) = \varepsilon^{-n}\big[\psi(\varepsilon^{-1}(x+y)) - \psi(\varepsilon^{-1}x)\big]. \tag{12.29}$$

Thus

$$\begin{aligned}\|T_y(\psi_\varepsilon * u) - \psi_\varepsilon * u\|_{\mathfrak{H}^1} &= \sup_{t>0,\varphi\in\mathcal{F}} \|(\psi_{\varepsilon y} - \psi_\varepsilon) * \varphi_t * u\|_{L^1} \\ &\le \|\psi_{\varepsilon y} - \psi_\varepsilon\|_{L^\infty}\|u\|_{\mathfrak{H}^1} \\ &\le C|y|\varepsilon^{-n-1}\|u\|_{\mathfrak{H}^1},\end{aligned} \tag{12.30}$$

which finishes the proof.

It is clear that we can replace $\mathfrak{H}^1$ by $\mathfrak{h}^1$ in Proposition 12.7 and Corollary 12.8, obtaining, for $u \in \mathfrak{h}^1(\mathbb{R}^n)$,

$$\|\psi_\varepsilon * u - u\|_{\mathfrak{h}^1} \longrightarrow 0, \tag{12.31}$$

as $\varepsilon \to 0$, and

$$\|T_y u - u\|_{\mathfrak{h}^1} \longrightarrow 0, \tag{12.32}$$

as $|y| \to 0$.

We can also approximate by cut-offs:

Proposition 12.9. *Fix $\chi \in C_0^\infty(\mathbb{R}^n)$, so that $\chi(x) = 1$ for $|x| \le 1$, 0 for $|x| \ge 2$, and $0 \le \chi \le 1$. Set $\chi_R(x) = \chi(x/R)$. Then, given $u \in \mathfrak{h}^1(\mathbb{R}^n)$, we have*

$$\lim_{R\to\infty} \|u - \chi_R u\|_{\mathfrak{h}^1} = 0. \tag{12.33}$$

Proof. Clearly, $\mathcal{G}^b(u - \chi_R u)(x) = 0$, for $|x| \le R - 1$, so

$$\lim_{R\to\infty} G^b(u - \chi_R u)(x) = 0, \quad \forall\, x \in \mathbb{R}^n.$$

To get (12.33), we would like to appeal to the dominated convergence theorem. In fact, the estimates (12.17)–(12.19) (with $g = 1 - \chi_R$) give

$$\mathcal{G}^b(u - \chi_R u)(x) \le \mathcal{G}^b u(x) + Av(x), \quad \forall\, R \ge 1, \tag{12.34}$$

where $A = \|\nabla\chi\|_{L^\infty}$, and $v(x)$ is given by (12.19), with $r = 1$, so $v \in L^1(\mathbb{R}^n)$. Thus dominated convergence does give

$$\lim_{R\to\infty} \|\mathcal{G}^b(u - \chi_R u)\|_{L^1} = 0, \tag{12.35}$$

and the proof is done.

Together with (12.31), this gives

Corollary 12.10. *The space $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathfrak{h}^1(\mathbb{R}^n)$.*

A slightly more elaborate argument shows that

$$\mathcal{D}_0 = \Big\{u \in C_0^\infty(\mathbb{R}^n) : \int u(x)\,dx = 0\Big\} \tag{12.36}$$

is dense in $\mathfrak{H}^1(\mathbb{R}^n)$; see [Sem].

One significant measure of how much smaller $\mathfrak{H}^1(\mathbb{R}^n)$ is than $L^1(\mathbb{R}^n)$ is the following identification of an element of the dual of $\mathfrak{H}^1(\mathbb{R}^n)$ that does not belong to $L^\infty(\mathbb{R}^n)$.

Proposition 12.11. *We have*

$$\Big|\int f(x)\ \log|x|\,dx\Big| \le C\|f\|_{\mathfrak{H}^1}. \tag{12.37}$$

Proof. Let $\lambda(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $\lambda(x) = 1$ for $|x| \le 1,\ \lambda(x) = 0$ for $|x| \ge 2$. Set

$$\ell(x) = -\sum_{j=1}^\infty \lambda(2^j x) + \sum_{j=0}^\infty \big(1 - \lambda(2^{-j}x)\big). \tag{12.38}$$

It is easy to check that

$$\log|x| - (\log 2)\ell(x) \in L^\infty(\mathbb{R}^n). \tag{12.39}$$

Thus it suffices to estimate $\int f(x)\ell(x)\,dx$. We have

$$\Big|\int f(x)\ell(x)\,dx\Big| \le \sum_{j=-\infty}^\infty \Big|\int f(x)\lambda(2^j x)\,dx\Big|. \tag{12.40}$$

We claim that, for each $j \in \mathbb{Z}$,

$$\Big|\int f(x)\lambda(2^j x)\,dx\Big| \le C\,2^{-jn} \inf_{B_{2^{-j}}(0)} \mathcal{G}f. \tag{12.41}$$

In fact, given $j \in \mathbb{Z},\ z \in B_{2^{-j}}(0)$, we can write

$$\int f(x)2^{jn}\lambda(2^j x)\,dx = K\varphi_r * f(z), \tag{12.42}$$

with $r = 2^{2-j}$, $K = K(\lambda, n)$, for some $\varphi \in \mathcal{F}$; say $\varphi(x)$ is a multiple of a translate of $\lambda(4x)$. Consequently, with $S_j = B_{2^{-j}}(0)$, we have

$$\Big|\int f(x)\ell(x)\,dx\Big| \le C \sum_{j=-\infty}^{\infty} \int_{S_j \setminus S_{j+1}} \mathcal{G}f = C\|f\|_{\mathfrak{H}^1}. \tag{12.43}$$

By Corollary 12.8, we have the following:

Corollary 12.12. *Given $f \in \mathfrak{H}^1(\mathbb{R}^n)$,*

$$\log * f \in C(\mathbb{R}^n). \tag{12.44}$$

The result (12.37) is a very special case of the fact that the dual of $\mathfrak{H}^1(\mathbb{R}^n)$ is naturally isomorphic to a space of functions called BMO($\mathbb{R}^n$). This was established in [FS]. The special case given above is the only case we will use in this book. More about this duality and its implications for analysis can be found in the treatise [S3]. Also, [S3] has other important information about Hardy spaces, including a study of singular integral operators on these spaces.

The next result is a variant of the div-curl lemma (discussed in Exercises for § 6), due to [CLMS]. It states that a certain function that obviously belongs to $L^1(\mathbb{R}^n)$ actually belongs to $\mathfrak{H}^1(\mathbb{R}^n)$. Together with Corollary 12.12, this produces a useful tool for PDE. An application will be given in § 12B of Chap. 14. The proof below follows one of L. Evans and S. Muller, given in [Ev2].

Proposition 12.13. *If $u \in L^2(\mathbb{R}^n, \mathbb{R}^n)$, $v \in H^1(\mathbb{R}^n)$, and div $u = 0$, then $u \cdot \nabla v \in \mathfrak{H}^1(\mathbb{R}^n)$.*

Proof. Clearly, $u \cdot \nabla v \in L^1(\mathbb{R}^n)$. Now, with $\varphi \in C_0^\infty(\mathbb{R}^n)$, supported in the unit ball, set $\varphi_r(y) = r^{-n}\varphi\big(r^{-1}(x-y)\big)$. We have

$$\int (u \cdot \nabla v)\varphi_r\,dy = -\int_{B_r(x)} (v - v_{x,r})u \cdot \nabla\varphi_r\,dy, \tag{12.45}$$

since div $u = 0$. Thus, with $C_0 = \|\nabla\varphi\|_{L^\infty}$,

$$\Big|\int (u \cdot \nabla v)\varphi_r\,dy\Big| \le \frac{C_0}{r} \int_{B_r(x)} |u - v_{x,r}| \cdot |u|\,dy. \tag{12.46}$$

Take

$$p \in \Big(2, \frac{2n}{n-2}\Big), \quad q = \frac{p}{p-1} \in (1,2). \tag{12.47}$$

Then

$$\begin{aligned}\Big|\int (u\cdot\nabla v)\varphi_r\,dy\Big| &\le \frac{C_0}{r}\Bigg(\int\limits_{B_r(x)} |v-v_{x,r}|^p\,dy\Bigg)^{1/p}\Bigg(\int\limits_{B_r(x)} |u|^q\,dy\Bigg)^{1/q} \\ &\le \frac{C_0}{r^a}\Bigg(\int\limits_{B_r(x)} |\nabla v|^\rho\,dy\Bigg)^{1/\rho}\Bigg(\int\limits_{B_r(x)} |u|^q\,dy\Bigg)^{1/q},\end{aligned} \tag{12.48}$$

where $\rho = pn/(p+n) < 2$ and $a = n+1$. Consequently,

$$\begin{aligned}\Big|\int (u\cdot\nabla v)\varphi_r\,dy\Big| &\le C_0\mathcal{M}\big(|\nabla v|^\rho\big)^{1/\rho}\mathcal{M}\big(|u|^q\big)^{1/q} \\ &\le C_0\big\{\mathcal{M}\big(|\nabla v|^\rho\big)^{2/\rho} + \mathcal{M}\big(|u|^q\big)^{2/q}\big\}.\end{aligned} \tag{12.49}$$

By Corollary 12.3, we have $\big\|\mathcal{M}\big(|\nabla v|^\rho\big)\big\|_{L^{2/\rho}} \le C\big\||\nabla v|^\rho\big\|_{L^{2/\rho}}$, and so

$$\int \mathcal{M}\big(|\nabla v|^\rho\big)^{2/\rho}\,dx \le C\int |\nabla v|^2\,dx.$$

Similarly,

$$\int \mathcal{M}\big(|u|^q\big)^{2/q}\,dx \le C\int |u|^2\,dx.$$

Hence

$$\|u\cdot\nabla v\|_{\mathfrak{H}^1} = \sup_{\varphi\in\mathcal{F},r>0}\Big\|\int (u\cdot\nabla v)\varphi_r\,dy\Big\|_{L^1} \le C\big(\|\nabla v\|^2 + \|u\|_{L^2}^2\big). \tag{12.50}$$

We next establish a localized version of Proposition 12.13.

Proposition 12.14. *Let $\Omega\subset\mathbb{R}^n$ be open. If $u\in L^2(\Omega,\mathbb{R}^n)$, div $u=0$, and $v\in H^1(\Omega)$, then $u\cdot\nabla v\in\mathfrak{H}^1_{loc}(\Omega)$.*

Proof. We may as well suppose $n>1$. Take any $\overline{\mathcal{O}}\subset\Omega$, diffeomorphic to a ball. It suffices to show that $u\cdot\nabla v$ is equal on $\overline{\mathcal{O}}$ to an element of $\mathfrak{H}^1(\mathbb{R}^n)$. Say $\overline{\mathcal{O}}\subset\subset\overline{U}\subset\subset\Omega$, with $\overline{U}$ also diffeomorphic to a ball. Pick $\chi\in C_0^\infty(U)$, $\chi=1$ on $\overline{\mathcal{O}}$.

Let $\tilde{u}\in L^2(\Omega,\Lambda^{n-1})$ correspond to u via the volume element on Ω. Then $d\tilde{u}=0$. We use the Hodge decomposition of $L^2(U,\Lambda^{n-1})$, with absolute boundary condition:

$$\tilde{u} = d\delta G^A\tilde{u} + \delta d G^A\tilde{u} + P_h^A\tilde{u} \quad \text{on } u. \tag{12.51}$$

Since $d\tilde{u} = 0$, we have by (9.48) of Chap. 5 that $\delta d G^A \tilde{u} = 0$. Also, given $n > 1$, $\mathcal{H}^{n-1}(\overline{U}) = 0$, so $P_h^A \tilde{u} = 0$, too, and so

$$\tilde{u} = d\tilde{w}, \quad \tilde{w} \in H^1(U, \Lambda^{n-2}). \tag{12.52}$$

Having this, we define a vector field u_0 on $\mathbb{R}^n$ so that $\tilde{u}_0 = d(\chi\tilde{w})$, and we set $v_0 = \chi v$. It follows that $u_0,\ v_0$ satisfy the hypotheses of Proposition 12.13, so $u_0 \cdot \nabla v_0 \in \mathfrak{H}^1(\mathbb{R}^n)$. But $u_0 \cdot \nabla v_0 = u \cdot \nabla v$ on $\overline{\mathcal{O}}$, so the proof is done.

Let us finally mention that while we have only briefly alluded to the space BMO, it has also proven to be of central importance, especially since the work of [FS]. More about the role of BMO in paradifferential operator calculus can be found in [T2]. Also, Proposition 12.13 can be deduced from a commutator estimate involving BMO, as explained in [CLMS]; see also [AT].

Exercises

We say $x \in \mathbb{R}^n$ is a Lebesgue point for $f \in L^1(\mathbb{R}^n)$ provided

$$\lim_{r \to 0} \frac{1}{\operatorname{vol}(B_r)} \int\limits_{B_r(x)} |f(y) - f(x)|\, dy = 0.$$

In Exercises 1 and 2, we establish that, given $f \in L^1(\mathbb{R}^n)$, a.e. $x \in \mathbb{R}^n$ is a Lebesgue point of f.

1. Set

$$\widetilde{\mathcal{M}}(f)(x) = \sup_{r > 0} \frac{1}{\operatorname{vol}(B_r)} \int\limits_{B_r(x)} |f(y) - f(x)|\, dy.$$

Show that, for all $x \in \mathbb{R}^n$,

$$\widetilde{\mathcal{M}}(f)(x) \le \mathcal{M}(f)(x) + |f(x)|.$$

2. Given $\lambda > 0$, let

$$E_\lambda = \Big\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{1}{\operatorname{vol}(B_r)} \int\limits_{B_r(x)} |f(y) - f(x)|\, dy > \lambda \Big\}.$$

Take $\varepsilon > 0$, and take $g \in C_0^\infty(\mathbb{R}^n)$ so that $\|f - g\|_{L^1} < \varepsilon$. Show that E_λ is unchanged if f is replaced by $f - g$. Deduce that

$$E_\lambda \subset \Big\{ x : \mathcal{M}(f - g)(x) > \frac{1}{2}\lambda \Big\} \cup \Big\{ x : |f(x) - g(x)| > \frac{1}{2}\lambda \Big\},$$

and hence, via Proposition 12.1,

$$\operatorname{meas}(E_\lambda) \le \frac{C}{\lambda} \|f - g\|_{L^1} \le \frac{C\varepsilon}{\lambda}.$$

Deduce that $\operatorname{meas}(E_\lambda) = 0$, $\forall\, \lambda > 0$, and hence a.e. $x \in \mathbb{R}^n$ is a Lebesgue point for f.

3. Now verify that (12.25) holds whenever x is a Lebesgue point of u.
4. If $u : \mathbb{R}^2 \to \mathbb{R}^2$, show that

$$u \in H^1(\mathbb{R}^2) \Longrightarrow \det Du \in \mathfrak{H}^1(\mathbb{R}^2).$$

(*Hint*: Compute div w, when $w = (\partial_y u_1, -\partial_x u_2)$.)

5. If $u : \mathbb{R}^2 \to \mathbb{R}^3$, show that

$$u \in H^1(\mathbb{R}^2) \Longrightarrow u_x \times u_y \in \mathfrak{H}^1(\mathbb{R}^2).$$

(*Hint*. Show that the first argument of $u_x \times u_y$ is $\det Dv$, where $v = (u_2, u_3)$.)

A. Variations on complex interpolation

Let X and Y be Banach spaces, assumed to be linear subspaces of a Hausdorff locally convex space V (with continuous inclusions). We say (X, Y, V) is a compatible triple. For $\theta \in (0, 1)$, the classical complex interpolation space $[X, Y]_\theta$, introduced in Chap. 4 and much used in this chapter, is defined as follows. First, $Z = X + Y$ gets a natural norm; for $v \in X + Y$,

$$\|v\|_Z = \inf\{\|v_1\|_X + \|v_2\|_Y : v = v_1 + v_2,\ v_1 \in X,\ v_2 \in Y\}. \tag{A.1}$$

One has $X + Y \approx X \oplus Y/L$, where $L = \{(v, -v) : x \in X \cap Y\}$ is a closed linear subspace, so $X + Y$ is a Banach space. Let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, with closure $\overline{\Omega}$. Define $\mathcal{H}_\Omega(X, Y)$ to be the space of functions $f : \overline{\Omega} \to Z = X + Y$, continuous on $\overline{\Omega}$, holomorphic on Ω (with values in $X + Y$), satisfying $f : \{\operatorname{Im} z = 0\} \to X$ continuous, $f : \{\operatorname{Im} z = 1\} \to Y$ continuous, and

$$\|u(z)\|_Z \le C, \quad \|u(iy)\|_X \le C, \quad \|u(1 + iy)\|_Y \le C, \tag{A.2}$$

for some $C < \infty$, independent of $z \in \overline{\Omega}$ and $y \in \mathbb{R}$. Then, for $\theta \in (0, 1)$,

$$[X, Y]_\theta = \{u(\theta) : u \in \mathcal{H}_\Omega(X, Y)\}. \tag{A.3}$$

One has

$$[X, Y]_\theta \approx \mathcal{H}_\Omega(X, Y)/\{u \in \mathcal{H}_\Omega(X, Y) : u(\theta) = 0\}, \tag{A.4}$$

giving $[X, Y]_\theta$ the sructure of a Banach space. Here

$$\|u\|_{\mathcal{H}_\Omega(X,Y)} = \sup_{z\in\overline{\Omega}} \|u(z)\|_Z + \sup_y \|u(iy)\|_X + \sup_y \|u(1 + iy)\|_Y. \tag{A.5}$$

If I is an interval in $\mathbb{R}$, we say a family of Banach spaces $X_s,\ s \in I$ (subspaces of V) forms a complex interpolation scale provided that for $s, t \in I,\ \theta \in (0, 1)$,

$$[X_s, X_t]_\theta = X_{(1-\theta)s+\theta t}. \tag{A.6}$$

Examples of such scales include L^p-Sobolev spaces $X_s = H^{s,p}(M)$, $s \in \mathbb{R}$, provided $p \in (1,\infty)$, as shown in § 6 of this chapter, the case $p = 2$ having been done in Chap. 4. It turns out that (A.6) fails for Zygmund spaces $X_s = C^s_*(M)$, but an analogous identity holds for some closely related interpolation functors, which we proceed to introduce.

If (X, Y, V) is a compatible triple, as defined in above, we define $\mathcal{H}_\Omega(X, Y, V)$ to be the space of functions $u : \overline{\Omega} \to X + Y = Z$ such that

$$u : \Omega \longrightarrow Z \quad \text{is holomorphic,} \tag{A.7}$$

$$\|u(z)\|_Z \le C, \quad \|u(iy)\|_X \le C, \quad \|u(1+iy)\|_Y \le C, \tag{A.8}$$

and

$$u : \overline{\Omega} \longrightarrow V \quad \text{is continuous.} \tag{A.9}$$

For such u, we again use the norm (A.5). Note that the only difference with $\mathcal{H}_\Omega(X, Y)$ is that we are relaxing the continuity hypothesis for u on $\overline{\Omega}$. $\mathcal{H}_\Omega(X, Y, V)$ is also a Banach space, and we have a natural isometric inclusion

$$\mathcal{H}_\Omega(X, Y) \hookrightarrow \mathcal{H}_\Omega(X, Y, V). \tag{A.10}$$

Now for $\theta \in (0, 1)$ we set

$$[X, Y]_{\theta;V} = \{u(\theta) : u \in \mathcal{H}_\Omega(X, Y, V)\}. \tag{A.11}$$

Again this space gets a Banach space structure, via

$$[X, Y]_{\theta;V} \approx \mathcal{H}_\Omega(X, Y, V)/\{u \in \mathcal{H}_\Omega(X, Y, V) : u(\theta) = 0\}, \tag{A.12}$$

and there is a natural continuous injection

$$[X, Y]_\theta \hookrightarrow [X, Y]_{\theta;V}. \tag{A.13}$$

Sometimes this is an isomorphism. In fact, sometimes $[X, Y]_\theta = [X, Y]_{\theta;V}$ for practically all reasonable choices of V. For example, one can verify this for $X = L^p(\mathbb{R}^n)$, $Y = H^{s,p}(\mathbb{R}^n)$, the L^p-Sobolev space, with $p \in (1,\infty)$, $s \in (0,\infty)$. On the other hand, there are cases where equality in (A.10) does not hold, and where $[X, Y]_{\theta;V}$ is of greater interest than $[X, Y]_\theta$.

We next define $[X, Y]^b_\theta$. In this case we assume X and Y are Banach spaces and $Y \subset X$ (continuously). We take Ω as above, and set $\widetilde{\Omega} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z \le 1\}$, i.e., we throw in the right boundary but not the left boundary. We then define $\mathcal{H}^b_\Omega(X, Y)$ to be the space of functions $u : \widetilde{\Omega} \to X$ such that

$$\begin{gathered} u : \Omega \longrightarrow X \quad \text{is holomorphic,} \\ \|u(z)\|_X \le C, \quad \|u(1+iy)\|_Y \le C, \\ u : \widetilde{\Omega} \longrightarrow X \quad \text{is continuous.} \end{gathered} \tag{A.14}$$

Note that the essential difference between $\mathcal{H}_\Omega(X,Y)$ and the space we have just introduced is that we have completely dropped any continuity requirement at $\{\operatorname{Re} z = 0\}$. We also do not require continuity from $\{\operatorname{Re} z = 1\}$ to Y. The space $\mathcal{H}^b_\Omega(X,Y)$ is a Banach space, with norm

$$\|u\|_{\mathcal{H}^b_\Omega(X,Y)} = \sup_{z\in\widetilde{\Omega}} \|u(z)\|_X + \sup_y \|u(1+iy)\|_Y. \tag{A.15}$$

Now, for $\theta \in (0,1)$, we set

$$[X,Y]^b_\theta = \{u(\theta) : u \in \mathcal{H}^b_\Omega(X,Y)\}, \tag{A.16}$$

with the same sort of Banach space structure as arose in (A.4) and (A.12). We have continuous injections

$$[X,Y]_\theta \hookrightarrow [X,Y]_{\theta;X} \hookrightarrow [X,Y]^b_\theta. \tag{A.17}$$

Our next task is to extend the standard result on operator interpolation from the setting of $[X,Y]_\theta$ to that of $[X,Y]_{\theta;V}$ and $[X,Y]^b_\theta$.

Proposition A.1. *Let (X_j, Y_j, V_j) be compatible triples, $j = 1,2$. Assume that $T : V_1 \to V_2$ is continuous and that*

$$T : X_1 \longrightarrow X_2, \quad T : Y_1 \longrightarrow Y_2, \tag{A.18}$$

continuously. (Continuity is automatic, by the closed graph theorem.) Then, for each $\theta \in (0,1)$,

$$T : [X_1,Y_1]_{\theta;V_1} \longrightarrow [X_2,Y_2]_{\theta;V_2}. \tag{A.19}$$

Furthermore, if $Y_j \subset X_j$ (continuously) and T is a continuous linear map satisfying (A.18), then for each $\theta \in (0,1)$,

$$T : [X_1,Y_1]^b_\theta \longrightarrow [X_2,Y_2]^b_\theta. \tag{A.20}$$

Proof. Given $f \in [X_1,Y_2]_{\theta;V}$, pick $u \in \mathcal{H}_\Omega(X_1,Y_1,V_1)$ such that $f = u(\theta)$. Then we have

$$\mathcal{T} : \mathcal{H}_\Omega(X_1,Y_1,V_1) \to \mathcal{H}_\Omega(X_2,Y_2,V_2), \quad (\mathcal{T}u)(z) = Tu(z), \tag{A.21}$$

and hence

$$Tf = (\mathcal{T}u)(\theta) \in [X_2,Y_2]_{\theta;V_2}. \tag{A.22}$$

This proves (A.19). The proof of (A.20) is similar.

Remark: In case $V = X + Y$, with the weak topology, $[X, Y]_{\theta;V}$ is what is denoted $(X, Y)^w_\theta$ in [JJ], and called the weak complex interpolation space.

Alternatives to (A.6) for a family X_s of Banach spaces include

$$[X_s, X_t]_{\theta;V} = X_{(1-\theta)s+\theta t} \tag{A.23}$$

and

$$[X_s, X_t]^b_\theta = X_{(1-\theta)s+\theta t}. \tag{A.24}$$

Here, as before, we take $\theta \in (0, 1)$. It is an exercise, using results of § 6, to show that both (A.23) and (A.24), as well as (A.6), hold when $X_s = H^{s,p}(M)$, given $p \in (1, \infty)$, where M can be $\mathbb{R}^n$ or a compact Riemannian manifold. We now discuss the situation for Zygmund spaces.

We start with Zygmund spaces on the torus $\mathbb{T}^n$. We recall from § 8 that the Zygmund space $C^r_*(\mathbb{T}^n)$ is defined for $r \in \mathbb{R}$, as follows. Take $\varphi \in C_0^\infty(\mathbb{R}^n)$, radial, satisfying $\varphi(\xi) = 1$ for $|\xi| \le 1$. Set $\varphi_k(\xi) = \varphi(2^{-k}\xi)$. Then set $\psi_0 = \varphi$, $\psi_k = \varphi_k - \varphi_{k-1}$ for $k \in \mathbb{N}$, so $\{\psi_k : k \ge 0\}$ is a Littlewood–Paley partition of unity. We define $C^r_*(\mathbb{T}^n)$ to consist of $f \in \mathcal{D}'(\mathbb{T}^n)$ such that

$$\|f\|_{C^r_*} = \sup_{k \ge 0} 2^{kr} \|\psi_k(D) f\|_{L^\infty} < \infty. \tag{A.25}$$

With $\Lambda = (I - \Delta)^{1/2}$ and $s, t \in \mathbb{R}$, we have

$$\Lambda^{s+it} : C^r_*(\mathbb{T}^n) \longrightarrow C^{r-s}_*(\mathbb{T}^n). \tag{A.26}$$

By material developed in § 8,

$$r \in \mathbb{R}^+ \setminus \mathbb{Z}^+ \Longrightarrow C^r_*(\mathbb{T}^n) = C^r(\mathbb{T}^n), \tag{A.27}$$

where, if $r = k + \alpha$ with $k \in \mathbb{Z}^+$ and $0 < \alpha < 1$, $C^r(\mathbb{T}^n)$ consists of functions whose derivatives of order $\le k$ are Hölder continuous of exponent α.

We aim to show the following.

Proposition A.2. *If* $r < s < t$ *and* $0 < \theta < 1$, *then*

$$[C^s_*(\mathbb{T}^n), C^t_*(\mathbb{T}^n)]_{\theta;C^r_*(\mathbb{T}^n)} = C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n), \tag{A.28}$$

and

$$[C^s_*(\mathbb{T}^n), C^t_*(\mathbb{T}^n)]^b_\theta = C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n). \tag{A.29}$$

Proof. First, suppose $f \in [C^s_*, C^t_*]_{\theta;C^r_*}$, so $f = u(\theta)$ for some $u \in \mathcal{H}_\Omega(C^s_*, C^t_*, C^r_*)$. Then consider

$$v(z) = e^{z^2} \Lambda^{(t-s)z} \Lambda^s u(z). \tag{A.30}$$

Bounds of the type (A.8) on u, together with (8.13) in the torus setting, yield

$$\|v(iy)\|_{C^0_*},\ \|v(1+iy)\|_{C^0_*} \le C, \tag{A.31}$$

with C independent of $y \in \mathbb{R}$. In other words,

$$\|\psi_k(D)v(z)\|_{L^\infty} \le C, \quad \operatorname{Re} z = 0, 1, \tag{A.32}$$

with C independent of Im z and k. Also, for each $k \in \mathbb{Z}^+$, $\psi_k(D)v : \overline{\Omega} \to L^\infty(\mathbb{T}^n)$ continuously, so the maximum principle implies

$$\|\psi_k(D)\Lambda^{(t-s)\theta}\Lambda^s f\|_{L^\infty} \le C, \tag{A.33}$$

independent of $k \in \mathbb{Z}^+$. This gives $\Lambda^{(1-\theta)s+\theta t} f \in C^0_*$, hence $f \in C_*^{(1-\theta)s+\theta t}(\mathbb{T}^n)$.

Second, suppose $f \in C_*^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. Set

$$u(z) = e^{z^2}\Lambda^{(\theta-z)(t-s)} f. \tag{A.34}$$

Then $u(\theta) = e^{\theta^2} f$. We claim that

$$u \in \mathcal{H}_\Omega(C^s_*, C^t_*, C^r_*), \tag{A.35}$$

as long as $r < s < t$. Once we establish this, we will have the reverse containment in (A.28). Bounds of the form

$$\|u(z)\|_{C^s_*} \le C, \quad \|u(1+iy)\|_{C^t_*} \le C \tag{A.36}$$

follow from (8.13), and are more than adequate versions of (A.8). It remains to establish that

$$u : \overline{\Omega} \longrightarrow C^r_*(\mathbb{T}^n), \ \text{ continuously.} \tag{A.37}$$

Indeed, we know $u : \overline{\Omega} \to C^s_*(\mathbb{T}^n)$ is bounded. It is readily verified that

$$u : \overline{\Omega} \longrightarrow \mathcal{D}'(\mathbb{T}^n), \ \text{ continuously,} \tag{A.38}$$

and that

$$r < s \Longrightarrow C^s_*(\mathbb{T}^n) \hookrightarrow C^r_*(\mathbb{T}^n) \ \text{ is compact.} \tag{A.39}$$

The result (A.37) follows from these observations. Thus the proof of (A.28) is complete.

We turn to the proof of (A.29). If $u \in \mathcal{H}^b_\Omega(C^s_*, C^t_*)$, form $v(z)$ as in (A.30), and for $\varepsilon \in (0,1]$ set

$$v_\varepsilon(z) = e^{-\varepsilon\Lambda} v(z), \quad v_\varepsilon : \widetilde{\Omega} \to C^0_*(\mathbb{T}^n) \text{ bounded and continuous} \tag{A.40}$$

(with bound that might depend on ε). We have

$$\psi_k(D) v_\varepsilon(\varepsilon + iy) = e^{(\varepsilon+iy)^2} \psi_k(D) e^{-\varepsilon\Lambda} \Lambda^{(t-s)\varepsilon} \Lambda^{i(t-s)y} \Lambda^s u(z). \tag{A.41}$$

Now $\{\Lambda^s u(z) : z \in \widetilde{\Omega}\}$ is bounded in $C^0_*(\mathbb{T}^n)$, and the operator norm of $\Lambda^{i(t-s)y}$ on $C^0_*(\mathbb{T}^n)$ is exponentially bounded in $|y|$. We have

$$\{e^{-\varepsilon\Lambda}\Lambda^{\varepsilon(t-s)} : 0 < \varepsilon \le 1\} \text{ bounded in } \mathrm{OPS}^0_{1,0}(\mathbb{T}^n), \tag{A.42}$$

hence bounded in operator norm on $C^0_*(\mathbb{T}^n)$. We deduce that

$$\|\psi_k(D) v_\varepsilon(\varepsilon + iy)\|_{L^\infty} \le C, \tag{A.43}$$

independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. The hypothesis on u also implies

$$\|\psi_k(D) v_\varepsilon(1 + iy)\|_{L^\infty} \le C, \tag{A.44}$$

independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. Now the maximum principle applies. Given $\theta \in (0,1)$,

$$\|\psi_k(D) e^{-\varepsilon\Lambda} v(\theta)\|_{L^\infty} \le C, \tag{A.45}$$

independent of ε. Taking $\varepsilon \searrow 0$ yields $v(\theta) \in C^0_*(\mathbb{T}^n)$, hence $u(\varepsilon) \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$.

This proves one inclusion in (A.29). The proof of the reverse inclusion is similar to that for (A.28). Given $f \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$, take $u(z)$ as in (A.34). The claim is that $u \in \mathcal{H}^b_\Omega(C^s_*, C^t_*)$. We already have (A.36), and the only thing that remains is to check that

$$u : \widetilde{\Omega} \longrightarrow C^s_*(\mathbb{T}^n) \text{ continuously,} \tag{A.46}$$

and this is straightforward. (What fails is continuity of $u : \overline{\Omega} \to C^s_*(\mathbb{T}^n)$ at the left boundary of $\overline{\Omega}$.)

Remark: In contrast to (A.28)–(A.29), one has

$$[C^s_*(\mathbb{T}^n), C^t_*(\mathbb{T}^n)]_\theta = \text{ closure of } C^\infty(\mathbb{T}^n) \text{ in } C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n). \tag{A.47}$$

Related results are given in [Tri].

If $\mathrm{OPS}^m_{1,0}(\mathbb{T}^n)$ denotes the class of pseudodifferential operators on $\mathbb{T}^n$ with symbols in $S^m_{1,0}$, then for all $s, m \in \mathbb{R}$,

$$P \in \mathrm{OPS}^m_{1,0}(\mathbb{T}^n) \Longrightarrow P : C^s_*(\mathbb{T}^n) \to C^{s-m}_*(\mathbb{T}^n). \tag{A.48}$$

Cf. Proposition 8.6. Using coordinate invariance of $OPS^m_{1,0}$ and of $C^r(\mathbb{T}^n)$ for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, we deduce invariance of $C^s_*(\mathbb{T}^n)$ under diffeomorphisms, for all $s \in \mathbb{R}$.

From here, we can develop the spaces $C^s_*(M)$ on a compact Riemannian manifold M and the spaces $C^s_*(\overline{M})$ on a compact manifold with boundary. These developments are done in § 8.

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14

Nonlinear Elliptic Equations

Introduction

Methods of the calculus of variations applied to problems in geometry and classical continuum mechanics often lead to elliptic PDE that are not linear. We discuss a number of examples and some of the developments that have arisen to treat such problems.

The simplest nonlinear elliptic problems are the semilinear ones, of the form $Lu = f(x, D^{m-1}u)$, where L is a linear elliptic operator of order m and the nonlinear term $f(x, D^{m-1}u)$ involves derivatives of u of order $\leq m-1$. In § 1 we look at semilinear equations of the form

$$\Delta u = f(x, u), \tag{0.1}$$

on a compact, Riemannian manifold M, with or without boundary. The Dirichlet problem for (0.1) is solvable provided $\partial_u f(x, u) \geq 0$ if each connected component of M has a nonempty boundary. If M has no boundary, this condition does not always imply the solvability of (0.1), but one can solve this equation if one requires $f(x, u)$ to be positive for $u > a_1$ and negative for $u < a_0$. We use three approaches to (0.1): a variational approach, minimizing a function defined on a certain function space, the "method of continuity," solving a one-parameter family of equations of the type (0.1), and a variant of the method of continuity that involves a Leray–Schauder fixed-point theorem. This fixed-point theorem is established in Appendix B, at the end of this chapter.

A particular example of (0.1) is

$$\Delta u = k(x) - K(x)e^{2u}, \tag{0.2}$$

which arises when one has a 2-manifold with Gauss curvature $k(x)$ and wants to multiply the metric tensor by the conformal factor e^{2u} and obtain $K(x)$ as the Gauss curvature. The condition $\partial_u f(x, u) \geq 0$ implies that $K(x) \leq 0$ in (0.2).

In § 2 we study (0.2) on a compact, Riemannian 2-fold without boundary, given $K(x) < 0$. The Gauss–Bonnet formula implies that $\chi(M) < 0$ is a necessary

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condition for solvability in this case; the main result of § 2 is that this is also a sufficient condition. When you take $K \equiv -1$, this establishes the uniformization theorem for compact Riemann surfaces of negative Euler characteristic. When $\chi(M) = 0$, one takes $K = 0$ and (0.2) is linear. The remaining case of this uniformization theorem, $\chi(M) = 2$, is treated in Chap. 10, § 9.

The next topic is local solvability of nonlinear elliptic PDE. We establish this via the inverse function theorem for C^1-maps on a Banach space. We treat underdetermined as well as determined elliptic equations. We obtain solutions in § 3 with a high but finite degree of regularity. In some cases such solutions are actually C^∞. In § 4 we establish higher regularity for solutions to elliptic PDE that are already known to have a reasonably high degree of smoothness. This result suffices for applications made in § 3, though PDE encountered further on will require much more powerful regularity results.

In § 5 we establish the theorem of J. Nash, on isometric imbeddings of compact Riemannian manifolds in Euclidean space, largely following the ingenious simplification of M. Günther [Gu1], allowing one to apply the inverse function theorem for C^1-maps on a Banach space. Again, the regularity result of § 4 applies, allowing one to obtain a C^∞-isometric imbedding.

In § 6 we introduce the venerable problem of describing minimal surfaces. We establish a number of classical results, in particular the solution to the Plateau problem, producing a (generalized) minimal surface, as the image of the unit disc under a harmonic and essentially conformal map, taking the boundary of the disc homeomorphically onto a given simple closed curve.

In § 7 we begin to study the quasi-linear elliptic PDE satisfied by a function whose graph is a minimal surface. We use results of § 6 to establish some results on the Dirichlet problem for the minimal surface equation, and we note several questions about this Dirichlet problem whose solutions are not simple consequences of the results of § 6, such as boundary regularity. These questions serve as guides to the results of boundary problems for quasi-linear elliptic PDE derived in the next three sections.

In § 8 we apply the paradifferential operator calculus developed in Chap. 13, § 10, to obtain regularity results for nonlinear elliptic boundary problems. We concentrate on second-order PDE (possibly systems) on a compact manifold with boundary $\overline{M}$ and obtain higher regularity for a solution u, assumed a priori to belong to $C^{2+r}(\overline{M})$, for some $r > 0$, for a completely nonlinear elliptic PDE, or to $C^{1+r}(\overline{M})$, in the quasi-linear case. To check how much these results accomplish, we recall the minimal surface equation and note a gap between the regularity of a solution needed to apply the main result (Theorem 8.4) and the regularity a solution is known to possess as a consequence of results in § 7.

Section 9 is devoted to filling that gap, in the scalar case, by the famous DeGiorgi–Nash–Moser theory. We follow mainly J. Moser [Mo2], together with complementary results of C. B. Morrey on nonhomogeneous equations. Morrey's results use spaces now known as Morrey spaces, which are discussed in Appendix A at the end of this chapter.

With the regularity results of §§ 8 and 9 under our belt, we resume the study of the Dirichlet problem for quasi-linear elliptic PDE in the scalar case, in § 10, with particular attention to the minimal surface equation. We note that the Dirichlet problem for general boundary data is not solvable unless there is a restriction on the domain on which a solution u is sought. This has to do with the fact that the minimal surface equation is not "uniformly elliptic." We give examples of some uniformly elliptic PDE, modeling stretched membranes, for which the Dirichlet problem has a solution for general smooth data, on a general, smooth, bounded domain. We do not treat the most general scalar, second-order, quasi-linear elliptic PDE, though our treatment does include cases of major importance. More material can be found in [GT] and [LU].

In § 11 we return to the variational method, introduced in § 1, and prove that a variety of functionals

$$I(u) = \int_{\Omega} F(x, u, \nabla u)\, dV(x) \tag{0.3}$$

possess minima in sets

$$V = \{u \in H^1(\Omega) : u = g \text{ on } \partial\Omega\}. \tag{0.4}$$

The analysis includes cases both of real-valued u and of u taking values in $\mathbb{R}^N$. The latter case gives rise to $N \times N$ elliptic systems, and some regularity results for quasi-linear elliptic systems are established in § 12. Sometimes solutions are not smooth everywhere, but we can show that they are smooth on the complement of a closed set $\Sigma \subset \Omega$ of Hausdorff dimension $< n - 2$ ($n = \dim \Omega$). Results of this nature are called "partial regularity" results.

In § 13 we establish results on linear elliptic equations in nondivergence form, due to N. Krylov and M. Safonov, which take the place of DeGiorgi–Nash–Moser estimates in the study of certain fully nonlinear equations, done in § 14. In § 15 we apply this to equations of the Monge–Ampere type.

In § 16 we obtain some results for nonlinear elliptic equations for functions of two variables that are stronger than results available for functions of more variables.

One attack on second-order, scalar, nonlinear elliptic PDE that has been very active recently is the "viscosity method." We do not discuss this method here; one can consult the review article [CIL] for material on this.

1. A class of semilinear equations

In this section we look at equations of the form

$$\Delta u = f(x, u) \quad \text{on } M, \tag{1.1}$$

where M is a Riemannian manifold, either compact or the interior of a compact manifold $\overline{M}$ with smooth boundary. We first consider the Dirichlet boundary condition

$$u\big|_{\partial M} = g, \tag{1.2}$$

where $\overline{M}$ is connected and has nonempty boundary. We suppose $f \in C^\infty(\overline{M} \times \mathbb{R})$. We will treat (1.1)–(1.2) under the hypothesis that

$$\frac{\partial f}{\partial u} \geq 0. \tag{1.3}$$

Other cases will be considered later in this section. Suppose $F(x,u) = \int_0^u f(x,s)\,ds$, so

$$f(x,u) = \partial_u F(x,u). \tag{1.4}$$

Then (1.3) is the hypothesis that $F(x,u)$ is a convex function of u. Let

$$I(u) = \frac{1}{2}\|du\|^2_{L^2(M)} + \int\limits_M F\big(x,u(x)\big)\,dV(x). \tag{1.5}$$

We will see that a solution to (1.1)–(1.2) is a critical point of I on the space of functions u on M, equal to g on ∂M.

We will make the following *temporary* restriction on F:

$$\text{For } |u| \geq K,\ \partial_u f(x,u) = 0, \tag{1.6}$$

so $F(x,u)$ is linear in u for $u \geq K$ and for $u \leq -K$. Thus, for some constant L,

$$|\partial_u\, F(x,u)| \leq L \quad \text{on } \overline{M} \times \mathbb{R}. \tag{1.7}$$

Let

$$V = \{u \in H^1(M) : u = g \text{ on } \partial M\}. \tag{1.8}$$

Lemma 1.1. *Under the hypotheses (1.3)–(1.7), we have the following results about the functional $I : V \to \mathbb{R}$:*

$$I \text{ is strictly convex}; \tag{1.9}$$

$$I \text{ is Lipschitz continuous}, \tag{1.10}$$

with the norm topology on V;

$$I \text{ is bounded below;} \tag{1.11}$$

and

$$I(v) \to +\infty, \quad as \ \|v\|_{H^1} \to \infty. \tag{1.12}$$

Proof. (1.9) is trivial. (1.10) follows from

$$|F(x,u) - F(x,v)| \le L|u-v|, \tag{1.13}$$

which follows from (1.7). The convexity of $F(x,u)$ in u implies

$$F(x,u) \ge -C_0|u| - C_1. \tag{1.14}$$

Hence

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|du\|^2 - C_0\|u\|_{L^1} - C_1' \\ &\ge \frac{1}{4}\|du\|_{L^2}^2 + \frac{1}{2}B\|u\|_{L^2}^2 - C\|u\|_{L^2} - C', \end{aligned} \tag{1.15}$$

since

$$\frac{1}{2}\|du\|_{L^2}^2 \ge B\|u\|_{L^2}^2 - C'', \quad \text{for } u \in V. \tag{1.16}$$

The last line in (1.15) clearly implies (1.11) and (1.12).

Proposition 1.2. *Under the hypotheses (1.3)–(1.7), $I(u)$ has a unique minimum on V.*

Proof. Let $\alpha_0 = \inf_V I(u)$. By (1.11), α_0 is finite. Pick R such that $K = V \cap B_R(0) \neq \emptyset$, where $B_R(0)$ is the ball of radius R centered at 0 in $H^1(M)$, and such that $\|u\|_{H^1} \ge R \Rightarrow I(u) \ge \alpha_0 + 1$, which is possible by (1.12). Note that K is a closed, convex, bounded subset of $H^1(M)$. Let

$$K_\varepsilon = \{u \in K : \alpha_0 \le I(u) \le \alpha_0 + \varepsilon\}. \tag{1.17}$$

For each $\varepsilon > 0$, K_ε is a closed, convex subset of K. It follows that K_ε is weakly closed in K, which is weakly compact. Hence

$$\bigcap_{\varepsilon>0} K_\varepsilon = K_0 \neq \emptyset. \tag{1.18}$$

Now $\inf I(u)$ is assumed on K_0. By the strict convexity of $I(u)$, K_0 consists of a single point.

If u is the unique point in K_0 and $v \in C_0^\infty(M)$, then $u + sv \in V$, for all $s \in \mathbb{R}$, and $I(u+sv)$ is a smooth function of s which is minimal at $s = 0$, so

$$0 = \frac{d}{ds} I(u + sv)\Big|_{s=0} = (-\Delta u, v) + \int_M f\big(x, u(x)\big)v(x)\, dV(x). \tag{1.19}$$

Hence (1.1) holds. We have the following regularity result:

Proposition 1.3. *For $k = 1, 2, 3, \dots$, if $g \in H^{k+1/2}(\partial M)$, then any solution $u \in V$ to (1.1)–(1.2) belongs to $H^{k+1}(M)$. Hence, if $g \in C^\infty(\partial M)$, then $u \in C^\infty(\overline{M})$.*

Proof. We start with $u \in H^1(M)$. Then the right side of (1.1) belongs to $H^1(M)$ if $f(x, u)$ satisfies (1.6). This gives $u \in H^2(M)$, provided $g \in H^{3/2}(\partial M)$. Additional regularity follows inductively.

We have uniqueness of the element $u \in V$ minimizing $I(u)$, under the hypotheses (1.3)–(1.7). In fact, under the hypothesis (1.3), there is uniqueness of solutions to (1.1)–(1.2) which are sufficiently smooth, as a consequence of the following application of the maximum principle.

Proposition 1.4. *Let u and $v \in C^2(M) \cap C(\overline{M})$ satisfy (1.1), with $u = g$ and $v = h$ on ∂M. If (1.3) holds, then*

$$\sup_M\, (u - v) \le \sup_{\partial M}\, (g - h) \vee 0, \tag{1.20}$$

where $a \vee b = \max(a, b)$ and

$$\sup_M |u - v| \le \sup_{\partial M} |g - h|. \tag{1.21}$$

Proof. Let $w = u - v$. Then, by (1.3),

$$\Delta w = \lambda(x) w, \quad w\big|_{\partial M} = g - h, \tag{1.22}$$

where

$$\lambda(x) = \frac{f(x, u) - f(x, v)}{u - v} \ge 0.$$

If $\mathcal{O} = \{x \in M : w(x) \ge 0\}$, then $\Delta w \ge 0$ on $\mathcal{O}$, so the maximum principle applies on $\mathcal{O}$, yielding (1.20). Replacing w by $-w$ gives (1.20) with the roles of u and v, and of g and h, reversed, and (1.21) follows.

One application will be the following first step toward relaxing the hypothesis (1.6).

Corollary 1.5. *Let $f(x, 0) = \varphi(x) \in C^\infty(\overline{M})$. Take $g \in C^\infty(\partial M)$, and let $\Phi \in C^\infty(\overline{M})$ be the solution to*

$$\Delta \Phi = \varphi \text{ on } M, \quad \Phi = g \text{ on } \partial M. \tag{1.23}$$

Then, under the hypothesis (1.3), a solution u to (1.1)–(1.2) satisfies

$$\sup_M u \le \sup_M \Phi + \big(\sup_M (-\Phi) \vee 0\big) \tag{1.24}$$

and

$$\sup_M |u| \le \sup_M 2|\Phi|. \tag{1.25}$$

Proof. We have

$$\Delta(u-\Phi) = f(x,u) - f(x,0) = \lambda(x)u, \tag{1.26}$$

with $\lambda(x) = [f(x,u) - f(x,0)]/u \ge 0$. Thus $\Delta(u-\Phi) \ge 0$ on $\mathcal{O} = \{x \in M : u(x) > 0\}$, so

$$\sup_{\mathcal{O}} (u-\Phi) = \sup_{\partial\mathcal{O}} (u-\Phi) \le \sup_M (-\Phi) \vee 0.$$

This gives (1.24). Also $\Delta(\Phi - u) \ge 0$ on $\mathcal{O}^- = \{x \in M : u(x) < 0\}$, so

$$\sup_{\mathcal{O}^-} (\Phi - u) = \sup_{\partial\mathcal{O}^-} (\Phi - u) \le \sup_M \Phi \vee 0,$$

which together with (1.24) gives (1.25).

We can now prove the following result on the solvability of (1.1)–(1.2).

Theorem 1.6. *Suppose $f(x,u)$ satisfies (1.3). Given $g \in C^\infty(\partial M)$, there is a unique solution $u \in C^\infty(\overline{M})$ to (1.1)–(1.2).*

Proof. Let $f_j(x,u)$ be smooth, satisfying

$$f_j(x,u) = f(x,u), \quad \text{for } |u| \le j, \tag{1.27}$$

and be such that (1.3)–(1.7) hold for each f_j, with $K = K_j$. We have solutions $u_j \in C^\infty(\overline{M})$ to

$$\Delta u_j = f_j(x,u_j), \quad u_j\big|_{\partial M} = g. \tag{1.28}$$

Now $f_j(x,0) = f(x,0) = \varphi(x)$, independent of j, and the estimate (1.25) holds for all u_j, so

$$\sup_M |u_j| \le \sup_M 2|\Phi|, \tag{1.29}$$

where Φ is defined by (1.23). Thus the sequence (u_j) stabilizes for large j, and the proof is complete.

We next discuss a geometrical problem that can be solved using the results developed above. A more substantial variant of this problem will be tackled in the next section. The problem we consider here is the following. Let $\overline{M}$ be a connected, compact, two-dimensional manifold, with nonempty boundary. We suppose that we are given a Riemannian metric g on $\overline{M}$, and we desire to construct a conformally related metric whose Gauss curvature $K(x)$ is a given function on $\overline{M}$. As shown in (3.46) of Appendix C, if $k(x)$ is the Gauss curvature of g and if $g' = e^{2u}g$, then the Gauss curvature of g' is given by

$$K(x) = \big(-\Delta u + k(x)\big)e^{-2u}, \tag{1.30}$$

where Δ is the Laplace operator for the metric g. Thus we want to solve the PDE

$$\Delta u = k(x) - K(x)e^{2u} = f(x,u), \tag{1.31}$$

for u. This is of the form (1.1). The hypothesis (1.3) is satisfied provided $K(x) \le 0$. Thus Theorem 1.6 yields the following.

Proposition 1.7. *If $\overline{M}$ is a connected, compact 2-manifold with nonempty boundary ∂M, g a Riemannian metric on $\overline{M}$, and $K \in C^\infty(\overline{M})$ a given function satisfying*

$$K(x) \le 0 \text{ on } M, \tag{1.32}$$

then there exists $u \in C^\infty(\overline{M})$ such that the metric $g' = e^{2u}g$ conformal to g has Gauss curvature K. Given any $v \in C^\infty(\partial M)$, there is a unique such u satisfying $u = v$ on ∂M.

Results of this section do not apply if $K(x)$ is allowed to be positive somewhere; we refer to [KaW] and [Kaz] for results that do apply in that case.

If one desires to make (M, g) conformally equivalent to a flat metric, that is, one with $K(x) = 0$, then (1.31) becomes the linear equation

$$\Delta u = k(x). \tag{1.33}$$

This can be solved whenever M is connected with nonempty boundary, with u prescribed on ∂M. As shown in Proposition 3.1 of Appendix C, when the curvature vanishes, one can choose local coordinates so that the metric tensor becomes δ_{jk}. This could provide an alternative proof of the existence of local isothermal coordinates, which is established by a different argument in Chap. 5, § 10. However, the following logical wrinkle should be pointed out. The derivation of the formula (1.30) in § 3 of Appendix C made use of a reduction to the case $g_{jk} = e^{2v}\delta_{jk}$ and therefore relied on the existence of local isothermal coordinates. Now, one could grind out a direct proof of (1.30) without using this reduction, thus smoothing out this wrinkle.

We next tackle the equation (1.1) when M is compact, without boundary. For now, we retain the hypothesis (1.3), $\partial f/\partial u \geq 0$. Without a boundary for M, we have a hard time bounding u, since (1.16) fails for constant functions on M. In fact, the equation (1.31) cannot be solved when $K(x) = -1$, $k(x) = 1$, and $M = S^2$, so some further hypotheses are necessary. We will make the following hypothesis: For some $a_j \in \mathbb{R}$,

$$u < a_0 \Rightarrow f(x, u) < 0, \quad u > a_1 \Rightarrow f(x, u) > 0. \tag{1.34}$$

If $\partial f/\partial u > 0$, this is equivalent to the existence of a function $u = \varphi(x)$ such that $f\big(x, \varphi(x)\big) = 0$. We see how this hypothesis controls the size of a solution.

Proposition 1.8. *If u solves (1.1) and M is compact, then*

$$a_0 \leq u(x) \leq a_1, \tag{1.35}$$

provided (1.34) holds.

Proof. If u is maximal at x_0, then $\Delta u(x_0) \leq 0$, so $f\big(x_0, u(x_0)\big) \leq 0$, and so (1.34) implies $u \leq a_1$. The other inequality in (1.35) follows similarly.

To get an existence result out of this estimate, we use a technique known as the method of continuity. We show that, for each $\tau \in [0, 1]$, there is a smooth solution to

$$\Delta u = (1 - \tau)(u - b) + \tau f(x, u) = f_\tau(x, u), \tag{1.36}$$

where we pick $b = (a_0 + a_1)/2$. Clearly, this equation is solvable when $\tau = 0$. Let J be the largest interval in $[0, 1]$, containing 0, with the property that (1.36) is solvable for all $\tau \in J$. We wish to show that $J = [0, 1]$. First note that, for any $\tau \in [0, 1]$,

$$u < a_0 \Rightarrow f_\tau(x, u) < 0, \quad u > a_1 \Rightarrow f_\tau(x, u) > 0, \tag{1.37}$$

so any solution $u = u_\tau$ to (1.36) must satisfy

$$a_0 \leq u_\tau(x) \leq a_1. \tag{1.38}$$

Using this, we can show that J is *closed* in $[0, 1]$. In fact, let $u_j = u_{\tau_j}$ solve (1.36) for $\tau_j \in J$, $\tau_j \nearrow \sigma$. We have $\|u_j\|_{L^\infty} \leq a < \infty$ by (1.38), so $g_j(x) = f_{\tau_j}\big(x, u_j(x)\big)$ is bounded in $C(M)$. Thus elliptic regularity for the Laplace operator yields

$$\|u_j\|_{C^r(M)} \leq b_r < \infty, \tag{1.39}$$

for any $r < 2$. This yields a C^r-bound for g_j, and hence (1.39) holds for any $r < 4$. Iterating, we get u_j bounded in $C^\infty(M)$. Any limit point $u \in C^\infty(M)$ solves (1.36) with $\tau = \sigma$, so J is closed.

We next show that J is *open* in [0, 1]. That is, if $\tau_0 \in J,\ \tau_0 < 1$, then, for some $\varepsilon > 0,\ [\tau_0, \tau_0 + \varepsilon) \subset J$. To do this, fix k large and define

$$\Psi : [0,1] \times H^k(M) \longrightarrow H^{k-2}(M), \quad \Psi(\tau, u) = \Delta u - f_\tau(x, u). \tag{1.40}$$

This map is C^1, and its derivative with respect to the second argument is

$$D_2\Psi(\tau_0, u)v = Lv, \tag{1.41}$$

where

$$L : H^k(M) \longrightarrow H^{k-2}(M)$$

is given by

$$Lv = \Delta v - A(x)v, \quad A(x) = 1 - \tau_0 + \tau_0\, \partial_u f(x, u). \tag{1.42}$$

Now, if f satisfies (1.3), then $A(x) \geq 1 - \tau_0$, which is > 0 if $\tau_0 < 1$. Thus L is an invertible operator. The inverse function theorem implies that $\Psi(\tau, u) = 0$ is solvable for $|\tau - \tau_0| < \varepsilon$. We thus have the following:

Proposition 1.9. *If M is a compact manifold without boundary and if $f(x, u)$ satisfies the conditions (1.3) and (1.34), then the PDE (1.1) has a smooth solution. If (1.3) is strengthened to $\partial_u f(x, u) > 0$, then the solution is unique.*

The only point left to establish is uniqueness. If u and v are two solutions, then, as in (1.22), we have for $w = u - v$ the equation

$$\Delta w = \lambda(x)w, \quad \lambda(x) = \big[f(x, u) - f(x, v)\big]/(u - v) \geq 0.$$

Thus

$$-\|\nabla w\|_{L^2}^2 = \int \lambda(x)|w(x)|^2\, dV,$$

which implies $w = 0$ if $\lambda(x) > 0$ on M.

Note that if we only have $\lambda(x) \geq 0$, then w must be constant (if M is connected), and that constant must be 0 if $\lambda(x) > 0$ on an open subset of M, so cases of nonuniqueness are rather restricted, under the hypotheses of Proposition 1.9. The reader can formulate further uniqueness results.

It is possible to obtain solutions to (1.1) without the hypothesis (1.3) if we retain the hypothesis (1.34). To do this, first alter $f(x, u)$ on $u \leq a_0$ and on $u \geq a_1$ to a smooth $g(x, u)$ satisfying $g(x, u) = -\kappa_0 < 0$ for $u \leq a_0 - \delta$ and $g(x, u) = \kappa_1 > 0$ for $u \geq a_1 + \delta$, where δ is some positive number. We want to show that, for each $\tau \in [0, 1]$, the equation

(1.43) $$\Delta u = (1-\tau)(u-b) + \tau g(x,u) = g_\tau(x,u)$$

is solvable, with solution satisfying (1.38). Convert (1.43) to

(1.44) $$u = (\Delta - 1)^{-1}\big(g_\tau(x,u) - u\big) = \Phi_\tau(u).$$

Now each Φ_τ is a continuous and compact map on the Banach space $C(M)$:

(1.45) $$\Phi_\tau : C(M) \longrightarrow C(M),$$

with continuous dependence on τ. For solvability we can use the Leray-Schauder fixed-point theorem, proved in Appendix B at the end of this chapter. Note that any solution to (1.44) is also a solution to (1.43) and hence satisfies (1.38). In particular,

(1.46) $$u = \Phi_\tau(u) \Longrightarrow \|u\|_{C(M)} \le A = \max\big(|a_0|, |a_1|\big).$$

Since $\Phi_0(u) = -(\Delta - 1)^{-1} b = b$, which is independent of u, it follows from Theorem B.5 that (1.44) is solvable for all $\tau \in [0,1]$. We have the following improvement of Proposition 1.9.

Theorem 1.10. *If M is a compact manifold without boundary and if the function $f(x,u)$ satisfies the condition (1.34), then the equation (1.1) has a smooth solution, satisfying $a_0 \le u(x) \le a_1$.*

The equation (1.31) for the conformal factor needed to adjust the curvature of a 2-manifold to a desired $K(x)$ satisfies the hypotheses of Theorem 1.10 (even those of Proposition 1.9) in the special case when $k(x) < 0$ and $K(x) < 0$, yielding a special case of a result to be proved in § 2, where the assumption that $k(x) < 0$ is replaced by $\chi(M) < 0$. In some cases, Theorem 1.10 also applies to equations for such conformal factors in higher dimensions. When $\dim M = n \ge 3$, we alter the metric by

(1.47) $$g' = u^{4/(n-2)} g.$$

The scalar curvatures σ and S of the metrics g and g' are then related by

(1.48) $$S = u^{-\alpha}(\sigma u - \gamma \Delta u), \quad \gamma = 4\frac{n-1}{n-2}, \quad \alpha = \frac{n+2}{n-2},$$

where Δ is the Laplacian for the metric g. Hence, obtaining the scalar curvature S for g' is equivalent to solving

(1.49) $$\gamma \Delta u = \sigma(x) u - S(x) u^\alpha,$$

for a smooth positive function u. Note that $\alpha > 1$ and $\gamma > 1$. For $n = 3$, we have $\gamma = 8$ and $\alpha = 5$.

Note that (1.34) holds, for some a_j satisfying $0 < a_0 < a_1 < \infty$, provided both $\sigma(x)$ and $S(x)$ are negative on M. Thus we have the next result:

Proposition 1.11. *Let M be a compact manifold of dimension $n \geq 2$. Let g be a Riemannian metric on M with scalar curvature σ. If both σ and S are negative functions in $C^\infty(M)$, then there exists a conformally equivalent metric g' on M with scalar curvature S.*

An important special case of Proposition 1.11 is that if M has a metric with negative scalar curvature, then that metric can be conformally altered to one with *constant* negative scalar curvature. There is a very significant generalization of this result, first stated by H. Yamabe. Namely, for any compact manifold with a Riemannian metric g, there is a conformally equivalent metric with constant scalar curvature. This result, known as the solution to the Yamabe problem, was established by R. Schoen [Sch], following progress by N. Trudinger and T. Aubin.

Note that (1.3) also holds in the setting of Proposition 1.11; thus to prove this latter result, we could appeal as well to Proposition 1.9 as to Theorem 1.10. Here is a generalization of (1.49) to which Theorem 1.10 applies in some cases where Proposition 1.9 does not:

$$\gamma \Delta u = B(x)u^\beta + \sigma(x)u - A(x)u^\alpha, \quad \beta < 1 < \alpha. \tag{1.50}$$

It is possible that $\beta < 0$. Then we have (1.34), for some $a_j > 0$, and hence the solvability of (1.50), for some positive $u \in C^\infty(M)$, provided $A(x)$ and $B(x)$ are both negative on M, for any $\sigma \in C^\infty(M)$. If we assume $A < 0$ on M but only $B \leq 0$ on M, we still have (1.34), and hence the solvability of (1.50), provided $\sigma(x) < 0$ on $\{x \in M : B(x) = 0\}$.

An equation of the form (1.50) arises in Chap. 18, in a discussion of results of J. York and N. O'Murchadha, describing permissible first and second fundamental forms for a compact, spacelike hypersurface of a Ricci-flat spacetime, in the case when the mean curvature is a given constant. See (9.28) of Chap. 18.

Exercises

1. Assume $f(x,u)$ is smooth and satisfies (1.6). Define $F(x,u)$ and $I(u)$ as in (1.4) and (1.5). Show that I has the strict convexity property (1.9) on the space V given by (1.8), as long as

$$\frac{\partial}{\partial u} f(x,u) \geq -\lambda_0, \tag{1.51}$$

where λ_0 is the smallest eigenvalue of $-\Delta$ on M, with Dirichlet conditions on ∂M. Extend Proposition 1.2 to cover this case, and deduce that the Dirichlet problem (1.1)–(1.2) has a unique solution $u \in C^\infty(\overline{M})$, for any $g \in C^\infty(\partial M)$, when $f(x,u)$ satisfies these conditions.

2. Extend Theorem 1.6 to the case where $f(x,u)$ satisfies (1.51) instead of (1.3). (*Hint*: To obtain sup norm estimates, use the variants of the maximum principle indicated in Exercises 5–7 of § 2, Chap. 5.)

3. Let $\operatorname{spec}(-\Delta) = \{\lambda_j\}$, where $0 < \lambda_0 < \lambda_1 < \cdots$. Suppose there is a pair $\lambda_j < \lambda_{j+1}$ and $\varepsilon > 0$ such that

$$-\lambda_{j+1} + \varepsilon \le \frac{\partial}{\partial u} f(x,u) \le -\lambda_j - \varepsilon,$$

for all x, u. Show that, for $g \in C^\infty(\partial M)$, the boundary problem (1.1)–(1.2) has a unique solution $u \in C^\infty(\overline{M})$.
(*Hint*: With $\mu = (\lambda_j + \lambda_{j+1})/2,\ u = v + g,\ g \in C^\infty(\overline{M})$, rewrite (1.1)–(1.2) as

$$(\Delta + \mu)v = f(x, v+g) + \mu v - G, \quad v\big|_{\partial M} = 0,$$

where $G = (\Delta + \mu)g$, or

$$v = (\Delta + \mu)^{-1}\big[f(x, v+g) + \mu v\big] - g = \Phi(v). \tag{1.52}$$

Apply the contraction mapping principle.)

4. In the context of Exercise 3, this time assume

$$-\lambda_{j+1} + \varepsilon \le \frac{\partial}{\partial u} f(x,u) \le -\lambda_{j-1} - \varepsilon,$$

so $\partial f/\partial u$ might assume the value $-\lambda_j$. Take $\mu = (\lambda_{j-1} + \lambda_{j+1})/2$, let P_0 be the orthogonal projection of $L^2(M)$ on the λ_j eigenspace of $-\Delta$, and let $P_1 = I - P_0$. Writing

$$u - g = v = P_0 v + P_1 v = v_0 + v_1,$$

convert (1.1)–(1.2) to a system

$$\begin{aligned} v_1 &= (\Delta + \mu)^{-1} P_1\Big[f(x, v_0 + v_1 + g) + \mu v_1\Big] - P_1 g, \\ v_0 &= (\mu - \lambda_j)^{-1} P_0\Big[f(x, v_0 + v_1 + g) + \mu v_0\Big] - P_0 g. \end{aligned} \tag{1.53}$$

Given v_0, the first equation in (1.53) has a unique solution, $v_1 = \Xi(v_0)$, by the argument in Exercise 3. Thus the solvability of (1.1)–(1.2) is converted to the solvability of

$$v_0 = (\mu - \lambda_j)^{-1} P_0\Big[f\big(x, v_0 + \Xi(v_0) + g\big) + \mu v_0\Big] - P_0 g = \Psi(v_0). \tag{1.54}$$

Here, Ψ is a nonlinear operator on a finite-dimensional space. (Essentially, on the real line if λ_j is a simple eigenvalue of $-\Delta$.) Examine various cases, where there will or will not be solutions, perhaps more than one in number.

5. Given a Riemanian manifold M of dimension $n \ge 3$, with metric g and Laplace operator Δ, define the "conformal Laplacian" on functions:

$$Lf = \Delta f - \gamma_n^{-1}\sigma(x) f, \quad \gamma_n^{-1} = \frac{n-2}{4(n-1)}, \tag{1.55}$$

where $\sigma(x)$ is the scalar curvature of (M, g). If $g' = u^{4/(n-2)} g$ as in (1.47), and (M, g') has scalar curvature $S(x)$, set

$$\widetilde{L} f = \widetilde{\Delta} f - \gamma_n^{-1} S(x) f, \tag{1.56}$$

where $\widetilde{\Delta}$ is the Laplace operator for the metric g'. Show that

$$L(uf) = u^{4/(n-2)} u \widetilde{L} f. \tag{1.57}$$

(*Hint*: First show that $\Delta(uf) - uu^{4/(n-2)}\widetilde{\Delta} f = (\Delta u) f$. Then use the identity (1.49).)

6. Assume M is compact and connected. Let λ_0 be the smallest eigenvalue of $-L = -\Delta + \gamma_n^{-1}\sigma(x)$. A λ_0-eigenfunction v of L is nowhere vanishing (by Proposition 2.9 of Chap. 8). Assume $v(x) > 0$ on M. Form the new metric $\widetilde{g} = v^{4/(n-2)} g$. Show that the scalar curvature $\widetilde{S}$ of $(M, \widetilde{g})$ is given by

$$\widetilde{S}(x) = \lambda_0 v^{-4/(n-2)}, \tag{1.58}$$

which is positive everywhere if $\lambda_0 > 0$, negative everywhere if $\lambda_0 < 0$, and zero if $\lambda_0 = 0$.

7. Establish existence for an $\ell \times \ell$ system

$$\Delta u = f(x, u),$$

where M is a compact Riemannian manifold and $f : M \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ satisfies the condition that, for some $A < \infty$,

$$|u| \geq A \Longrightarrow f(x, u) \cdot u > 0.$$

(*Hint*: Replace f by τf, and let $0 \leq \tau \leq 1$. Show that any solution to such a system satisfies $|u(x)| < A$.)

8. Let $\overline{\Omega}$ be a compact, connected Riemannian manifold with nonempty boundary. Consider

$$\Delta u + f(x, u) = 0, \quad u\big|_{\partial\Omega} = g, \tag{1.59}$$

for some real-valued u; assume $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$, $g \in C^\infty(\partial\Omega)$. Assume there is an *upper solution* $\overline{u}$ and a *lower solution* $\underline{u}$, in $C^2(\Omega) \cap C(\overline{\Omega})$, satisfying

$$\Delta \overline{u} + f(x, \overline{u}) \leq 0, \quad \overline{u}\big|_{\partial\Omega} \geq g,$$
$$\Delta \underline{u} + f(x, \underline{u}) \geq 0, \quad \underline{u}\big|_{\partial\Omega} \leq g.$$

Also assume $\underline{u} \leq \overline{u}$ on $\overline{\Omega}$.
Under these hypotheses, show that there exists a solution $u \in C^\infty(\overline{\Omega})$ to (1.59), such that $\underline{u} \leq u \leq \overline{u}$.
One approach. Let $K = \{v \in C(\overline{\Omega}) : \underline{u} \leq v \leq \overline{u}\}$, which is a closed, bounded, convex set in $C(\overline{\Omega})$. Pick $\lambda > 0$ so that $|\partial_u f(x, u)| \leq \lambda$, for $\min \underline{u} \leq u \leq \max \overline{u}$. Let $\Phi(v) = w$ be the solution to

$$\Delta w - \lambda w = -\lambda v - f(x, v), \quad w\big|_{\partial\Omega} = g.$$

Show that $\Phi : K \to K$ continuously and that $\Phi(K)$ is relatively compact in K. Deduce that Φ has a fixed point $u \in K$.
Second approach. If $u_0 = \underline{u}$ and $u_{j+1} = \Phi(u_j)$, show that

$$\underline{u} = u_0 \leq u_1 \leq \cdots \leq u_j \leq \cdots \leq \overline{u}$$

and that $u_j \nearrow u$, solving (1.59).

2. Surfaces with negative curvature

In this section we examine the possibility of imposing a given Gauss curvature $K(x) < 0$ on a compact surface M without boundary, by conformally altering a given metric g, whose Gauss curvature is $k(x)$. As noted in § 1, if g and g' are conformally related,

$$g' = e^{2u} g, \tag{2.1}$$

then K and k are related by

$$K(x) = e^{-2u}(-\Delta u + k(x)), \tag{2.2}$$

where Δ is the Laplace operator for the original metric g, so we want to solve the PDE

$$\Delta u = k(x) - K(x) e^{2u}. \tag{2.3}$$

This is not possible if M is diffeomorphic to the sphere S^2 or the torus $\mathbb{T}^2$, by virtue of the Gauss–Bonnet formula (proved in § 5 of Appendix C):

$$\int_M k \, dV = \int_M K e^{2u} \, dV = 2\pi \chi(M), \tag{2.4}$$

where dV is the area element on M, for the original metric g, and $\chi(M)$ is the Euler characteristic of M. We have

$$\chi(S^2) = 2, \quad \chi(\mathbb{T}^2) = 0. \tag{2.5}$$

For us to be able to arrange that $K < 0$ be the curvature of M, it is necessary for $\chi(M)$ to be negative. This is the only obstruction; following [Bgr], we will establish the following.

Theorem 2.1. *If M is a compact surface satisfying $\chi(M) < 0$, with given Riemannian metric g, then for any negative $K \in C^\infty(M)$, the equation (2.3) has a solution, so M has a metric, conformal to g, with Gauss curvature $K(x)$.*

We will produce the solution to (2.3) as an element where the function

$$F(u) = \int_M \Big(\frac{1}{2} |du|^2 + k(x) u \Big) \, dV \tag{2.6}$$

on the set

$$S = \Big\{ u \in H^1(M) : \int_M K(x) e^{2u} \, dV = 2\pi \chi(M) \Big\} \tag{2.7}$$

achieves a minimum. Note that the Gauss–Bonnet formula is built into (2.7), since a metric $g' = e^{2u}g$ has volume element $e^{2u}dV$. While providing an obstruction to specifying $K(x)$, the Gauss–Bonnet formula also provides an aid in making a prescription of $K(x) < 0$ when it is possible to do so, as we will see below.

Lemma 2.2. *The set S is a nonempty C^1-submanifold of $H^1(M)$ if $K < 0$ and $\chi(M) < 0$.*

Proof. Set

$$\Phi(u) = e^{2u}. \tag{2.8}$$

By Trudinger's inequality,

$$\Phi : H^1(M) \longrightarrow L^p(M), \tag{2.9}$$

for all $p < \infty$. Take $p = 1$. We see that Φ is differentiable at each $u \in H^1(M)$ and

$$D\Phi(u)v = 2e^{2u}v, \quad D\Phi(u) : H^1(M) \to L^1(M). \tag{2.10}$$

Furthermore,

$$\begin{aligned}\big\|\big(D\Phi(u) - D\Phi(w)\big)v\big\|_{L^1(M)} &\le 2\int_M |v| \cdot |e^{2u} - e^{2w}|\,dV \\ &\le 2\Big(\int |v|^4\,dV\Big)^{1/4}\Big(\int |u-w|^4\,dV\Big)^{1/4}\Big(\int e^{4|u|+4|w|}\,dV\Big)^{1/2} \\ &\le C\|v\|_{H^1} \cdot \|u-w\|_{H^1} \cdot \exp\big[C\big(\|u\|_{H^1} + \|w\|_{H^1}\big)\big],\end{aligned} \tag{2.11}$$

so the map $\Phi : H^1(M) \to L^1(M)$ is a C^1-map. Consequently,

$$J(u) = \int_M Ke^{2u}\,dV \Longrightarrow J : H^1(M) \to \mathbb{R} \text{ is a } C^1\text{-map}. \tag{2.12}$$

Furthermore, $DJ(u) = 2K\,e^{2u}$, as an element of $H^{-1}(M) \approx \mathcal{L}(H^1(M), \mathbb{R})$, so $DJ(u) \neq 0$ on S. The implicit function theorem then implies that S is a C^1-submanifold of $H^1(M)$. If $K < 0$ and $\chi(M) < 0$, it is clear that there is a *constant* function in S, so $S \neq \emptyset$.

Lemma 2.3. *Suppose $F : S \to \mathbb{R}$, defined by (2.6), assumes a minimum at $u \in S$. Then u solves the PDE (2.3), provided the hypotheses of Theorem 2.1 hold.*

Proof. Clearly, $F : S \to \mathbb{R}$ is a C^1-map. If $\gamma(s)$ is any C^1-curve in S with $\gamma(0) = u,\ \gamma'(0) = v$, we have

$$\begin{aligned} 0 = \frac{d}{ds} F(u+sv)\big|_{s=0} &= \int_M \big[(du, dv) + k(x)v\big]\, dV \\ &= \int_M \big(-\Delta u + k(x)\big) v\, dV. \end{aligned} \tag{2.13}$$

The condition that v is tangent to S at u is

$$\int_M K e^{2(u+sv)}\, dV = 2\pi\chi(M) + O(s^2), \tag{2.14}$$

which is equivalent to

$$\int_M vKe^{2u}\, dV = 0. \tag{2.15}$$

Thus, if $u \in S$ is a minimum for F, we have

$$v \in H^1(M),\ \int_M vKe^{2u}\, dV = 0 \Longrightarrow \int_M \big(-\Delta u + k(x)\big) v\, dV = 0,$$

and hence $-\Delta u + k(x)$ is parallel to Ke^{2u} in $H^1(M)$; that is,

$$-\Delta u + k(x) = \beta K e^{2u}, \tag{2.16}$$

for some constant β. Integrating and using the Gauss-Bonnet theorem yield $\beta = 1$ if $\chi(M) \neq 0$.

By Trudinger's estimate, the right side of (2.16) belongs to $L^2(M)$, so $u \in H^2(M)$. This implies $e^{2u} \in H^2(M)$, and an easy inductive argument gives $u \in C^\infty(M)$.

Our task is now to show that F has a minimum on S, given $K < 0$ and $\chi(M) < 0$. Let us write, for any $u \in H^1(M)$,

$$u = u_0 + \alpha, \tag{2.17}$$

where $\alpha = (\text{Area } M)^{-1} \int_M u\, dV$ is the mean value of u, and

$$u_0 \in \overline{H}(M) = \Big\{v \in H^1(M) : \int_M v\, dV = 0\Big\}. \tag{2.18}$$

Then u belongs to S if and only if

$$e^{2\alpha} \int_M K e^{2u_0}\, dV = 2\pi\chi(M),$$

or equivalently,

$$\alpha = \frac{1}{2} \log \left[2\pi\chi(M) \Big/ \int K e^{2u_0}\, dV \right]. \tag{2.19}$$

Thus, for $u \in S$,

$$\begin{aligned} F(u) = & \int_M \left(\frac{1}{2} |du_0|^2 + k u_0 \right) dV \\ & + \pi\chi(M) \left\{ \log 2\pi |\chi(M)| - \log \left| \int_M K e^{2u_0}\, dV \right| \right\}. \end{aligned} \tag{2.20}$$

Lemma 2.4. *If* $\chi(M) < 0$ *and* $K < 0$*, then* $\inf_S F(u) = a > -\infty$.

Proof. By (2.20), we need to estimate

$$-\chi(M) \log \left| \int_M K e^{2u_0}\, dV \right|$$

from below. Indeed, granted that $K(x) \le -\delta < 0$,

$$\int K e^{2u_0}\, dV \le -\delta \int e^{2u_0}\, dV.$$

Since $e^x \ge 1 + x$, we have $\int e^{2u_0}\, dV \ge \int dV + \int 2u_0\, dV =$ area M, so

$$\int_M K e^{2u_0}\, dV \ge -\delta A \quad (A = \text{ Area } M),$$

and hence

$$-\chi(M) \log \left| \int_M K e^{2u_0}\, dV \right| \ge |\chi(M)| \log |\delta A| \ge b > -\infty. \tag{2.21}$$

Thus, for $u \in S$,

$$F(u) \ge \int_M \left(\frac{1}{2} |du_0|^2 + k u_0 \right) dV - C_2, \tag{2.22}$$

with C_2 independent of $u_0 \in H^1(M)$. Now, since $\|u_0\|_{L^2} \le C\|du_0\|_{L^2}$,

$$\Big|\int_M ku_0\, dV\Big| \le C_3\varepsilon\|du_0\|_{L^2}^2 + \frac{C_4}{\varepsilon}, \tag{2.23}$$

with C_3 and C_4 independent of ε. Taking $\varepsilon = 1/2C_3$, we get $F(u) \ge -C_3C_4 - C_2$, which proves the lemma.

We are now in a position to prove the main existence result.

Proposition 2.5. *If M and K are as in Theorem 2.1, then F achieves a minimum at a point $u \in S$, which consequently solves (2.3).*

Proof. Pick $u_n \in S$ so that $a + 1 \ge F(u_n) \searrow a$. If we use (2.22) and (2.23), with $\varepsilon = 1/4C_3$, we have

$$a + 1 \ge \frac{1}{4}\|du_{n0}\|_{L^2}^2 - C_5, \tag{2.24}$$

where $u_{n0} = u_n -$ mean value. But the mean value of u_n is

$$\frac{1}{2}\log\Big[2\pi\chi(M)\Big/\int_M Ke^{2u_{n0}}\, dV\Big],$$

which is bounded from above by the proof of Lemma 2.4. Hence

$$u_n \text{ is bounded in } H^1(M). \tag{2.25}$$

Passing to a subsequence, we have an element $u \in H^1(M)$ such that

$$u_n \longrightarrow u \quad \text{weakly in } H^1(M). \tag{2.26}$$

By Proposition 4.3 of Chap. 12, $e^{2u_n} \to e^{2u}$ in $L^1(M)$, in norm, so $u \in S$. Now (2.26) implies that $\int_M k(x)u_n\, dV \to \int_M k(x)u\, dV$ and that

$$\int_M |du|^2\, dV \le \liminf_{n\to\infty} \int_M |du_n|^2\, dV, \tag{2.27}$$

so $F(u) \le a = \int_S F(v)$, and the existence proof is completed.

The most important special case of Theorem 2.1 is the case $K = -1$. For any compact surface with $\chi(M) < 0$, given a Riemannian metric g, it is conformally equivalent to a metric for which $K = -1$. The universal covering surface

$$\widetilde{M} \longrightarrow M, \tag{2.28}$$

endowed with the lifted metric, also has curvature -1. A basic theorem of differential geometry is that any two complete, simply connected Riemannian manifolds, with the same constant curvature (and the same dimension), are isometric. See the exercises for dimension 2. For a proof in general, see [ChE]. One model surface of curvature -1 is the *Poincaré disk*,

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = \{z \in \mathbb{C} : |z| < 1\}, \tag{2.29}$$

with metric

$$ds^2 = 4(1 - x^2 - y^2)^{-2}\big(dx^2 + dy^2\big). \tag{2.30}$$

This was discussed in § 5 of Chap. 8. Any compact surface M with negative Euler characteristic is conformally equivalent to the quotient of $\mathcal{D}$ by a discrete group Γ of isometries. If M is orientable, all the elements of Γ preserve orientation.

A group of orientation-preserving isometries of $\mathcal{D}$ is provided by the group G of linear fractional transformations, where

$$T_g z = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.31}$$

for

$$g \in G = \mathrm{SU}(1,1) = \left\{ \begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix} : u, v \in \mathbb{C},\ |u|^2 - |v|^2 = 1 \right\}. \tag{2.32}$$

It is easy to see that G acts transitively on $\mathcal{D}$; that is, for any $z_1, z_2 \in \mathcal{D}$, there exists $g \in G$ such that $T_g z_1 = z_2$. We claim $\{T_g : G \in G\}$ exhausts the group of orientation-preserving isometries of $\mathcal{D}$. In fact, let T be such an isometry of $\mathcal{D}$; say $T(0) = z_0$. Pick $g \in G$ such that $T_g z_0 = 0$. Then $T_g \circ T$ is an orientation-preserving isometry of $\mathcal{D}$, fixing 0, and it is easy to deduce that $T_g \circ T$ must be a rotation, which is given by an element of G.

Since each element of G defines a holomorphic map of $\mathcal{D}$ to itself, we have the following result, a major chunk of the *uniformization theorem* for compact Riemann surfaces:

Proposition 2.6. *If M is a compact Riemann surface, $\chi(M) < 0$, then there is a holomorphic covering map of M by the unit disk $\mathcal{D}$.*

Let us take a brief look at the case $\chi(M) = 0$. We claim that any metric g on such M is conformally equivalent to a *flat* metric g', that is, one for which $K = 0$. Note that the PDE (2.3) is linear in this case; we have

$$\Delta u = k(x). \tag{2.33}$$

This equation can be solved on M if and only if

$$\int_M k(x)\, dV = 0, \tag{2.34}$$

which, by the Gauss–Bonnet formula (2.4) holds precisely when $\chi(M) = 0$. In this case, the universal covering surface $\widetilde{M}$ of M inherits a flat metric, and it must be isometric to Euclidean space. Consequently, in analogy with Proposition 2.6, we have the following:

Proposition 2.7. *If M is a compact Riemann surface, $\chi(M) = 0$, then M is holomorphically equivalent to the quotient of $\mathbb{C}$ by a discrete group of translations.*

By the characterization

$$\chi(M) = \dim H^0(M) - \dim H^1(M) + \dim H^2(M) = 2 - \dim H^1(M),$$

if M is a compact, connected Riemann surface, we must have $\chi(M) \le 2$. If $\chi(M) = 2$, it follows from the Riemann–Roch theorem that M is conformally equivalent to the standard sphere S^2 (see § 10 of Chap. 10). This implies the following.

Proposition 2.8. *If M is a compact Riemannian manifold homeomorphic to S^2, with Riemannian metric tensor g, then M has a metric tensor, conformal to g, with Gauss curvature $\equiv 1$.*

In other words, we can solve for $u \in C^\infty(M)$ the equation

$$\Delta u = k(x) - e^{2u}, \tag{2.35}$$

where $k(x)$ is the Gauss curvature of g. This result does not follow from Theorem 2.1. A PDE proof, involving a nonlinear parabolic equation, is given by [Chow], following work of [Ham]. An elliptic PDE proof, under the hypothesis that M has a metric with Gauss curvature $k(x) > 0$, has been given in Chap. 2 of [CK].

We end this section with a direct *linear* PDE proof of the following, which as noted above implies Proposition 2.8. This argument appeared in [MT].

Proposition 2.9. *If M is a compact Riemannian manifold homeomorphic to S^2, there is a conformal diffeomorphism $F : M \to S^2$ onto the standard Riemann sphere.*

Proof. Pick a Riemannian metric on M, compatible with its conformal structure. Then pick $p \in M$, and pick $h \in \mathcal{D}'(M)$, supported at p, given in local coordinates as a first-order derivative of δ_p (plus perhaps a multiple of δ_p), such that $\langle 1, h\rangle =$ 0. Hence there exists a solution $u \in \mathcal{D}'(M)$ to

$$\Delta u = h. \tag{2.36}$$

Then $u \in C^\infty(M \setminus p)$, and u is harmonic on $M \setminus p$ and has a $\operatorname{dist}(x, p)^{-1}$ type of singularity. Now, if M is homeomorphic to S^2, then $M \setminus p$ is simply connected, so u has a single-valued harmonic conjugate on $M \setminus p$, given by $v(x) = \int_q^x *du$, where we pick $q \in M \setminus p$. We see that v also has a $\operatorname{dist}(x, p)^{-1}$ type singularity. Then $f = u + iv$ is holomorphic on $M \setminus p$ and has a simple pole at p. From here it is straightforward that f provides a conformal diffeomorphism of M onto the standard Riemann sphere.

Actually, the bulk of [MT] dealt with an attack on the curvature equation (2.3), with M a planar domain and $K \equiv -1$, so the equation is

$$\Delta u = e^{2u} \quad \text{on} \quad \Omega \subset \mathbb{C}. \tag{2.37}$$

Here is one of the main results of [MT].

Proposition 2.10. *Assume $\Omega = \mathbb{C} \setminus S$, where S is a closed subset of $\mathbb{C}$ with more than one point. Then there exists a solution to (2.37) on Ω such that $e^{2u}(dx^2 + dy^2)$ is a complete metric on Ω with curvature $\equiv -1$.*

As with Proposition 2.6, this has as a corollary the following special case of the general uniformization theorem.

Corollary 2.11. *If $\Omega \subset \mathbb{C}$ is as in Proposition 2.10, there exists a holomorphic covering of Ω by the unit disk $\mathcal{D}$.*

Techniques employed in the proof of Proposition 2.10 include maximal principle arguments and barrier constructions. We refer to [MT] for further details.

Exercises

1. Let M be a complete, simply connected 2-manifold, with Gauss curvature $K = -1$. Fix $p \in M$, and consider

$$\operatorname{Exp}_p : \mathbb{R}^2 \approx T_pM \longrightarrow M.$$

Show that this is a diffeomorphism.
(*Hint*: The map is onto by completeness. Negative curvature implies no Jacobi fields vanishing at 0 and another point, so $D\operatorname{Exp}_p$ is everywhere nonsingular. Use simple connectivity of M to show that Exp_p must be one-to-one.)

2. For M as in Exercise 1, take geodesic polar coordinates, so the metric is

$$ds^2 = dr^2 + G(r, \theta)\, d\theta^2.$$

Use formula (3.37) of Appendix C, for the Gauss curvature, to deduce that

$$\partial_r^2 \sqrt{G} = \sqrt{G}$$

if $K = -1$. Show that

$$\sqrt{G}(0,\theta) = 0, \quad \partial_r \sqrt{G}(0,\theta) = 1,$$

and deduce that $\sqrt{G}(r,\theta) = \varphi(r)$ is the unique solution to

$$\varphi''(r) - \varphi(r) = 0, \quad \varphi(0) = 0, \ \varphi'(0) = 1.$$

Deduce that

$$G(r,\theta) = \sinh^2 r.$$

3. Using Exercise 2, deduce that any two complete, simply connected 2-manifolds with Gauss curvature $K = -1$ are isometric. Use (3.37) or (3.41) of Appendix C to show that the Poincaré disk (2.30) has this property.

3. Local solvability of nonlinear elliptic equations

We take a look at nonlinear PDE, of the form

$$f(x, D^m u) = g(x), \tag{3.1}$$

where, in the latter argument of f,

$$D^m u = \{D^\alpha u : |\alpha| \le m\}. \tag{3.2}$$

We suppose $f(x,\zeta)$ is smooth in its arguments, $x \in \Omega \subset \mathbb{R}^n$, and $\zeta = \{\zeta_\alpha : |\alpha| \le m\}$. The function u might take values in some vector space $\mathbb{R}^k$. Set

$$F(u) = f(x, D^m u), \tag{3.3}$$

so $F : C^\infty(\Omega) \to C^\infty(\Omega)$; F is the nonlinear differential operator. Let $u_0 \in C^m(\Omega)$. We say that the linearization of F at u_0 is $DF(u_0)$, which is a linear map from $C^m(\Omega)$ to $C(\Omega)$. (Sometimes less smooth u_0 can be considered.) We have

$$DF(u_0)v = \frac{\partial}{\partial s} F(u_0 + sv)\Big|_{s=0} = \sum_{|\beta| \le m} \frac{\partial f}{\partial \zeta_\beta}(x, D^m u_0)\, D^\beta v, \tag{3.4}$$

so $DF(u_0)$ is itself a linear differential operator of order m. We say the operator F is elliptic at u_0 if its linearization $DF(u_o)$ is an elliptic, linear differential operator.

An operator of the form (3.3) with

$$f(x, D^m u) = \sum_{|\alpha|=m} a_\alpha(x, D^{m-1}u) D^\alpha u + f_1(x, D^{m-1}u) \tag{3.5}$$

is said to be *quasi-linear*. In that case, the linearization at u_0 is

$$DF(u_0) = \sum_{|\alpha|=m} a_\alpha(x, D^{m-1}u_0)D^\alpha v + Lv, \tag{3.6}$$

where L is a linear differential operator of order $m-1$, with coefficients depending on $D^{m-1}u_0$. A nonlinear operator that is not quasi-linear is called *completely nonlinear*. The distinction is made because some aspects of the theory of quasi-linear operators are simpler than the general case.

An example of a completely nonlinear operator is the Monge–Ampere operator

$$F(u) = \det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = u_{xx}u_{yy} - u_{xy}^2, \tag{3.7}$$

with $(x, y) \in \Omega \subset \mathbb{R}^2$. In this case,

$$\begin{aligned} DF(u)v &= \operatorname{Tr}\left[\begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix} \begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} \right] \\ &= u_{yy}v_{xx} - 2u_{xy}v_{xy} + u_{xx}v_{yy}. \end{aligned} \tag{3.8}$$

Thus the linear operator $DF(u)$ acting on v is elliptic provided the matrix

$$\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} \tag{3.9}$$

is either positive-definite or negative-definite. Since, for u real-valued, this is a real symmetric matrix, we see that this condition holds precisely when $F(u) > 0$.

More generally, for $\Omega \subset \mathbb{R}^n$, we consider the Monge–Ampere operator

$$F(u) = \det H(u), \tag{3.7a}$$

where $H(u) = (\partial_j \partial_k u)$ is the Hessian matrix of second-order derivatives. In this case, we have

$$DF(u)v = \operatorname{Tr}\big[\mathcal{C}(u)H(v)\big], \tag{3.8a}$$

where $H(v)$ is the Hessian matrix for v and $\mathcal{C}(u)$ is the cofactor matrix of $H(u)$, satisfying

$$H(u)\mathcal{C}(u) = \big[\det H(u)\big]I.$$

In this setting we see that $DF(u)$ is a linear, second-order differential operator that is elliptic provided the matrix $\mathcal{C}(u)$ is either positive-definite or negative-definite, and this holds provided the Hessian matrix $H(u)$ is either positive-definite or negative-definite.

Having introduced the concepts above, we aim to establish the following local solvability result:

Theorem 3.1. *Let $g \in C^\infty(\Omega)$, and let $u_1 \in C^\infty(\Omega)$ satisfy*

$$F(u_1) = g(x), \quad \text{at } x = x_0, \tag{3.10}$$

where $F(u)$ is of the form (3.3). Suppose that F is elliptic at u_1. Then, for any ℓ, there exists $u \in C^\ell(\Omega)$ such that

$$F(u) = g \tag{3.11}$$

on a neighborhood of x_0.

We begin with a formal power-series construction to arrange that (3.11) hold to infinite order at x_0.

Lemma 3.2. *Under the hypotheses of Theorem 3.1, there exists $u_0 \in C^\infty(\Omega)$ such that*

$$F(u_0) - g(x) = O(|x - x_0|^\infty) \tag{3.12}$$

and

$$(u_0 - u_1)(x) = O(|x - x_0|^{m+1}). \tag{3.13}$$

Proof. Making a change of variable, we can suppose $x_0 = 0$. Denote coordinates near 0 in Ω by $(x, y) = (x_1, \dots, x_{n-1}, y)$. We write $u_0(x, y)$ as a formal power series in y:

$$u_0(x, y) = v_0(x) + v_1(x)y + \cdots + \frac{1}{k!}v_k(x)y^k + \cdots. \tag{3.14}$$

Set

$$v_0(x) = u_1(x, 0), \quad v_1(x) = \partial_y u_1(x, 0), \dots, v_{m-1}(x) = \partial_y^{m-1} u_1(x, 0). \tag{3.15}$$

Now the PDE $F(u) = g$ can be rewritten in the form

$$\frac{\partial^m u}{\partial y^m} = F^\#(x, y, D_x^m u, D_x^{m-1} D_y u, \dots, D_x^1 D_y^{m-1} u). \tag{3.16}$$

Then the equation for $v_m(x)$ becomes

$$v_m(x) = f^\#(x, 0, D_x^m v_0(x), \dots, D_x^1 v_{m-1}(x)). \tag{3.17}$$

Now, by (3.10), we have $v_m(0) = \partial_y^m u_1(0,0)$, so (3.13) is satisfied. Taking y-derivatives of (3.16) yields inductively the other coefficients $v_j(x),\ j \geq m+1$, and the lemma follows from this construction.

Note that if F is elliptic at u_1, then F continues to be elliptic at u_0, at least on a neighborhood of x_0; shrink Ω appropriately.

To continue the proof of Theorem 3.1, for $k > m + 1 + n/2$, we have that

$$F : H^k(\Omega) \longrightarrow H^{k-m}(\Omega) \tag{3.18}$$

is a C^1-map. We have

$$\mathcal{L} = DF(u_0) : H^k(\Omega) \longrightarrow H^{k-m}(\Omega). \tag{3.19}$$

Now, $\mathcal{L}$ is an elliptic operator of order m. We know from Chap. 5 that the Dirichlet problem is a regular boundary problem for the strongly elliptic operator $\mathcal{L}\mathcal{L}^*$. Furthermore, if Ω is a sufficiently small neighborhood of x_0, the map

$$\mathcal{L}\mathcal{L}^* : H^{k+m}(\Omega) \cap H_0^m(\Omega) \longrightarrow H^{k-m}(\Omega) \tag{3.20}$$

is invertible. Hence the map (3.19) is surjetive, so we can apply the implicit function theorem. For any neighborhood $\mathcal{B}_k$ of u_0 in $H^k(\Omega)$, the image of $\mathcal{B}_k$ under the map F contains a neighborhood $\mathcal{C}_k$ of $F(u_0)$ in $H^{k-m}(\Omega)$. Now if (3.12) holds, then any neighborhood of $r(x) = F(u_0) - g$ in $H^{k-m}(\Omega)$ contains functions that vanish on a neighborhood of x_0, so any neighborhood $\mathcal{C}_k$ of $F(u_0)$ contains functions equal to $g(x)$ on a neighborhood of x_0. This establishes the local solvability asserted in Theorem 3.1.

One would rather obtain a local solution $u \in C^\infty$ than just an ℓ-fold differentiable solution. This can be achieved by using elliptic regularity results that will be established in the next section.

We now discuss a refinement of Theorem 3.1.

Proposition 3.3. *If $u_1, g \in C^\infty(\Omega)$ satisfy the hypotheses of Theorem 3.1 at $x = x_0$, with F elliptic at u_1, then, for any ℓ, there exists $u \in C^\ell(\Omega)$ such that, on a neighborhood of x_0,*

$$F(u) = g \tag{3.21}$$

and, furthermore,

$$(u - u_1)(x) = O(|x - x_0|^{m+1}). \tag{3.22}$$

In the literature, one frequently sees a result weaker than (3.22). The desirability of having this refinement was pointed out to the author by R. Bryant. As before, results of the next section will give $u \in C^\infty(\Omega)$.

To begin the proof, we invoke Lemma 3.2, as before, obtaining u_0. Now, for $k > m + 1 + n/2$, set

$$\begin{aligned} \mathcal{V}_k &= \{u \in H^k(\Omega) : (u - u_0)(x) = O(|x - x_0|^{m+1})\}, \\ \mathcal{G}_{k-m} &= \{h \in H^{k-m}(\Omega) : h(x_0) = g(x_0)\}. \end{aligned} \tag{3.23}$$

Then

$$F : \mathcal{V}_k \longrightarrow \mathcal{G}_{k-m} \tag{3.24}$$

is a C^1-map, and we want to show that F maps a neighborhood of u_0 in $\mathcal{V}_k$ onto a neighborhood of $g_0 = F(u_0)$ in $\mathcal{G}_{k-m}$. We will again use the implicit function theorem. We want to show that the linear map

$$\mathcal{L} = DF(u_0) : \mathcal{V}_k^b \longrightarrow \mathcal{G}_{k-m}^b \tag{3.25}$$

is surjective, where

$$\begin{aligned} \mathcal{V}_k^b &= \{v \in H^k(\Omega) : D^\beta v(x_0) = 0 \text{ for } |\beta| \le m\}, \\ \mathcal{G}_{k-m}^b &= \{h \in H^{k-m}(\Omega) : h(x_0) = 0\} \end{aligned} \tag{3.26}$$

are the tangent spaces to $\mathcal{V}_j$ and $\mathcal{G}_{k-m}$, at u_0 and g_0, respectively.

By the previous argument involving (3.19) and (3.20), we know that, for any given $h \in \mathcal{G}_{k-m}^b$, we can find $v_1 \in H^k(\Omega)$ such that $\mathcal{L}v_1 = h$, perhaps after shrinking Ω. To prove the surjectivity in (3.25), we need to find $v \in H^k(\Omega)$ such that $\mathcal{L}v = 0$ and such that $v - v_1 = O(|x - x_0|^{m+1})$, so that $v_1 - v \in \mathcal{V}_k^b$ and $\mathcal{L}(v_1 - v) = h$. We will actually produce $v \in C^\infty(\Omega)$. To work on this problem, we will find it convenient to use the notion of the m-jet $J_0^m(v)$ of a function $v \in C^\infty(\Omega)$, at x_0, being the Taylor polynomial of order m for v about x_0. Note that

$$J_0^m(v) = J_0^m(v^\#) \Longleftrightarrow (v - v^\#)(x) = O(|x - x_0|^{m+1}), \tag{3.27}$$

given that $v, v^\# \in C^\infty(\Omega)$. The existence of the function v we seek here is guaranteed by the following assertion.

Lemma 3.4. *Given an elliptic operator $\mathcal{L}$ of order m, as above, let*

$$\mathcal{J} = \{J_0^m(v) : \mathcal{L}v(x_0) = 0\} \tag{3.28}$$

and

$$\mathcal{S} = \{J_0^m(v) : v \in C^\infty(\Omega),\ \mathcal{L}v = 0 \text{ on } \Omega\}. \tag{3.29}$$

Clearly, $\mathcal{S} \subset \mathcal{J}$. If Ω is a sufficiently small neighborhood of x_0, then $\mathcal{S} = \mathcal{J}$.

Proof. This result is a simple special case of our goal, Proposition 3.3; the beginning of the proof here just retraces arguments from the beginning of that proof. Namely, let $v_1 \in C^\infty(\Omega)$ have m-jet in $\mathcal{J}$, hence satisfying $\mathcal{L}v_1(x_0) = 0$. Then Lemma 3.2 applies, so there exists v_0 such that

$$J_0^m(v_0) = J_0^m(v_1) \ \text{ and } \ \mathcal{L}v_0 = O(|x - x_0|^\infty). \tag{3.30}$$

Set $h_0 = \mathcal{L}v_0$. Suppose Ω is shrunk so far that $\mathcal{L}\mathcal{L}^*$ in (3.20) is an isomorphism. Now, for any $\varepsilon > 0$, there exists $h_1 \in C^\infty(\overline{\Omega})$ such that

$$h_1 = h_0 \text{ near } x_0, \quad \|h_1\|_{H^\ell(\Omega)} < \varepsilon. \tag{3.31}$$

Then the Dirichlet problem

$$\mathcal{L}\mathcal{L}^*\tilde{w} = h_1 \text{ on } \Omega, \quad \tilde{w} \in H_0^m(\Omega)$$

has a unique solution $\tilde{w}$ satisfying estimates

$$\|\tilde{w}\|_{H^{\ell+2m}(\Omega)} \leq C_\ell \|h_1\|_{H^\ell(\Omega)}. \tag{3.32}$$

Fix $\ell > n/2$. By Sobolev's imbedding theorem, $w = \mathcal{L}^*\tilde{w}$ satisfies

$$\|w\|_{C^m(\Omega)} \leq C^\# \|w\|_{H^{\ell+m}(\Omega)}. \tag{3.33}$$

In light of this, we have

$$\|w\|_{C^m(\Omega)} \leq C_\ell^\# \varepsilon, \quad \mathcal{L}w = h_1 \text{ on } \Omega, \tag{3.34}$$

so $v = v_1 - w$ defines an element in $\mathcal{S}$, provided Ω is shrunk to Ω_1, on which $h_1 = h_0$ in (3.31). Furthermore, $J_0^m(v)$ differs from $J_0^m(v_1)$ by $J_0^m(w)$, which is small (i.e., proportional to ε). Since $\mathcal{S}$ is a linear subspace of the finite-dimensional space $\mathcal{J}$, this approximability yields the identity $\mathcal{S} = \mathcal{J}$ and proves the lemma.

From the lemma, as we have seen, it follows that the map (3.25) is a surjective linear map between two Hilbert spaces, so the implicit function theorem therefore applies to the map F in (3.24). In other words, F maps a neighborhood of u_0 in $\mathcal{V}_k$ onto a neighborhood of $g_0 = F(u_0)$ in $\mathcal{G}_{k-m}$. As in the proof of Theorem 3.1, we see that any neighborhood of $r(x) = F(u_0) - g$ in $\mathcal{G}_{k-m}^b$ contains functions that vanish on a neighborhood of x_0, so any neighborhood of $F(u_0)$ in $\mathcal{G}_{k-m}$ contains functions equal to $g(x)$ on a neighborhood of x_0. This completes the proof of Proposition 3.3.

In some geometrical problems, it is useful to extend the notion of ellipticity. A differential operator of the form (3.3) is said to be *underdetermined elliptic* at u_0 provided $DF(u_0)$ has surjective symbol.

Proposition 3.5. *If $F(u_1)$ satisfies $F(u_1) = g$ at $x = x_0$, and if F is underdetermined elliptic at u_1, then, for any ℓ, there exists $u \in C^\ell(\Omega)$ such that $F(u) = g$ on a neighborhood of x_0 and such that $(u - u_1)(x) = O(|x - x_0|^{m+1})$.*

Proof. We produce u in the form $u = u_1 + u_2$, where we want

$$F(u_1 + u_2) = g \text{ near } x_0, \quad u_2(x) = O(|x - x_0|^{m+1}). \tag{3.35}$$

We will find u_2 in the form $u_2 = \mathcal{L}^* w$, where $\mathcal{L} = DF(u_1)$. Thus we want to find $w \in C^{\ell+m}(\Omega)$ satisfying

$$\Phi(w) = F(u_1 + \mathcal{L}^* w) = g \text{ near } x_0, \quad w(x) = O(|x - x_0|^{2m+1}). \tag{3.36}$$

Now $\Phi(w)$ is strongly elliptic of order $2m$ at w_1 and $\Phi(w_1) = 0$ at x_0 if $w_1 = 0$. Thus the existence of w satisfying (3.36) follows from Proposition 3.3, and the proof is finished.

We will apply the local existence theory to establish the following classical local isometric imbedding result.

Proposition 3.6. *Let M be a 2-dimensonal Riemannian manifold. If $p_0 \in M$ and the Gauss curvature $K(p_0) > 0$, then there is a neighborhood $\mathcal{O}$ of p_0 in M that can be smoothly isometrically imbedded in $\mathbb{R}^3$.*

The proof involves constructing a smooth, real-valued function u on $\mathcal{O}$ such that $du(p_0) = 0$ and such that $g_1 = g - du^2$ is a flat metric on $\mathcal{O}$, where g is the given metric tensor on M. Assuming this can be accomplished, then by the fundamental property of curvature (Proposition 3.1 of Appendix C), we can take coordinates (x, y) on $\mathcal{O}$ (after possibly shrinking $\mathcal{O}$) such that $g_1 = dx^2 + dy^2$. Thus $g = dx^2 + dy^2 + du^2$, which implies that $(x, y, u) : \mathcal{O} \to \mathbb{R}^3$ provides the desired local isometric imbedding.

Thus our task is to find such a function u. We need a formula for the Gauss curvature K_1 of $\mathcal{O}$, with metric tensor $g_1 = g - du^2$. A lengthy but finite computation from the fundamental formulas given in § 3 of Appendix C yields

$$\big(1 - |\nabla u|^2\big)^2 K_1 = \big(1 - |\nabla u|^2\big) K - \det H_g(u). \tag{3.37}$$

Here, $|\nabla u|^2 = g^{jk} u_{;j} u_{;k}$, and $H_g(u)$ is the Hessian of u relative to the Levi-Civita connection of g:

$$H_g(u) = \big(u^{;j}{}_{;k}\big). \tag{3.38}$$

This is the tensor field of type (1,1) associated to the tensor field $\nabla^2 u$ of type (0,2), such as defined by (2.3)–(2.4) of Appendix C, or equivalently by (3.27) of Chap. 2. In normal coordinates centered at $p \in M$, we have $H_g(u) = (\partial_j \partial_k u)$, at p.

Therefore, g_1 is a flat metric if and only if u satisfies the PDE

$$\det H_g(u) = \big(1 - |\nabla u|^2\big) K. \tag{3.39}$$

By the sort of analysis done in (3.7)–(3.9), we see that this equation is elliptic, provided $K > 0$ and $|\nabla u| < 1$. Thus Proposition 3.3 applies, to yield a local solution $u \in C^\ell(\mathcal{O})$, for arbitrarily large ℓ, provided the metric tensor g is smooth. As mentioned above, results of § 4 will imply that $u \in C^\infty(\mathcal{O})$.

If $K(p_0) < 0$, then (3.39) will be hyperbolic near p_0, and results of Chap. 16 will apply, to produce an analogue of Proposition 3.6 in that case. No matter what the value of $K(p_0)$, if the metric tensor g is real analytic, then the nonlinear Cauchy–Kowalewsky theorem, proved in § 4 of Chap. 16, will apply, yielding in that case a real analytic, local isometric imbedding of M into $\mathbb{R}^3$.

If M is compact (diffeomorphic to S^2) and has a metric with $K > 0$ everywhere, then in fact M can be *globally* isometrically imbedded in $\mathbb{R}^3$. This result is established in [Ni2] and [Po]. Of course, it is not true that a given compact Riemannian 2-manifold M can be globally isometrically imbedded in $\mathbb{R}^3$ (for example, if $K < 0$), but it can always be isometrically imbedded in $\mathbb{R}^N$ for sufficiently large N. In fact, this is true no matter what the dimension of M. This important result of J. Nash will be proved in § 5 of this chapter.

Exercises

1. Given the formula (3.8a) for the linearization of $F(u) = \det H(u)$, show that the symbol of $DF(u)$ is given by

$$\sigma_{DF(u)}(x, \xi) = -\mathcal{C}(u)\xi \cdot \xi. \tag{3.40}$$

2. Let a surface $M \subset \mathbb{R}^3$ be given by $x_3 = u(x_1, x_2)$. Given $K(x_1, x_2)$, to construct u such that the Gauss curvature of M at $(x_1, x_2, u(x_1, x_2))$ is equal to $K(x_1, x_2)$ is to solve

$$\det H(u) = \big(1 + |\nabla u|^2\big)^2 K. \tag{3.41}$$

   See (4.29) of Appendix C. If one is given a smooth $K(x_1, x_2) > 0$, then this PDE is elliptic. Applying Proposition 3.3, what geometrical properties of M can you prescribe at a given point and guarantee a local solution?
3. Verify (3.37). Compare with formula (**) on p. 210 of [Spi], Vol. 5.
4. Show that, in local coordinates on a 2-dimensional Riemannian manifold, the left side of (3.39) is given by

$$\det\big(u^{;j}{}_{;k}\big) = g^{-1} \det(\partial_j \partial_k u) + A^{jk}(x, \nabla u)\, \partial_j \partial_k u + Q(\nabla u, \nabla u),$$

   where $g = \det(g_{jk})$,

$$A^{jk}(x, \nabla u) = \pm g^{jk} \sigma^{j'\ell}{}_{k'} \partial_\ell u,$$

with "+" if $j = k$, "−" if $j \neq k$, j' and k' the indices complementary to j and k, and

$$\sigma^{j\ell}{}_k = \partial_k g^{j\ell} + \Gamma^j{}_{mk} g^{m\ell},$$

and

$$Q(\nabla u, \nabla u) = \det(\tau^j{}_k), \quad \tau^j{}_k = \sigma^{j\ell}{}_k \partial_\ell u.$$

4. Elliptic regularity I (interior estimates)

Here we will discuss two methods of establishing regularity of solutions to nonlinear elliptic PDE. The first is to consider regularity for a linear elliptic differential operator of order m

$$A(x, D) = \sum_{|\alpha| \le m} a_\alpha(x)\ D^\alpha, \tag{4.1}$$

whose coefficients have limited regularity. The second method will involve use of paradifferential operators. For both methods, we will make use of the Hölder spaces $C^s(\mathbb{R}^n)$ and Zygmund spaces $C^s_*(\mathbb{R}^n)$, discussed in § 8 of Chap. 13. Material in this section largely follows the exposition in [T].

Let us suppose $a_\alpha(x) \in C^s(\mathbb{R}^n),\ s \in (0, \infty) \setminus \mathbb{Z}$. Then $A(x, \xi)$ belongs to the symbol space $C^s_* S^m_{1,0}$, as defined in § 9 of Chap. 13. Recall that $p(x, \xi) \in C^s_* S^m_{1,\delta}$, provided

$$|D^\alpha_\xi p(x, \xi)| \le C_\alpha \langle \xi \rangle^{m-|\alpha|} \tag{4.2}$$

and

$$\| D^\alpha_\xi p(\cdot, \xi) \|_{C^s_*(\mathbb{R}^n)} \le C^\alpha \langle \xi \rangle^{m-|\alpha|+\delta s}. \tag{4.3}$$

We would like to establish regularity results for elliptic $A(x, \xi) \in C^s_* S^m_{1,0}$, by pseudodifferential operator techniques. It is not so convenient to work with an operator with symbol $A(x, \xi)^{-1}$. Rather, we will decompose $A(x, \xi)$ into a sum

$$A(x, \xi) = A^\#(x, \xi) + A^b(x, \xi), \tag{4.4}$$

in such a way that a good parametrix can be constructed for $A^\#(x, D)$, while $A^b(x, D)$ is regarded as a remainder term to be estimated. Pick $\delta \in (0, 1)$. As shown in Proposition 9.9 of Chap. 13, any $A(x, \xi) \in C^s_* S^m_{1,0}$ can be written in the form (4.4), with

$$A^\#(x, \xi) \in S^m_{1,\delta}, \quad A^b(x, \xi) \in C^s_* S^{m-\delta s}_{1,\delta}. \tag{4.5}$$

To $A^b(x, D)$ we apply Proposition 9.10 of Chap. 13, which, we recall, states that

$$p(x,\xi) \in C^s_* S^\mu_{1,\delta} \Longrightarrow p(x, D) : C^{\mu+r}_* \longrightarrow C^r_*, \quad -(1-\delta)s < r < s. \tag{4.6}$$

Consequently,

$$A^b(x, D) : C^{m+r-\delta s}_* \longrightarrow C^r_*, \quad -(1-\delta)s < r < s. \tag{4.7}$$

Now let $p(x, D) \in OPS^{-m}_{1,\delta}$ be a parametrix for $A^\#(x, D)$, which is elliptic. Hence, mod C^∞,

$$p(x, D)A(x, D)u = u + p(x, D)A^b(x, D)u, \tag{4.8}$$

so if

$$A(x, D)u = f, \tag{4.9}$$

then, mod C^∞,

$$u = p(x, D)f - p(x, D)A^b(x, D)u. \tag{4.10}$$

In view of (4.7), we see that when (4.10) is satisfied,

$$u \in C^{m+r-\delta s}_*, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*. \tag{4.11}$$

We then have the following.

Proposition 4.1. *Let $A(x,\xi) \in C^s_* S^m_{1,0}$ be elliptic, and suppose u solves (4.9). Assuming*

$$s > 0, \quad 0 < \delta < 1 \quad \textit{and} \quad -(1-\delta)s < r < s, \tag{4.12}$$

we have

$$u \in C^{m+r-\delta s}, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*. \tag{4.13}$$

Note that, for $|\alpha| = m$, $D^\alpha u \in C^{r-\delta s}_*$, and $r - \delta s$ could be negative. However, $a_\alpha(x)D^\alpha u$ will still be well defined for $a_\alpha \in C^s$. Indeed, if (4.6) is applied to the special case of a multiplication operator, we have

$$a \in C^s, \ u \in C^\sigma_* \Longrightarrow au \in C^\sigma_*, \quad \text{for } -s < \sigma < s. \tag{4.14}$$

Note that the range of r in (4.12) can be rewritten as $-s < r - \delta s < (1-\delta)s$. If we set $r - \delta s = -s + \varepsilon$, this means $0 < \varepsilon < (2-\delta)s$, so we can rewrite (4.13) as

$$u \in C^{m-s+\varepsilon},\ f \in C^r_* \Longrightarrow u \in C^{m+r}_*,\ \text{ provided } \varepsilon > 0,\ r < s, \tag{4.15}$$

as long as the relation $r = -(1-\delta)s + \varepsilon$ holds. Letting δ range over $(0, 1)$, we see that this will hold for any $r \in (-s+\varepsilon, \varepsilon)$. However, if $r \in [\varepsilon, s)$, we can first obtain from the hypothesis (4.15) that $u \in C^{m+\rho}_*$, for any $\rho < \varepsilon$. This improves the a priori regularity of u by almost s units. This argument can be iterated repeatedly, to yield:

Theorem 4.2. *If $A(x, \xi) \in C^s S^m_{1,0}$ is elliptic and u solves (4.9), then (assuming $s > 0$)*

$$\begin{gathered} u \in C^{m-s+\varepsilon},\ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \\ \textit{provided } \varepsilon > 0 \textit{ and } -s < r < s. \end{gathered} \tag{4.16}$$

We can sharpen this up to obtain the following Schauder regularity result:

Theorem 4.3. *Under the hypotheses above,*

$$u \in C^{m-s+\varepsilon},\ f \in C^s_* \Longrightarrow u \in C^{m+s}_*. \tag{4.17}$$

Proof. Applying (4.16), we can assume $u \in C^{m+r}_*$ with $s - r > 0$ arbitrarily small. Now if we invoke Proposition 9.7 of Chap. 13, which says

$$p(x, \xi) \in C^r S^m_{1,1} \Longrightarrow p(x, D) : C^{m+r+\varepsilon}_* \longrightarrow C^r_*, \tag{4.18}$$

for all $\varepsilon > 0$, we can supplement 4.7 with

$$A^b(x, D) : C^{m+s-\delta s+\varepsilon}_* \longrightarrow C^s_*, \quad \varepsilon > 0. \tag{4.19}$$

If $\delta > 0$, and if $\varepsilon > 0$ is picked small enough, we can write $m+s-\delta s+\varepsilon = m+r$ with $r < s$, so, under the hypotheses of (4.17), the right side of (4.8) belongs to C^{m+s}_*, proving the theorem. We note that a similar argument also produces the regularity result:

$$u \in H^{m-s+\varepsilon,p},\ f \in C^s_* \Longrightarrow u \in C^{m+s}_*. \tag{4.20}$$

We now apply these results to solutions to the *quasi-linear* elliptic PDE

$$\sum_{|\alpha| \le m} a_\alpha(x, D^{m-1}u)\, D^\alpha u = f. \tag{4.21}$$

As long as $u \in C^{m-1+s}$, $a_\alpha(x, D^{m-1}u) \in C^s$. If also $u \in C^{m-s+\varepsilon}$, we obtain (4.16) and (4.17). If $r > s$, using the conclusion $u \in C^{m+s}_*$, we obtain $a_\alpha(x, D^{m-1}u) \in C^{s+1}$, so we can reapply (4.16) and (4.17) for further regularity, obtaining the following:

Theorem 4.4. *If u solves the quasi-linear elliptic PDE (4.21), then*

$$u \in C^{m-1+s} \cap C^{m-s+\varepsilon}, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \tag{4.22}$$

provided $s > 0, \ \varepsilon > 0$, *and* $-s < r$. *Thus*

$$u \in C^{m-1+s}, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \tag{4.23}$$

provided

$$s > \frac{1}{2}, \quad r > s - 1. \tag{4.24}$$

We can sharpen Theorem 4.4 a bit as follows. Replace the hypothesis in (4.22) by

$$u \in C^{m-1+s} \cap H^{m-1+\sigma,p}, \tag{4.25}$$

with $p \in (1, \infty)$. Recall that Proposition 9.10 of Chap. 13 gives both (4.6) and, for $p \in (1, \infty)$,

$$\begin{aligned} p(x,\xi) \in C^s_* S^m_{1,\delta} \Longrightarrow p(x,D) : H^{r+m,p} \longrightarrow H^{r,p}, \\ -(1-\delta)s < r < s. \end{aligned} \tag{4.26}$$

Parallel to (4.14), we have

$$a \in C^s, \ u \in H^{\sigma,p} \Longrightarrow au \in H^{\sigma,p}, \ \text{ for } -s < \sigma < s, \tag{4.27}$$

as a consequence of (4.26), so we see that the left side of (4.21) is well defined provided $s + \sigma > 1$. We have (4.8) and, by (4.26),

$$A^b(x,D) : H^{m+r-\delta s,p} \longrightarrow H^{r,p}, \ \text{ for } -(1-\delta)s < r < s, \tag{4.28}$$

parallel to (4.7). Thus, if (4.25) holds, we obtain

$$p(x,D)A^b(x,D)u \in H^{m-1+\sigma+\delta s,p}, \tag{4.29}$$

provided $-(1-\delta)s < \delta s - 1 + \sigma < s$, i.e., provided

$$s + \sigma > 1 \ \text{ and } \ -1 + \sigma + \delta s < s. \tag{4.30}$$

Thus, if $f \in H^{\rho,p}$ with $\rho > \sigma - 1$, we manage to improve the regularity of u over the hypothesized (4.25). One way to record this gain is to use the Sobolev imbedding theorem:

$$H^{m-1+\sigma+\delta s,p} \subset H^{m-1+\sigma,p_1}, \quad p_1 = \frac{pn}{n-\delta s} > p\Big(1 + \frac{\delta s p}{n}\Big). \tag{4.31}$$

If we assume $f \in C^r_*$ with $r > \sigma - 1$, we can iterate this argument sufficiently often to obtain $u \in C^{m-1+\sigma-\varepsilon}$, for arbitrary $\varepsilon > 0$. Now we can arrange $s + \sigma > 1+\varepsilon$, so we are now in a position to apply Theorem 4.4. This proves the following:

Theorem 4.5. *If u solves the quasi-linear elliptic PDE (4.21), then*

$$u \in C^{m-1+s} \cap H^{m-1+\sigma,p}, \ f \in C^r_* \Longrightarrow u \in C^{m+r}_*, \tag{4.32}$$

provided $1 < p < \infty$ *and*

$$s > 0, \quad s + \sigma > 1, \quad r > \sigma - 1. \tag{4.33}$$

Note that if $u \in H^{m,p}$ for some $p > n$, then $u \in C^{m-1+s}$ for $s = 1-n/p > 0$, and then (4.32) applies, with $\sigma = 1$, or even with $\sigma = n/p + \varepsilon$.

We next obtain a result regarding the regularity of solutions to a completely nonlinear elliptic system

$$F(x, D^m u) = f. \tag{4.34}$$

We could apply Theorems 4.2 and 4.3 to the equation for $u_j = \partial u/\partial x_j$:

$$\sum_{|\alpha|\le m} \frac{\partial F}{\partial \zeta_\alpha}(x, D^m u) D^\alpha u_j = -F_{x_j}(x, D^m u) + \frac{\partial f}{\partial x_j} = f_j. \tag{4.35}$$

Suppose $u \in C^{m+s}$, $s > 0$, so the coefficients $a_\alpha(x) = (\partial F/\partial \zeta_\alpha)(x, D^m u) \in C^s$. If $f \in C^r_*$, then $f_j \in C^s + C^{r-1}_*$. We can apply Theorems 4.2 and 4.3 to u_j provided $u \in C^{m+1-s+\varepsilon}$, to conclude that $u \in C^{m+s+1}_* \cup C^{m+r}_*$. This implication can be iterated as long as $s + 1 < r$, until we obtain $u \in C^{m+r}_*$.

This argument has the drawback of requiring too much regularity of u, namely that $u \in C^{m+1-s+\varepsilon}$ as well as $u \in C^{m+s}$. We can fix this up by considering difference quotients rather than derivatives $\partial_j u$. Thus, for $y \in \mathbb{R}^n$, $|y|$ small, set

$$v_y(x) = |y|^{-1}\big[u(x + y) - u(x)\big];$$

v_y satisfies the PDE

$$\sum_{|\alpha|\le m} \Phi_{\alpha y}(x) D^\alpha v_y(x) = G_y(x, D^m u), \tag{4.36}$$

where

$$\Phi_{\alpha y}(x) = \int_0^1 (\partial F/\partial \zeta_\alpha)\big(x, tD^m u(x) + (1 - t)D^m u(x + y)\big)\, dt \tag{4.37}$$

and G_y is an appropriate analogue of the right side of (4.35). Thus $\Phi_{\alpha y}$ is in C^s, uniformly as $|y| \to 0$, if $u \in C^{m+s}$, while this hypothesis also gives a uniform bound on the C^{m-1+s}-norm of v_y. Now, for each y, Theorems 4.2 and 4.3 apply to v_y, and one can get an *estimate* on $\|v_y\|_{C^{m+\rho}}$, $\rho = \min(s, r-1)$, *uniform* as $|y| \to 0$. Therefore, we have the following.

Theorem 4.6. *If u solves the elliptic PDE (4.34), then*

$$u \in C^{m+s}, \; f \in C_*^r \Longrightarrow u \in C_*^{m+r}, \tag{4.38}$$

provided

$$0 < s < r. \tag{4.39}$$

We shall now give a second approach to regularity results for nonlinear elliptic PDE, making use of the paradifferential operator calculus developed in § 10 of Chap. 13. In addition to giving another perspective on interior estimates, this will also serve as a warm-up for the work on boundary estimates in § 8.

If F is smooth in its arguments, then, as shown in (10.53)–(10.55) of Chap. 13,

$$F(x, D^m u) = \sum_{|\alpha| \le m} M_\alpha(x, D) D^\alpha u + F(x, D^m \Psi_0(D) u), \tag{4.40}$$

where $F\big(x, D^m \Psi_0(D) u\big) \in C^\infty$ and

$$M_\alpha(x, \xi) = \sum_k m_k^\alpha(x) \psi_{k+1}(\xi), \tag{4.41}$$

with

$$m_k^\alpha(x) = \int_0^1 \frac{\partial F}{\partial \zeta_\alpha} \big(\Psi_k(D) D^m u + t \psi_{k+1}(D) D^m u\big)\, dt. \tag{4.42}$$

As shown in Proposition 10.7 of Chap. 13, we have, for $r \ge 0$,

$$u \in C^{m+r} \Longrightarrow M_\alpha(x, \xi) \in \mathcal{A}_0^r S_{1,1}^0 \subset S_{1,1}^0 \cap C^r S_{1,0}^0. \tag{4.43}$$

We recall from (10.31) of Chap. 13 that

$$p(x, \xi) \in \mathcal{A}_0^r S_{1,\delta}^m \Longleftrightarrow \|D_\xi^\alpha p(\cdot, \xi)\|_{C^{r+s}} \le C_{\alpha s} \langle \xi \rangle^{m-|\alpha|+\delta s}, \quad s \ge 0. \tag{4.44}$$

Consequently, if we set

$$M(u; x, D) = \sum_{|\alpha| \le m} M_\alpha(x, D) D^\alpha, \tag{4.45}$$

we obtain

Proposition 4.7. *If* $u \in C^{m+r}$, $r \geq 0$, *then*

$$F(x, D^m u) = M(u; x, D)u + R, \tag{4.46}$$

with $R \in C^\infty$ *and*

$$M(u; x, \xi) \in \mathcal{A}_0^r S_{1,1}^m \subset S_{1,1}^m \cap C^r S_{1,0}^m. \tag{4.47}$$

Decomposing each $M_\alpha(x, \xi)$, we have, by (10.60)–(10.61) of Chap. 13,

$$M(u; x, \xi) = M^\#(x, \xi) + M^b(x, \xi), \tag{4.48}$$

with

$$M^\#(x, \xi) \in \mathcal{A}_0^r S_{1,\delta}^m \subset S_{1,\delta}^m \tag{4.49}$$

and

$$M^b(x, \xi) \in C^r S_{1,\delta}^{m-\delta r} \cap \mathcal{A}_0^r S_{1,1}^m \subset S_{1,1}^{m-r\delta}. \tag{4.50}$$

Let us explicitly recall that (4.49) implies

$$D_x^\beta M^\#(x, \xi) \in \begin{array}{ll} S_{1,\delta}^m, & |\beta| \leq r, \\ S_{1,\delta}^{m+\delta(|\beta|-r)}, & |\beta| \geq r. \end{array} \tag{4.51}$$

Note that the linearization of $F(x, D^m u)$ at u is given by

$$Lv = \sum_{|\alpha| \leq m} \tilde{M}_\alpha(x) D^\alpha v, \tag{4.52}$$

where

$$\tilde{M}_\alpha(x) = \frac{\partial F}{\partial \zeta_\alpha}(x, D^m u). \tag{4.53}$$

Comparison with (4.40)–(4.42) gives (for $u \in C^{m+r}$)

$$M(u; x, \xi) - L(x, \xi) \in C^r S_{1,1}^{m-r}, \tag{4.54}$$

by the same analysis as in the proof of the $\delta = 1$ case of (9.35) of Chap. 13. More generally, the difference in (4.54) belongs to $C^r S_{1,\delta}^{m-r\delta}$, $0 \leq \delta \leq 1$. Thus $L(x, \xi)$ and $M(u; x, \xi)$ have many qualitative properties in common.

Consequently, given $u \in C^{m+r}$, the operator $M^\#(x, D) \in OPS_{1,\delta}^m$ is microlocally elliptic in any direction $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ that is noncharacteristic for $F(x, D^m u)$, which by definition means noncharacteristic for L. In particular, $M^{\#}(x, D)$ is elliptic if $F(x, D^m u)$ is. Now if

$$F(x, D^m u) = f \tag{4.55}$$

is elliptic and $Q \in OPS^{-m}_{1,\delta}$ is a parametrix for $M^{\#}(x, D)$, we have

$$u = Q(f - M^b(x, D)u), \quad \mod C^\infty. \tag{4.56}$$

By (4.50) we have

$$QM^b(x, D) : H^{m-r\delta+s,p} \longrightarrow H^{m+s,p}, \quad s > 0. \tag{4.57}$$

(In fact $s > -(1 - \delta)r$ suffices.) We deduce that

$$u \in H^{m-\delta r+s,p}, \ f \in H^{s,p} \Longrightarrow u \in H^{m+s,p}, \tag{4.58}$$

granted $r > 0$, $s > 0$, and $p \in (1, \infty)$. There is a similar implication, with Sobolev spaces replaced by Hölder (or Zygmund) spaces. This sort of implication can be iterated, leading to a second proof of Theorem 4.6. We restate the result, including Sobolev estimates, which could also have been obtained by the first method used to prove Theorem 4.6.

Theorem 4.8. *Suppose, given $r > 0$, that $u \in C^{m+r}$ satisfies (4.55) and this PDE is elliptic. Then, for each $s > 0$, $p \in (1, \infty)$,*

$$f \in H^{s,p} \Longrightarrow u \in H^{m+s,p} \ \text{ and } \ f \in C^s_* \Longrightarrow u \in C^{m+s}_*. \tag{4.59}$$

By way of further comparison with the methods used earlier in this section, we now rederive Theorem 4.5, on regularity for solutions to a quasi-linear elliptic PDE. Note that, in the quasi-linear case,

$$F(x, D^m u) = \sum_{|\alpha| \le m} a_\alpha(x, D^{m-1}u) D^\alpha u = f, \tag{4.60}$$

the construction above gives $F(x, D^m u) = M(u; x, D)u + R_0(u)$ with the property that, for $r \ge 0$,

$$u \in C^{m+r} \Longrightarrow M(u; x, \xi) \in C^{r+1} S^m_{1,0} \cap S^m_{1,1} + C^r S^{m-1}_{1,0} \cap S^{m-1}_{1,1}. \tag{4.61}$$

Of more interest to us now is that, for $0 < r < 1$,

$$u \in C^{m-1+r} \Longrightarrow M(u; x, \xi) \in C^r S^m_{1,0} \cap S^m_{1,1} + S^{m-r}_{1,1}, \tag{4.62}$$

which follows from (10.23) of Chap. 13. Thus we can decompose the term in $C^r S^m_{1,0} \cap S^m_{1,1}$ via symbol smoothing, as in (10.60)–(10.61) of Chap. 13, and throw the term in $S^{m-r}_{1,1}$ into the remainder, to get

$$M(u; x, \xi) = M^{\#}(x, \xi) + M^b(x, \xi), \tag{4.63}$$

with

$$M^{\#}(x, \xi) \in S^m_{1,\delta}, \quad M^b(x, \xi) \in S^{m-r\delta}_{1,1}. \tag{4.64}$$

If $P(x, D) \in OPS^{-m}_{1,\delta}$ is a parametrix for the elliptic operator $M^{\#}(x, D)$, then whenever $u \in C^{m-1+r} \cap H^{m-1+\rho,p}$ is a solution to (4.60), we have, mod C^∞,

$$u = P(x, D)f - P(x, D)M^b(x, D)u. \tag{4.65}$$

Now

$$P(x, D)M^b(x, D) : H^{m-1+\rho,p} \longrightarrow H^{m-1+\rho+r\delta,p} \quad \text{if } r + \rho > 1, \tag{4.66}$$

by the last part of (4.64). As long as this holds, we can iterate this argument and obtain Theorem 4.5, with a shorter proof than the one given before.

Next we look at one example of a quasi-linear elliptic system in divergence form, with a couple of special features. One is that we will be able to assume less regularity a priori on u than in results above. The other is that the lower-order terms have a more significant impact on the analysis than above. After analyzing the following system, we will show how it arises in the study of the Ricci tensor.

We consider second-order elliptic systems of the form

$$\sum \partial_j a_{jk}(x, u)\partial_k u + B(x, u, \nabla u) = f. \tag{4.67}$$

We assume that $a_{jk}(x, u)$ and $B(x, u, p)$ are smooth in their arguments and that

$$|B(x, u, p)| \le C\langle p\rangle^2. \tag{4.68}$$

Proposition 4.9. *Assume that a solution u to (4.67) satisfies*

$$\nabla u \in L^q, \quad \textit{for some } q > n, \quad \textit{hence } u \in C^r, \tag{4.69}$$

for some $r \in (0, 1)$. Then, if $p \in (q, \infty)$ and $s \ge -1$, we have

$$f \in H^{s,p} \Longrightarrow u \in H^{s+2,p}. \tag{4.70}$$

To begin the proof of Proposition 4.9, we write

$$\sum_k a_{jk}(x, u)\,\partial_k u = A_j(u; x, D)u \tag{4.71}$$

mod C^∞, with

$$u \in C^r \Longrightarrow A_j(u;x,\xi) \in C^r S^1_{1,0} \cap S^1_{1,1} + S^{1-r}_{1,1}, \tag{4.72}$$

as established in Chap. 13. Hence, given $\delta \in (0,1)$,

$$\begin{gathered} A_j(u;x,\xi) = A^\#_j(x,\xi) + A^b_j(x,\xi), \\ A^\#_j(x,\xi) \in S^1_{1,\delta}, \quad A^b_j(x,\xi) \in S^{1-r\delta}_{1,1}. \end{gathered} \tag{4.73}$$

It follows that we can write

$$\sum \partial_j a_{jk}(x,u)\, \partial_k u = P^\# u + P^b u, \tag{4.74}$$

with

$$P^\# = \sum \partial_j A^\#_j(x,D) \in OPS^2_{1,\delta}, \quad \text{elliptic}, \tag{4.75}$$

and

$$P^b = \sum \partial_j A^b_j(x,D). \tag{4.76}$$

By Theorem 9.1 of Chap. 13, we have

$$A^b_j(x,D) : H^{1-r\delta+\mu,p'} \longrightarrow H^{\mu,p'}, \quad \text{for } \mu > 0,\ 1 < p' < \infty. \tag{4.77}$$

In particular (taking $\mu = r\delta,\ p' = q$),

$$\nabla u \in L^q \Longrightarrow P^b u \in H^{-1+r\delta,q}. \tag{4.78}$$

Now, if

$$E^\# \in OPS^{-2}_{1,\delta} \tag{4.79}$$

denotes a parametrix of $P^\#$, we have, mod C^∞,

$$u = E^\# f - E^\# B(x,u,\nabla u) - E^\# P^b u, \tag{4.80}$$

and we see that under the hypothesis (4.69), we have some control over the last term:

$$E^\# P^b u \in H^{1+r\delta,q} \subset H^{1,\tilde{q}}, \quad \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{r\delta}{n}. \tag{4.81}$$

Note also that under our hypothesis on $B(x, u, p)$,

$$\nabla u \in L^q \Longrightarrow B(x, u, \nabla u) \in L^{q/2}. \tag{4.82}$$

Now, by Sobolev's imbedding theorem,

$$E^\# B(x, u, \nabla u) \in H^{1,\tilde{p}}, \tag{4.83}$$

with $\tilde{p} = q/(2 - q/n)$ if $q < 2n$ and for all $\tilde{p} < \infty$ if $q \geq 2n$. Note that $\tilde{p} > q(1 + a/n)$ if $q = n + a$. This treats the middle term on the right side of (4.80). Of course, the hypothesis on f yields

$$E^\# f \in H^{s+2,p}, \quad s + 2 \geq 1, \tag{4.84}$$

which is just where we want to place u.

Having thus analyzed the three terms on the right side of (4.80), we have

$$u \in H^{1,q^\#}, \quad q^\# = \min(\tilde{p}, p, \tilde{q}). \tag{4.85}$$

Iterating this argument a finite number of times, we get

$$u \in H^{1,p}. \tag{4.86}$$

If $s = -1$ in (4.70), our work is done.

If $s > -1$ in (4.70), we proceed as follows. We already have $u \in H^{1,p}$, so $\nabla u \in L^p$. Thus, on the next pass through estimates of the form (4.78)–(4.83), we obtain

$$\begin{aligned} E^\# P^b u &\in H^{1+r\delta,p}, \\ E^\# B(x, u, \nabla u) &\in H^{2,p/2} \subset H^{2-n/p,p}, \end{aligned} \tag{4.87}$$

and hence

$$u \in H^{1+\sigma,p}, \quad \sigma = \min\left(r\delta, 1 - \frac{n}{p}, 1 + s\right). \tag{4.88}$$

We can iterate this sort of argument a finite number of times until the conclusion in (4.70) is reached.

Further results on elliptic systems of the form (4.67) will be given in § 12B. We now apply Proposition 4.9 to estimates involving the Ricci tensor. Consider a Riemannian metric g_{jk} defined on the unit ball $B_1 \subset \mathbb{R}^n$. We will work under the following hypotheses:

(i) For some constants $a_j \in (0, \infty)$, there are estimates

$$0 < a_0 I \leq \big(g_{jk}(x)\big) \leq a_1 I. \tag{4.89}$$

(ii) The coordinates $x_1, \dots, x_n$ are harmonic, namely

$$\Delta x_\ell = 0. \tag{4.90}$$

Here, Δ is the Laplace operator determined by the metric g_{jk}. In general,

$$\Delta v = g^{jk}\partial_j\partial_k v - \lambda^\ell \partial_\ell v, \quad \lambda^\ell = g^{jk}\Gamma^\ell{}_{jk}. \tag{4.91}$$

Note that $\Delta x_\ell = -\lambda^\ell$, so the coordinates are harmonic if and only if $\lambda^\ell = 0$. Thus, in harmonic coordinates,

$$\Delta v = g^{jk}\,\partial_j\partial_k v. \tag{4.92}$$

We will also assume some bounds on the Ricci tensor, and we desire to see how this influences the regularity of g_{jk} in these coordinates. Generally, as can be derived from formulas in § 3 of Appendix C, the Ricci tensor is given by

$$\begin{aligned} \operatorname{Ric}_{jk} &= \frac{1}{2} g^{\ell m}\big[-\partial_\ell\partial_m g_{jk} - \partial_j\partial_k g_{\ell m} \\ &\qquad + \partial_k\partial_m g_{\ell j} + \partial_\ell\partial_j g_{km}\big] + M_{jk}(g,\nabla g) \\ &= -\frac{1}{2} g^{\ell m}\partial_\ell\partial_m g_{jk} + \frac{1}{2} g_{j\ell}\partial_k\lambda^\ell + \frac{1}{2} g_{k\ell}\partial_j\lambda^\ell + H_{jk}(g,\nabla g), \end{aligned} \tag{4.93}$$

with λ^ℓ as in (4.91). In harmonic coordinates, we obtain

$$-\frac{1}{2}\sum \partial_j g^{jk}(x)\,\partial_k g_{\ell m} + Q_{\ell m}(g,\nabla g) = \operatorname{Ric}_{\ell m}, \tag{4.94}$$

and $Q_{\ell m}(g,\nabla g)$ is a quadratic form in ∇g, with coefficients that are smooth functions of g, as long as (4.89) holds. Also, when (4.89) holds, the equation (4.94) is elliptic, of the form (4.67). Thus Proposition 4.9 implies the following.

Proposition 4.10. *Assume the metric tensor satisfies hypotheses (i) and (ii). Also assume that, on B_1,*

$$\nabla g_{jk} \in L^q, \quad \text{for some } q > n, \tag{4.95}$$

and

$$Ric_{\ell m} \in H^{s,p}, \tag{4.96}$$

for some $p \in (q,\infty)$, $s \ge -1$. Then, on the ball $B_{9/10}$,

$$g_{jk} \in H^{s+2,p}. \tag{4.97}$$

In [DK] it was shown that if $g_{jk} \in C^2$, in harmonic coordinates, then, for $k \in \mathbb{Z}^+$, $\alpha \in (0,1)$, $\operatorname{Ric}_{\ell m} \in C^{k+\alpha} \Rightarrow g_{jk} \in C^{k+2+\alpha}$. Such results also follow by the methods used to prove Proposition 4.10. A result stronger than Proposition 4.10, using Morrey spaces, is proved in [T2].

Exercises

1. Consider the system $F(x, D^m u) = f$ when

$$F(x, D^m u) = \sum_{|\alpha| \le m} a_\alpha(x, D^j u)\, D^\alpha u,$$

for some j such that $0 \le j < m$. Assume this quasi-linear system is elliptic. Given $p, q \in (1, \infty)$, $r > 0$, assume

$$u \in C^{j+r} \cap H^{m-1+\rho,p}, \quad r + \rho > 1.$$

Show that

$$f \in H^{s,q} \Longrightarrow u \in H^{s+m,q}.$$

5. Isometric imbedding of Riemannian manifolds

In this section we will establish the following result.

Theorem 5.1. *If M is a compact Riemannian manifold, there exists a C^∞-map*

$$\Phi : M \longrightarrow \mathbb{R}^N, \tag{5.1}$$

which is an isometric imbedding.

This was first proved by J. Nash [Na1], but the proof was vastly simplified by M. Günther [Gu1]–[Gu3]. These works also deal with noncompact Riemannian manifolds and derive good bounds for N, but to keep the exposition simple we will not cover these results.

To prove Theorem 5.1, we can suppose without loss of generality that M is a torus $\mathbb{T}^k$. In fact, imbed M smoothly in some Euclidean space $\mathbb{R}^k$; M will sit inside some box; identify opposite faces to have $M \subset \mathbb{T}^k$. Then smoothly extend the Riemannian metric on M to one on $\mathbb{T}^k$.

If $\mathcal{R}$ denotes the set of smooth Riemannian metrics on $\mathbb{T}^k$ and $\mathcal{E}$ is the set of such metrics arising from smooth imbeddings of $\mathbb{T}^k$ into some Euclidean space, our goal is to prove

$$\mathcal{E} = \mathcal{R}. \tag{5.2}$$

Now $\mathcal{R}$ is clearly an open convex cone in the Fréchet space

$$V = C^\infty(\mathbb{T}^k, S^2 T^*)$$

of smooth, second-order, symmetric, covariant tensor fields. As a preliminary to demonstrating (5.2), we show that the subset $\mathcal{E}$ shares some of these properties.

Lemma 5.2. *$\mathcal{E}$ is a convex cone in V.*

Proof. If $g_0 \in \mathcal{E}$, it is obvious from scaling the imbedding producing g_0 that $\alpha g_0 \in \mathcal{E}$, for any $\alpha \in (0, \infty)$. Suppose also that $g_1 \in \mathcal{E}$. If these metrics g_j arise from imbeddings $\varphi_j : \mathbb{T}^k \to \mathbb{R}^{\nu_j}$, then $g_0 + g_1$ is a metric arising from the imbedding $\varphi_0 \oplus \varphi_1 : \mathbb{T}^k \to \mathbb{R}^{\nu_0+\nu_1}$. This proves the lemma.

Using Lemma 5.2 plus some functional analysis, we will proceed to establish that any Riemannian metric on $\mathbb{T}^k$ can be *approximated* by one in $\mathcal{E}$. First, we define some more useful objects. If $u : \mathbb{T}^k \to \mathbb{R}^m$ is any smooth map, let γ_u denote the symmetric tensor field on $\mathbb{T}^k$ obtained by pulling back the Euclidean metric on $\mathbb{R}^m$. In a natural local coordinate system on $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, arising from standard coordinates $(x_1, \dots, x_k)$ on $\mathbb{R}^k$,

$$\gamma_u = \sum_{i,j,\ell} \frac{\partial u_\ell}{\partial x_i} \frac{\partial u_\ell}{\partial x_j}\, dx_i \otimes dx_j. \tag{5.3}$$

Whenever u is an immersion, γ_u is a Riemannian metric; and if u is an imbedding, then γ_u is of course an element of $\mathcal{E}$. Denote by $\mathcal{C}$ the set of tensor fields on $\mathbb{T}^k$ of the form γ_u. By the same reasoning as in Lemma 5.2, $\mathcal{C}$ is a convex cone in V.

Lemma 5.3. *$\mathcal{E}$ is a dense subset of $\mathcal{R}$.*

Proof. If not, take $g \in \mathcal{R}$ such that $g \notin \overline{\mathcal{E}}$, the closure of $\mathcal{E}$ in V. Now $\overline{\mathcal{E}}$ is a closed, convex subset of V, so the Hahn–Banach theorem implies that there is a continuous linear functional $\ell : V \to \mathbb{R}$ such that $\ell(\overline{\mathcal{E}}) \le 0$ while $\ell(g) = a > 0$.

Let us note that $\mathcal{C} \subset \overline{\mathcal{E}}$ (and hence $\overline{\mathcal{C}} = \overline{\mathcal{E}}$). In fact, if $u : \mathbb{T}^k \to \mathbb{R}^m$ is any smooth map and $\varphi : \mathbb{T}^k \to \mathbb{R}^n$ is an imbedding, then, for any $\varepsilon > 0$, $\varepsilon\varphi \oplus u : \mathbb{T}^k \to \mathbb{R}^{n+m}$ is an imbedding, and $\gamma_{\varepsilon\varphi\oplus u} = \varepsilon^2\gamma_\varphi + \gamma_u \in \mathcal{E}$. Taking $\varepsilon \searrow 0$, we have $\gamma_u \in \overline{\mathcal{E}}$.

Consequently, the linear functional ℓ produced above has the property $\ell(\overline{\mathcal{C}}) \le 0$. Now we can represent ℓ as a $k \times k$ symmetric matrix of distributions ℓ_{ij} on $\mathbb{T}^k$, and we deduce that

$$\sum_{i,j} \langle \partial_i f\, \partial_j f, \ell_{ij} \rangle \le 0, \quad \forall f \in C^\infty(\mathbb{T}^k). \tag{5.4}$$

If we apply a Friedrichs mollifier J_ε, in the form of a convolution operator on $\mathbb{T}^k$, it follows easily that (5.4) holds with $\ell_{ij} \in \mathcal{D}'(\mathbb{T}^k)$ replaced by $\lambda_{ij} = J_\varepsilon \ell_{ij} \in C^\infty(\mathbb{T}^k)$. Now it is an exercise to show that if $\lambda_{ij} \in C^\infty(\mathbb{T}^k)$ satisfies both $\lambda_{ij} = \lambda_{ji}$ and the analogue of (5.4), then $\Lambda = (\lambda_{ij})$ is a negative-semidefinite, matrix-valued function on $\mathbb{T}^k$, and hence, for any positive-definite $G = (g_{ij}) \in C^\infty(\mathbb{T}^k, S^2T^*)$,

$$\sum_{i,j} \langle g_{ij}, \lambda_{ij} \rangle \le 0. \tag{5.5}$$

Taking $\lambda_{ij} = J_\varepsilon \ell_{ij}$ and passing to the limit $\varepsilon \to 0$, we have

$$\sum_{i,j} \langle g_{ij}, \ell_{ij} \rangle \le 0, \tag{5.6}$$

for any Riemannian metric tensor (g_{ij}) on $\mathbb{T}^k$. This contradicts the hypothesis that we can take $g \notin \overline{\mathcal{E}}$, so Lemma 5.3 is proved.

The following result, to the effect that $\mathcal{E}$ has nonempty interior, is the analytical heart of the proof of Theorem 5.1.

Lemma 5.4. *There exist a Riemannian metric $g_0 \in \mathcal{E}$ and a neighborhood U of 0 in V such that $g_0 + h \in \mathcal{E}$ whenever $h \in U$.*

We now prove (5.2), hence Theorem 5.1, granted this result. Let $g \in \mathcal{R}$, and take $g_0 \in \mathcal{E}$, given by Lemma 5.4. Then set $g_1 = g + \alpha(g - g_0)$, where $\alpha > 0$ is picked sufficiently small that $g_1 \in \mathcal{R}$. It follows that g is a convex combination of g_0 and g_1; that is, $g = ag_0 + (1-a)g_1$ for some $a \in (0, 1)$. By Lemma 5.4, we have an open set $U \subset V$ such that $g_0 + h \in \mathcal{E}$ whenever $h \in U$. But by Lemma 5.3, there exists $h \in U$ such that $g_1 - bh \in \mathcal{E}$, $b = a/(1-a)$. Thus $g = a(g_0 + h) + (1-a)(g_1 - bh)$ is a convex combination of elements of $\mathcal{E}$, so by Lemma 5.1, $g \in \mathcal{E}$, as desired.

We turn now to a proof of Lemma 5.4. The metric g_0 will be one arising from a *free* imbedding

$$u : \mathbb{T}^k \longrightarrow \mathbb{R}^\mu, \tag{5.7}$$

defined as follows.

Definition. *An imbedding as in (5.7) is free provided that the $k + k(k+1)/2$ vectors*

$$\partial_j u(x), \quad \partial_j \partial_k u(x) \tag{5.8}$$

are linearly independent in $\mathbb{R}^\mu$, for each $x \in \mathbb{T}^k$.

Here, we regard $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, so $u : \mathbb{R}^k \to \mathbb{R}^\mu$, invariant under the translation action of $\mathbb{Z}^k$ on $\mathbb{R}^k$, and $(x_1, \dots x_k)$ are the standard coordinates on $\mathbb{R}^k$. It is not hard to establish the existence of free imbeddings; see the exercises.

Now, given that u is a free imbedding and that (h_{ij}) is a smooth, symmetric tensor field that is small in some norm (stronger than the C^2-norm), we want to find $v \in C^\infty(\mathbb{T}^k, \mathbb{R}^\mu)$, small in a norm at least as strong as the C^1-norm, such that, with $g_0 = \gamma_u$,

$$\sum_{\ell} \partial_i(u_\ell + v_\ell)\partial_j(u_\ell + v_\ell) = g_{0ij} + h_{ij}, \tag{5.9}$$

or equivalently, using the dot product on $\mathbb{R}^\mu$,

$$\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v + \partial_i v \cdot \partial_j v = h_{ij}. \tag{5.10}$$

We want to solve for v. Now, such a system turns out to be highly underdetermined, and the key to success is to append convenient side conditions. Following [Gu3], we apply $\Delta - 1$ to (5.10), where $\Delta = \sum \partial_j^2$, obtaining

$$\begin{aligned} \partial_i \Big\{(\Delta - 1)(\partial_j u \cdot v) + \Delta v \cdot \partial_j v\Big\} &+ \partial_j \Big\{(\Delta - 1)(\partial_i u \cdot v) + \Delta v \cdot \partial_i v\Big\} \\ -2\Big\{(\Delta - 1)(\partial_i\partial_j u \cdot v) &+ \frac{1}{2}\partial_i v \cdot \partial_j v - \partial_i\partial_\ell v \cdot \partial_j \partial_\ell v \\ &+ \Delta v \cdot \partial_i\partial_j v + \frac{1}{2}(\Delta - 1)h_{ij}\Big\} = 0, \end{aligned} \tag{5.11}$$

where we sum over ℓ. Thus (5.10) will hold whenever v satisfies the new system

$$\begin{aligned} (\Delta - 1)\big(\zeta_i(x) \cdot v\big) &= -\Delta v \cdot \partial_i v, \\ (\Delta - 1)\big(\zeta_{ij}(x) \cdot v\big) &= -\frac{1}{2}(\Delta - 1)h_{ij} \\ &\quad + \Big(\partial_i\partial_\ell v \cdot \partial_j\partial_\ell v - \Delta v \cdot \partial_i\partial_j v - \frac{1}{2}\partial_i v \cdot \partial_j v\Big). \end{aligned} \tag{5.12}$$

Here we have set $\zeta_i(x) = \partial_i u(x)$, $\zeta_{ij}(x) = \partial_i\partial_j u(x)$, smooth $\mathbb{R}^\mu$-valued functions on $\mathbb{T}^k$.

Now (5.12) is a system of $k(k+3)/2 = \kappa$ equations in μ unknowns, and it has the form

$$(\Delta - 1)\big(\xi(x)v\big) + Q(D^2 v, D^2 v) = H = \Big(0, -\frac{1}{2}(\Delta - 1)h_{ij}\Big), \tag{5.13}$$

where $\xi(x) : \mathbb{R}^\mu \to \mathbb{R}^\kappa$ is *surjective* for each x, by the linear independence hypothesis on (5.8), and Q is a bilinear function of its arguments $D^2 v = \{D^\alpha v : |\alpha| \le 2\}$. This is hence an underdetermined system for v. We can obtain a determined system for a function w on $\mathbb{T}^k$ with values in $\mathbb{R}^\kappa$, by setting

$$v = \xi(x)^t w, \tag{5.14}$$

namely

$$(\Delta - 1)\big(A(x)w\big) + \widetilde{Q}(D^2 w, D^2 w) = H, \tag{5.15}$$

where, for each $x \in \mathbb{T}^k$,

$$A(x) = \xi(x)\xi(x)^t \in \operatorname{End}(\mathbb{R}^\kappa) \text{ is invertible.} \tag{5.16}$$

If we denote the left side of (5.15) by $F(w)$, the operator F is a nonlinear differential operator of order 2, and we have

$$DF(w)f = (\Delta - 1)\big(A(x)f\big) + B(D^2w, D^2 f), \tag{5.17}$$

where B is a bilinear function of its arguments. In particular,

$$DF(0)f = (\Delta - 1)\big(A(x)f\big). \tag{5.18}$$

We thus see that, for any $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$,

$$DF(0) : C^{r+2}(\mathbb{T}^k, \mathbb{R}^\kappa) \longrightarrow C^r(\mathbb{T}^k, \mathbb{R}^\kappa) \text{ is invertible.} \tag{5.19}$$

Consequently, if we fix $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, and if $H \in C^r(\mathbb{T}^k, \mathbb{R}^\kappa)$ has sufficiently small norm (i.e., if $(h_{ij}) \in C^{r+2}(\mathbb{T}^k, S^2T^*)$ has sufficiently small norm), then (5.15) has a unique solution $w \in C^{r+2}(\mathbb{T}^k, \mathbb{R}^\kappa)$ with small norm, and via (5.14) we get a solution $v \in C^{r+2}(\mathbb{T}^k, \mathbb{R}^\mu)$, with small norm, to (5.13). If the norm of v is small enough, then of course $u + v$ is also an imbedding.

Furthermore, if the C^{r+2}-norm of w is small enough, then (5.15) is an elliptic system for w. By the regularity result of Theorem 4.6, we can deduce that w is C^∞ (hence v is C^∞) if h is C^∞. This concludes the proof of Lemma 5.4, hence of Nash's imbedding theorem.

Exercises

In Exercises 1–3, let B be the unit ball in $\mathbb{R}^k$, centered at 0. Let (λ_{ij}) be a smooth, symmetric, matrix-valued function on B such that

$$\sum_{i,j} \int (\partial_i f)(x)\,(\partial_j f)(x)\,\lambda_{ij}(x)\,dx \le 0, \quad \forall f \in C_0^\infty(B). \tag{5.20}$$

1. Taking $f_\varepsilon \in C_0^\infty(B)$ of the form

$$f_\varepsilon(x) = f(\varepsilon^{-2}x_1, \varepsilon^{-1}x'), \quad 0 < \varepsilon < 1,$$

examine the behavior as $\varepsilon \searrow 0$ of (5.20), with f replaced by f_ε. Establish that $\lambda_{11}(0) \le 0$.
2. Show that the condition (5.20) is invariant under rotations of $\mathbb{R}^k$, and deduce that $\big(\lambda_{ij}(0)\big)$ is a negative-semidefinite matrix.
3. Deduce that $\big(\lambda_{ij}(x)\big)$ is negative-semidefinite for all $x \in B$.
4. Using the results above, demonstrate the implication (5.4) $\Rightarrow$ (5.5), used in the proof of Lemma 5.3.

5. Suppose we have a C^∞-imbedding $\varphi : \mathbb{T}^k \to \mathbb{R}^n$. Define a map

$$\psi : \mathbb{T}^k \longrightarrow \mathbb{R}^n \oplus S^2\mathbb{R}^n \approx \mathbb{R}^\mu, \quad \mu = n + \frac{1}{2}n(n+1),$$

to have components

$$\varphi_j(x),\ 1 \le j \le n, \quad \varphi_i(x)\varphi_j(x),\ 1 \le i \le j \le n.$$

Show that ψ is a free imbedding.

6. Using Leibniz' rule to expand derivatives of products, verify that (5.10) and (5.11) are equivalent, for $v \in C^\infty(\mathbb{T}^k, \mathbb{R}^\mu)$.
7. In [Na1] the system (5.10) was augmented with $\partial_i u \cdot v = 0$, yielding, instead of (5.12), the system

$$\begin{aligned} \zeta_i(x) \cdot v &= 0, \\ \zeta_{ij}(x) \cdot v &= \frac{1}{2}\big(\partial_i v \cdot \partial_j v - h_{ij}\big). \end{aligned} \tag{5.21}$$

What makes this system more difficult to solve than (5.12)?

6. Minimal surfaces

A *minimal surface* is one that is critical for the area functional. To begin, we consider a k-dimensional manifold M (generally with boundary) in $\mathbb{R}^n$. Let ξ be a compactly supported normal field to M, and consider the one-parameter family of manifolds $M_s \subset \mathbb{R}^n$, images of M under the maps

$$\varphi_s(x) = x + s\xi(x), \quad x \in M. \tag{6.1}$$

We want a formula for the derivative of the k-dimensional area of M_s, at $s = 0$. Let us suppose ξ is supported on a single coordinate chart, and write

$$A(s) = \int_\Omega \big\| \partial_1 X \wedge \cdots \wedge \partial_k X \big\| \, du_1 \cdots du_k, \tag{6.2}$$

where $\Omega \subset \mathbb{R}^k$ parameterizes M_s by $X(s,u) = X_0(u) + s\xi(u)$. We can also suppose this chart is chosen so that $\|\partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\| = 1$. Then we have

$$A'(0) =$$
$$\sum_{j=1}^k \int \big\langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \big\rangle \, du_1 \cdots du_k. \tag{6.3}$$

By the Weingarten formula (see (4.9) of Appendix C), we can replace $\partial_j \xi$ by $-A_\xi E_j$, where $E_j = \partial_j X_0$. Without loss of generality, for any fixed $x \in M$, we can assume that $E_1, \dots, E_k$ is an orthonormal basis of $T_x M$. Then

$$\big\langle E_1 \wedge \cdots \wedge A_\xi E_j \wedge \cdots \wedge E_k, E_1 \wedge \cdots \wedge E_k \big\rangle = \big\langle A_\xi E_j, E_j \big\rangle, \tag{6.4}$$

at x. Summing over j yields Tr $A_\xi(x)$, which is invariantly defined, so we have

$$A'(0) = -\int_M \operatorname{Tr} A_\xi(x)\, dA(x), \tag{6.5}$$

where $A_\xi(x) \in \operatorname{End}(T_x M)$ is the Weingarten map of M and $dA(x)$ the Riemannian k-dimensional area element. We say M is a minimal submanifold of $\mathbb{R}^n$ provided $A'(0) = 0$ for all variations of the form (6.1), for which the normal field ξ vanishes on ∂M.

If we specialize to the case where $k = n-1$ and M is an oriented hypersurface of $\mathbb{R}^n$, letting N be the "outward" unit normal to M, for a variation M_s of M given by

$$\varphi_s(x) = x + sf(x)N(x), \quad x \in M, \tag{6.6}$$

we hence have

$$A'(0) = -\int_M \operatorname{Tr} A_N(x)\, f(x)\, dA(x). \tag{6.7}$$

The criterion for a hypersurface M of $\mathbb{R}^n$ to be minimal is hence that Tr $A_N = 0$ on M.

Recall from § 4 of Appendix C that $A_N(x)$ is a symmetric operator on $T_x M$. Its eigenvalues, which are all real, are called the principal curvatures of M at x. Various symmetric polynomials in these principal curvatures furnish quantities of interest. The mean curvature $H(x)$ of M at x is defined to be the mean value of these principal curvatures, that is,

$$H(x) = \frac{1}{k} \operatorname{Tr} A_N(x). \tag{6.8}$$

Thus a hypersurface $M \subset \mathbb{R}^n$ is a minimal submanifold of $\mathbb{R}^n$ precisely when $H = 0$ on M.

Note that changing the sign of N changes the sign of A_N, hence of H. Under such a sign change, the mean curvature vector

$$\mathfrak{H}(x) = H(x)N(x) \tag{6.9}$$

is invariant. In particular, this is well defined whether or not M is orientable, and its vanishing is the condition for M to be a minimal submanifold. There is the following useful formula for the mean curvature of a hypersurface $M \subset \mathbb{R}^n$. Let $X : M \hookrightarrow \mathbb{R}^n$ be the isometric imbedding. We claim that

$$\mathfrak{H}(x) = \frac{1}{k}\Delta X, \tag{6.10}$$

with $k = n-1$, where Δ is the Laplace operator on the Riemannian manifold M, acting componentwise on X. This is easy to see at a point $p \in M$ if we translate and rotate $\mathbb{R}^n$ to make $p = 0$ and represent M as the image of $\mathbb{R}^k = \mathbb{R}^{n-1}$ under

$$Y(x') = \big(x', f(x')\big), \quad x' = (x_1, \dots, x_k),\ \nabla f(0) = 0. \tag{6.11}$$

Then one verifies that

$$\Delta X(p) = \partial_1^2 Y(0) + \cdots + \partial_k^2 Y(0) = \big(0, \dots, 0, \partial_1^2 f(0) + \cdots + \partial_k^2 f(0)\big),$$

and (6.10) follows from the formula

$$\langle A_N(0)X, Y\rangle = \sum_{i,j=1}^{k} \partial_i \partial_j f(0)\ X_i Y_j \tag{6.12}$$

for the second fundamental form of M at p, derived in (4.19) of Appendix C.

More generally, if $M \subset \mathbb{R}^n$ has dimension $k \le n-1$, we can define the mean curvature vector $\mathfrak{H}(x)$ by

$$\langle \mathfrak{H}(x), \xi\rangle = \frac{1}{k} \operatorname{Tr} A_\xi(x), \quad \mathfrak{H}(x) \perp T_x M, \tag{6.13}$$

so the criterion for M to be a minimal submanifold is that $\mathfrak{H} = 0$. Furthermore, (6.10) continues to hold. This can be seen by the same type of argument used above; represent M as the image of $\mathbb{R}^k$ under (6.11), where now $f(x') = (x_{k+1}, \dots, x_n)$. Then (6.12) generalizes to

$$\langle A_\xi(0)X, Y\rangle = \sum_{i,j=1}^{k} \langle \xi, \partial_i \partial_j f(0)\rangle\ X_i Y_j, \tag{6.14}$$

which yields (6.10). We record this observation.

Proposition 6.1. *Let $X : M \to \mathbb{R}^n$ be an isometric immersion of a Riemannian manifold into $\mathbb{R}^n$. Then M is a minimal submanifold of $\mathbb{R}^n$ if and only if the coordinate functions $x_1, \dots, x_n$ are harmonic functions on M.*

A two-dimensional minimal submanifold of $\mathbb{R}^n$ is called a minimal surface. The theory is most developed in this case, and we will concentrate on the two-dimensional case in the material below.

When $\dim M = 2$, we can extend Proposition 6.1 to cases where $X : M \to \mathbb{R}^n$ is not an isometric map. This occurs because, in such a case, the class of harmonic functions on M is invariant under conformal changes of metric. In fact, if Δ is the Laplace operator for a Riemannian metric g_{ij} on M and Δ_1 that for

$g_{1ij} = e^{2u} g_{ij}$, then, since $\Delta f = g^{-1/2}\,\partial_i(g^{ij} g^{1/2}\,\partial_j f)$ and $g_1^{ij} = e^{-2u} g^{ij}$, while $g_1^{1/2} = e^{ku} g^{1/2}$ (if $\dim M = k$), we have

$$\Delta_1 f = e^{-2u}\,\Delta f + e^{-ku}\langle df, de^{(k-2)u}\rangle = e^{-2u}\Delta f \quad \text{if } k = 2. \tag{6.15}$$

Hence $\ker \Delta = \ker \Delta_1$ if $k = 2$. We hence have the following:

Proposition 6.2. *If Ω is a Riemannian manifold of dimension 2 and $X : \Omega \to \mathbb{R}^n$ a smooth immersion, with image M, then M is a minimal surface provided X is harmonic and $X : \Omega \to M$ is conformal.*

In fact, granted that $X : \Omega \to M$ is conformal, M is minimal if and only if X is harmonic on Ω.

We can use this result to produce lots of examples of minimal surfaces, by the following classical device. Take Ω to be an open set in $\mathbb{R}^2 = \mathbb{C}$, with coordinates (u_1, u_2). Given a map $X : \Omega \to \mathbb{R}^n$, with components $x_j : \Omega \to \mathbb{R}$, form the complex-valued functions

$$\psi_j(\zeta) = \frac{\partial x_j}{\partial u_1} - i\frac{\partial x_j}{\partial u_2} = 2\frac{\partial}{\partial \zeta} x_j, \quad \zeta = u_1 + iu_2. \tag{6.16}$$

Clearly, ψ_j is holomorphic if and only if x_j is harmonic (for the standard flat metric on Ω), since $\Delta = 4(\partial/\partial\overline{\zeta})(\partial/\partial\zeta)$. Furthermore, a short calculation gives

$$\sum_{j=1}^{n} \psi_j(\zeta)^2 = \left|\partial_1 X\right|^2 - \left|\partial_2 X\right|^2 - 2i\;\partial_1 X \cdot \partial_2 X. \tag{6.17}$$

Granted that $X : \Omega \to \mathbb{R}^n$ is an immersion, the criterion that it be conformal is precisely that this quantity vanish. We have the following result.

Proposition 6.3. *If $\psi_1, \dots, \psi_n$ are holomorphic functions on $\Omega \subset \mathbb{C}$ such that*

$$\sum_{j=1}^{n} \psi_j(\zeta)^2 = 0 \quad \text{on } \Omega, \tag{6.18}$$

while $\sum |\psi_j(\zeta)|^2 \neq 0$ on Ω, then setting

$$x_j(u) = Re \int \psi_j(\zeta)\, d\zeta \tag{6.19}$$

defines an immersion $X : \Omega \to \mathbb{R}^n$ whose image is a minimal surface.

If Ω is not simply connected, the domain of X is actually the universal covering surface of Ω.

We mention some particularly famous minimal surfaces in $\mathbb{R}^3$ that arise in such a fashion. Surely the premier candidate for (6.18) is

$$\sin^2\zeta + \cos^2\zeta - 1 = 0. \tag{6.20}$$

Here, take $\psi_1(\zeta) = \sin\zeta$, $\psi_2(\zeta) = -\cos\zeta$, and $\psi_3(\zeta) = -i$. Then (6.19) yields

$$x_1 = (\cos u_1)(\cosh u_2), \quad x_2 = (\sin u_1)(\cosh u_2), \quad x_3 = u_2. \tag{6.21}$$

The surface obtained in $\mathbb{R}^3$ is called the *catenoid*. It is the surface of revolution about the x_3-axis of the curve $x_1 = \cosh x_3$ in the $(x_1 - x_3)$-plane. Whenever $\psi_j(\zeta)$ are holomorphic functions satisfying (6.18), so are $e^{i\theta}\psi_j(\zeta)$, for any $\theta \in \mathbb{R}$. The resulting immersions $X_\theta : \Omega \to \mathbb{R}^n$ give rise to a family of minimal surfaces $M_\theta \subset \mathbb{R}^n$, which are said to be *associated*. In particular, $M_{\pi/2}$ is said to be *conjugate* to $M = M_0$. When M_0 is the catenoid, defined by (6.21), the conjugate minimal surface arises from $\psi_1(\zeta) = i\ \sin\zeta$, $\psi_2(\zeta) = -i\ \cos\zeta$, and $\psi_3(\zeta) = 1$ and is given by

$$x_1 = (\sin u_1)(\sinh u_2), \quad x_2 = (\cos u_1)(\sinh u_2), \quad x_3 = u_1. \tag{6.22}$$

This surface is called the *helicoid*. We mention that associated minimal surfaces are locally isometric but generally not congruent; that is, the isometry between the surfaces does not extend to a rigid motion of the ambient Euclidean space.

The catenoid and helicoid were given as examples of minimal surfaces by Meusnier, in 1776.

One systematic way to produce triples of holomorphic functions $\psi_j(\zeta)$ satisfying (6.18) is to take

$$\psi_1 = \frac{1}{2}f(1-g^2), \quad \psi_2 = \frac{i}{2}f(1+g^2), \quad \psi_3 = fg, \tag{6.23}$$

for arbitrary holomorphic functions f and g on Ω. More generally, g can be meromorphic on Ω as long as f has a zero of order $2m$ at each point where g has a pole of order m. The resulting map $X : \Omega \to M \subset \mathbb{R}^3$ is called the Weierstrass–Enneper representation of the minimal surface M. It has an interesting connection with the Gauss map of M, which will be sketched in the exercises. The example arising from $f = 1, g = \zeta$ produces "Enneper's surface." This surface is immersed in $\mathbb{R}^3$ but not imbedded.

For a long time the only known examples of complete imbedded minimal surfaces in $\mathbb{R}^3$ of finite topological type were the plane, the catenoid, and the helicoid, but in the 1980s it was proved by [HM1] that the surface obtained by taking $g = \zeta$ and $f(\zeta) = \wp(\zeta)$ (the Weierstrass $\wp$-function) is another example. Further examples have been found; computer graphics have been a valuable aid in this search; see [HM2].

A natural question is how general is the class of minimal surfaces arising from the construction in Proposition 6.3. In fact, it is easy to see that every minimal $M \subset \mathbb{R}^n$ is at least locally representable in such a fashion, using the existence of local isothermal coordinates, established in § 10 of Chap. 5. Thus any $p \in M$ has a neighborhood $\mathcal{O}$ such that there is a conformal diffeomorphism $X : \Omega \to \mathcal{O}$, for some open set $\Omega \subset \mathbb{R}^2$. By Proposition 6.2 and the remark following it, if M is minimal, then X must be harmonic, so (6.16) furnishes the functions $\psi_j(\zeta)$ used in Proposition 6.3. Incidentally, this shows that any minimal surface in $\mathbb{R}^n$ is real analytic.

As for the question of whether the construction of Proposition 6.3 globally represents every minimal surface, the answer here is also "yes." A proof uses the fact that every noncompact Riemann surface (without boundary) is covered by either $\mathbb{C}$ or the unit disk in $\mathbb{C}$. This is a more complete version of the uniformization theorem than the one we established in § 2 of this chapter. A positive answer, for simply connected, compact minimal surfaces, with smooth boundary, is implied by the following result, which will also be useful for an attack on the Plateau problem.

Proposition 6.4. *If $\overline{M}$ is a compact, connected, simply connected Riemannian manifold of dimension 2, with nonempty, smooth boundary, then there exists a conformal diffeomorphism*

$$\Phi : \overline{M} \longrightarrow \overline{D}, \tag{6.24}$$

where $\overline{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

This is a slight generalization of the Riemann mapping theorem, established in § 4 of Chap. 5, and it has a proof along the lines of the argument given there. Thus, fix $p \in M$, and let $G \in \mathcal{D}'(M) \cap C^\infty(\overline{M} \setminus p)$ be the unique solution to

$$\Delta G = 2\pi\delta, \quad G = 0 \text{ on } \partial M. \tag{6.25}$$

Since M is simply connected, it is orientable, so we can pick a Hodge star operator, and $*dG = \beta$ is a smooth closed 1-form on $\overline{M} \setminus p$. If γ is a curve in M of degree 1 about p, then $\int_\gamma \beta$ can be calculated by deforming γ to be a small curve about p. The parametrix construction for the solution to (6.25), in normal coordinates centered at p, gives $G(x) \sim \log \operatorname{dist}(x, p)$, and one establishes that $\int_\gamma \beta = 2\pi$. Thus we can write $\beta = dH$, where H is a smooth function on $\overline{M} \setminus p$, well defined mod $2\pi\mathbb{Z}$. Hence $\Phi(x) = e^{G+iH}$ is a single-valued function, tending to 0 as $x \to p$, which one verifies to be the desired conformal diffeomorphism (6.24), by the same reasoning as used to complete the proof of Theorem 4.1 in Chap. 5.

An immediate corollary is that the argument given above for the local representation of a minimal surface in the form (6.19) extends to a global representation of a compact, simply connected minimal surface, with smooth boundary.

So far we have dealt with smooth surfaces, at least immersed in $\mathbb{R}^n$. The theorem of J. Douglas and T. Rado that we now tackle deals with "generalized" surfaces, which we will simply define to be the images of two-dimensional manifolds under smooth maps into $\mathbb{R}^n$ (or some other manifold). The theorem, a partial answer to the "Plateau problem," asserts the existence of an area-minimizing generalized surface whose boundary is a given simple, closed curve in $\mathbb{R}^n$.

To be precise, let γ be a smooth, simple, closed curve in $\mathbb{R}^n$, that is, a diffeomorphic image of S^1. Let

$$\begin{aligned} \mathfrak{X}_\gamma = \{\varphi \in C(\overline{D}, \mathbb{R}^n) \cap C^\infty(D, \mathbb{R}^n) :& \\ \varphi : S^1 \to \gamma \text{ monotone, and } \alpha(\varphi) < \infty\},& \end{aligned} \tag{6.26}$$

where α is the area functional:

$$\alpha(\varphi) = \int_D |\partial_1 \varphi \wedge \partial_2 \varphi| \, dx_1 dx_2. \tag{6.27}$$

Then let

$$\mathcal{A}_\gamma = \inf\{\alpha(\varphi) : \varphi \in \mathfrak{X}_\gamma\}. \tag{6.28}$$

The existence theorem of Douglas and Rado is the following:

Theorem 6.5. *There is a map $\varphi \in \mathfrak{X}_\gamma$ such that $\alpha(\varphi) = \mathcal{A}_\gamma$.*

We can choose $\varphi_\nu \in \mathfrak{X}_\gamma$ such that $\alpha(\varphi_\nu) \searrow \mathcal{A}_\gamma$, but $\{\varphi_\nu\}$ could hardly be expected to have a convergent subsequence unless some structure is imposed on the maps φ_ν. The reason is that $\alpha(\varphi) = \alpha(\varphi \circ \psi)$ for any C^∞-diffeomorphism $\psi : \overline{D} \to \overline{D}$. We say $\varphi \circ \psi$ is a *reparameterization* of φ. The key to success is to take φ_ν, which approximately minimize not only the area functional $\alpha(\varphi)$ but also the energy functional

$$\vartheta(\varphi) = \int_D |\nabla \varphi(x)|^2 \, dx_1 dx_2, \tag{6.29}$$

so that we will also have $\vartheta(\varphi_\nu) \searrow d_\gamma$, where

$$d_\gamma = \inf\{\vartheta(\varphi) : \varphi \in \mathfrak{X}_\gamma\}. \tag{6.30}$$

To relate these, we compare (6.29) and the area functional (6.27).

To compare integrands, we have

$$|\nabla \varphi|^2 = |\partial_1 \varphi|^2 + |\partial_2 \varphi|^2, \tag{6.31}$$

while the square of the integrand in (6.27) is equal to

$$
\begin{aligned}
|\partial_1\varphi \wedge \partial_2\varphi|^2 &= |\partial_1\varphi|^2|\partial_2\varphi|^2 - \langle \partial_1\varphi, \partial_2\varphi\rangle \\
&\le |\partial_1\varphi|^2|\partial_2\varphi|^2 \\
&\le \frac{1}{4}\big(|\partial_1\varphi|^2 + |\partial_2\varphi|^2\big)^2,
\end{aligned}
\tag{6.32}
$$

where equality holds if and only if

$$
|\partial_1\varphi| = |\partial_2\varphi| \ \text{ and } \ \langle \partial_1\varphi, \partial_2\varphi\rangle = 0. \tag{6.33}
$$

Whenever $\nabla\varphi \neq 0$, this is the condition that φ be conformal. More generally, if (6.33) holds, but we allow $\nabla\varphi(x) = 0$, we say that φ is essentially conformal. Thus, we have seen that, for each $\varphi \in \mathfrak{X}_\gamma$,

$$
\alpha(\varphi) \le \frac{1}{2}\vartheta(\varphi), \tag{6.34}
$$

with equality if and only if φ is essentially conformal. The following result allows us to transform the problem of minimizing $\alpha(\varphi)$ over $\mathfrak{X}_\gamma$ into that of minimizing $\vartheta(\varphi)$ over $\mathfrak{X}_\gamma$, which will be an important tool in the proof of Theorem 6.5. Set

$$
\mathfrak{X}_\gamma^\infty = \{\varphi \in C^\infty(\overline{D}, \mathbb{R}^n) : \varphi : S^1 \to \gamma \text{ diffeo.}\}. \tag{6.35}
$$

Proposition 6.6. *Given $\varepsilon > 0$, any $\varphi \in \mathfrak{X}_\gamma^\infty$ has a reparameterization $\varphi \circ \psi$ such that*

$$
\frac{1}{2}\vartheta(\varphi \circ \psi) \le \alpha(\varphi) + \varepsilon. \tag{6.36}
$$

Proof. We will obtain this from Proposition 6.4, but that result may not apply to $\varphi(\overline{D})$, so we do the following. Take $\delta > 0$ and define $\Phi_\delta : \overline{D} \to \mathbb{R}^{n+2}$ by $\Phi_\delta(x) = \big(\varphi(x), \delta x\big)$. For any $\delta > 0$, Φ_δ is a diffeomorphism of $\overline{D}$ onto its image, and if δ is very small, area $\Phi_\delta(\overline{D})$ is only a little larger than area $\varphi(D)$. Now, by Proposition 6.4, there is a conformal diffeomorphism $\Psi : \Phi_\delta(\overline{D}) \to \overline{D}$. Set $\psi = \psi_\delta = \big(\Psi \circ \Phi_\delta\big)^{-1} : \overline{D} \to \overline{D}$. Then $\Phi_\delta \circ \psi = \Psi^{-1}$ and, as established above, $(1/2)\vartheta(\Psi^{-1}) = \operatorname{Area}(\Psi^{-1}(\overline{D}))$, i.e.,

$$
\tfrac{1}{2}\vartheta(\Phi_\delta \circ \psi) = \operatorname{Area}\big(\Phi_\delta(\overline{D})\big). \tag{6.37}
$$

Since $\vartheta(\varphi \circ \psi) \le \vartheta(\Phi_\delta \circ \psi)$, the result (6.34) follows if δ is taken small enough.

One can show that

$$
\mathcal{A}_\gamma = \inf\{\alpha(\varphi) : \varphi \in \mathfrak{X}_\gamma^\infty\}, \quad d_\gamma = \inf\{\vartheta(\varphi) : \varphi \in \mathfrak{X}_\gamma^\infty\}. \tag{6.38}
$$

It then follows from Proposition 6.6 that $\mathcal{A}_\gamma = (1/2)d_\gamma$, and if $\varphi_\nu \in \mathfrak{X}_\gamma^\infty$ is chosen so that $\vartheta(\varphi_\nu) \to d_\gamma$, then a fortiori $\alpha(\varphi_\nu) \to \mathcal{A}_\gamma$.

There is still an obstacle to obtaining a convergent subsequence of such $\{\varphi_\nu\}$. Namely, the energy integral (6.29) is invariant under reparameterizations $\varphi \mapsto \varphi \circ \psi$ for which $\psi : \overline{D} \to \overline{D}$ is a conformal diffeomorphism. We can put a clamp on this by noting that, given any two triples of (distinct) points $\{p_1, p_2, p_2\}$ and $\{q_1, q_2, q_3\}$ in $S^1 = \partial D$, there is a unique conformal diffeomorphism $\psi : \overline{D} \to \overline{D}$ such that $\psi(p_j) = q_j, 1 \le j \le 3$. Let us now make one choice of $\{p_j\}$ on S^1—for example, the three cube roots of 1—and make one choice of a triple $\{q_j\}$ of distinct points in γ. The following key compactness result will enable us to prove Theorem 6.5.

Proposition 6.7. *For any $d \in (d_\gamma, \infty)$, the set*

$$\Sigma_d = \left\{\varphi \in \mathfrak{X}_\gamma^\infty : \varphi \text{ harmonic}, \varphi(p_j) = q_j, \text{ and } \vartheta(\varphi) \le d\right\} \tag{6.39}$$

is relatively compact in $C(\overline{D}, \mathbb{R}^n)$.

In view of the mapping properties of the Poisson integral, this result is equivalent to the relative compactness in $C(\partial D, \gamma)$ of

$$\mathcal{S}_K = \{u \in C^\infty(S^1, \gamma) \text{ diffeo.} : u(p_j) = q_j, \text{ and } \|u\|_{H^{1/2}(S^1)} \le K\}, \tag{6.40}$$

for any given $K < \infty$. For $u \in \mathcal{S}_K$, we have $\|u\|_{H^{1/2}(S^1)} \approx \|\mathrm{PI}\, u\|_{H^1(D)}$. To demonstrate this compactness, there is no loss of generality in taking $\gamma = S^1 \subset \mathbb{R}^2$ and $p_j = q_j$.

We will show that the oscillation of u over any arc $I \subset S^1$ of length 2δ is $\le CK/\sqrt{\log(1/\delta)}$. This modulus of continuity will imply the compactness, by Ascoli's theorem.

Pick a point $z \in S^1$. Let C_r denote the portion of the circle of radius r and center z which lies in $\overline{D}$. Thus C_r is an arc, of length $\le \pi r$. Let $\delta \in (0, 1)$. As r varies from δ to $\sqrt{\delta}$, C_r sweeps out part of an annulus, as illustrated in Fig. 6.1.

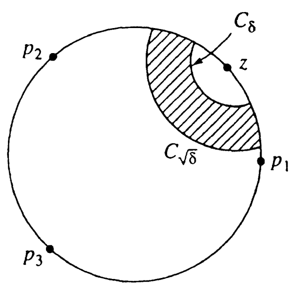

FIGURE 6.1 Annular Region in the Disk

We claim there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$\int_{C_\rho} |\nabla\varphi|\, ds \le K \sqrt{\frac{2\pi}{\log \frac{1}{\delta}}} \tag{6.41}$$

if $K = \|\nabla\varphi\|_{L^2(D)},\ \varphi = \mathrm{PI}\, u$. To establish this, let

$$\omega(r) = r \int_{C_r} |\nabla\varphi|^2\, ds.$$

Then

$$\int_\delta^{\sqrt{\delta}} \omega(r)\, \frac{dr}{r} = \int_\delta^{\sqrt{\delta}} \int_{C_r} |\nabla\varphi|^2\, ds\, dr = I \le K^2.$$

By the mean-value theorem, there exists $\rho \in [\delta, \sqrt{\delta}]$ such that

$$I = \omega(\rho) \int_\delta^{\sqrt{\delta}} \frac{dr}{r} = \frac{\omega(\rho)}{2} \log \frac{1}{\delta}.$$

For this value of ρ, we have

$$\rho \int_{C_\rho} |\nabla\varphi|^2\, ds = \frac{2I}{\log \frac{1}{\delta}} \le \frac{2K^2}{\log \frac{1}{\delta}}. \tag{6.42}$$

Then Cauchy's inequality yields (6.41), since length$(C_\rho) \le \pi\rho$.

This almost gives the desired modulus of continuity. The arc C_ρ is mapped by φ into a curve of length $\le K\sqrt{2\pi/\log(1/\delta)}$, whose endpoints divide γ into two segments, one rather short (if δ is small) and one not so short. There are two possibilities: $\varphi(z)$ is contained in either the short segment (as in Fig. 6.2) or the long segment (as in Fig. 6.3). However, as long as $\varphi(p_j) = p_j$ for three points p_j, this latter possibility cannot occur. We see that

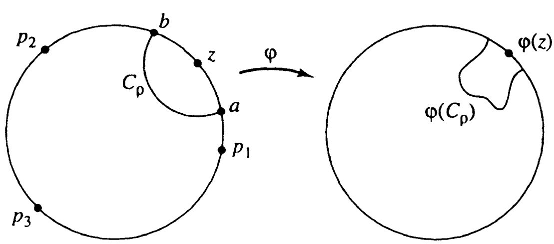

FIGURE 6.2 Mapping of an Arc

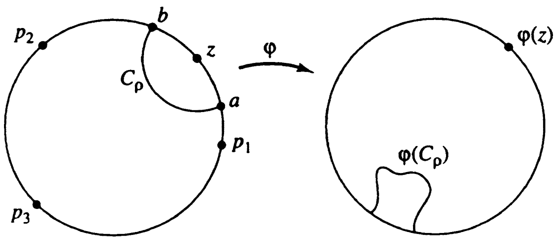

FIGURE 6.3 Alternative Mapping of an Arc

$$|u(a) - u(b)| \le K \sqrt{\frac{2\pi}{\log \frac{1}{\delta}}},$$

if a and b are the points where C_ρ intersects S^1. Now the monotonicity of u along S^1 guarantees that the total variation of u on the (small) arc from a to b in S^1 is $\le K\sqrt{2\pi / \log(1/\delta)}$. This establishes the modulus of continuity and concludes the proof.

Now that we have Proposition 6.7, we proceed as follows. Pick a sequence φ_ν in $\mathfrak{X}_\gamma^\infty$ such that $\vartheta(\varphi_\nu) \to d_\gamma$, so also $\alpha(\varphi_\nu) \to \mathcal{A}_\gamma$. Now we do not increase $\vartheta(\varphi_\nu)$ if we replace φ_ν by the Poisson integral of $\varphi_\nu\big|_{\partial D}$, and we do not alter this energy integral if we reparameterize via a conformal diffeomorphism to take $\{p_j\}$ to $\{q_j\}$. Thus we may as well suppose that $\varphi_\nu \in \Sigma_d$. Using Proposition 6.7 and passing to a subsequence, we can assume

$$\varphi_\nu \longrightarrow \varphi \quad \text{in } C(\overline{D}, \mathbb{R}^n), \tag{6.43}$$

and we can furthermore arrange

$$\varphi_\nu \longrightarrow \varphi \quad \text{weakly in } H^1(D, \mathbb{R}^n). \tag{6.44}$$

Of course, by interior estimates for harmonic functions, we have

$$\varphi_\nu \longrightarrow \varphi \quad \text{in } C^\infty(D, \mathbb{R}^n). \tag{6.45}$$

The limit function φ is certainly harmonic on D. By (6.44), we of course have

$$\vartheta(\varphi) \le \underline{\lim}_{\nu\to\infty} \vartheta(\varphi_\nu) = d_\gamma. \tag{6.46}$$

Now (6.34) applies to φ, so we have

$$\alpha(\varphi) \le \frac{1}{2}\vartheta(\varphi) \le \frac{1}{2} d_\gamma = \mathcal{A}_\gamma. \tag{6.47}$$

On the other hand, (6.43) implies that $\varphi : \partial D \to \gamma$ is monotone. Thus φ belongs to $\mathfrak{X}_\gamma$. Hence we have

$$\alpha(\varphi) = \mathcal{A}_\gamma. \tag{6.48}$$

This proves Theorem 6.5 and most of the following more precise result.

Theorem 6.8. *If γ is a smooth, simple, closed curve in $\mathbb{R}^n$, there exists a continuous map $\varphi : \overline{D} \to \mathbb{R}^n$ such that*

$$\vartheta(\varphi) = d_\gamma \ \text{ and } \ \alpha(\varphi) = \mathcal{A}_\gamma, \tag{6.49}$$

$$\varphi : D \longrightarrow \mathbb{R}^n \text{ is harmonic and essentially conformal}, \tag{6.50}$$

$$\varphi : S^1 \longrightarrow \gamma, \ \text{ homeomorphically.} \tag{6.51}$$

Proof. We have (6.49) from (6.46)–(6.48). By the argument involving (6.31) and (6.32), this forces φ to be essentially conformal. It remains to demonstrate (6.51).

We know that $\varphi : S^1 \to \gamma$, monotonically. If it fails to be a homeomorphism, there must be an interval $I \subset S^1$ on which φ is constant. Using a linear fractional transformation to map D conformally onto the upper half-plane $\Omega^+ \subset \mathbb{C}$, we can regard φ as a harmonic and essentially conformal map of $\Omega^+ \to \mathbb{R}^n$, constant on an interval I on the real axis $\mathbb{R}$. Via the Schwartz reflection principle, we can extend φ to a harmonic function

$$\varphi : \mathbb{C} \setminus (\mathbb{R} \setminus I) \longrightarrow \mathbb{R}^n.$$

Now consider the holomorphic function $\psi : \mathbb{C} \setminus (\mathbb{R} \setminus I) \to \mathbb{C}^n$, given by $\psi(\zeta) = \partial\varphi/\partial\zeta$. As in the calculations leading to Proposition 6.3, the identities

$$|\partial_1\varphi|^2 - |\partial_2\varphi|^2 = 0, \quad \partial_1\varphi \cdot \partial_2\varphi = 0, \tag{6.52}$$

which hold on Ω^+, imply $\sum_{j=1}^n \psi_j(\zeta)^2 = 0$ on Ω^+; hence this holds on $\mathbb{C} \setminus (\mathbb{R} \setminus I)$, and so does (6.52). But since $\partial_1\varphi = 0$ on I, we deduce that $\partial_2\varphi = 0$ on I, hence $\psi = 0$ on I, hence $\psi \equiv 0$. This implies that φ, being both $\mathbb{R}^n$-valued and antiholomorphic, must be constant, which is impossible. This contradiction establishes (6.51).

Theorem 6.8 furnishes a generalized minimal surface whose boundary is a given smooth, closed curve in $\mathbb{R}^n$. We know that φ is smooth on D. It has been shown by [Hild] that φ is C^∞ on $\overline{D}$ when the curve γ is C^∞, as we have assumed here. It should be mentioned that Douglas and others treated the Plateau problem for simple, closed curves γ that were not smooth. We have restricted attention to smooth γ for simplicity. A treatment of the general case can be found in [Nit1]; see also [Nit2].

There remains the question of the smoothness of the image surface $M = \varphi(D)$. The map $\varphi : D \to \mathbb{R}^n$ would fail to be an immersion at a point $z \in D$ where

$\nabla\varphi(z) = 0$. At such a point, the $\mathbb{C}^n$-valued holomorphic function $\psi = \partial\varphi/\partial\zeta$ must vanish; that is, each of its components must vanish. Since a holomorphic function on $D \subset \mathbb{C}$ that is not identically zero can vanish only on a discrete set, we have the following:

Proposition 6.9. *The map $\varphi : D \to \mathbb{R}^n$ parameterizing the generalized minimal surface in Theorem 6.8 has injective derivative except at a discrete set of points in D.*

If $\nabla\varphi(z) = 0$, then $\varphi(z) \in M = \varphi(D)$ is said to be a *branch point* of the generalized minimal surface M; we say M is a branched surface. If $n \geq 4$, there are indeed generalized minimal surfaces with branch points that arise via Theorem 6.8. Results of Osserman [Oss2], complemented by [Gul], show that if $n = 3$, the construction of Theorem 6.8 yields a smooth minimal surface, immersed in $\mathbb{R}^3$. Such a minimal surface need not be imbedded; for example, if γ is a knot in $\mathbb{R}^3$, such a surface with boundary equal to γ is certainly not imbedded. If γ is analytic, it is known that there cannot be branch points on the boundary, though this is open for merely smooth γ. An extensive discussion of boundary regularity is given in Vol. 2 of [DHKW].

The following result of Rado yields one simple criterion for a generalized minimal surface to have no branch points.

Proposition 6.10. *Let γ be a smooth, closed curve in $\mathbb{R}^n$. If a minimal surface with boundary γ produced by Theorem 6.8 has any branch points, then γ has the property that*

$$\text{for some } p \in \mathbb{R}^n, \text{ every hyperplane through } p \text{ intersects } \gamma \text{ in at least four points.} \tag{6.53}$$

Proof. Suppose $z_0 \in D$ and $\nabla\varphi(z_0) = 0$, so $\psi = \partial\varphi/\partial\zeta$ vanishes at z_0. Let $L(x) = \alpha \cdot x + c = 0$ be the equation of an arbitrary hyperplane through $p = \varphi(z_0)$. Then $h(x) = L\big(\varphi(x)\big)$ is a (real-valued) harmonic function on D, satisfying

$$\Delta h = 0 \text{ on } D, \quad \nabla h(z_0) = 0. \tag{6.54}$$

The proposition is then proved, by the following:

Lemma 6.11. *Any real-valued $h \in C^\infty(D) \cap C(\overline{D})$ having the property (6.54) must assume the value $h(z_0)$ on at least four points on ∂D.*

We leave the proof as an exercise for the reader.

The following result gives a condition under which a minimal surface constructed by Theorem 6.8 is the graph of a function.

Proposition 6.12. *Let $\mathcal{O}$ be a bounded convex domain in $\mathbb{R}^2$ with smooth boundary. Let $g : \partial\mathcal{O} \to \mathbb{R}^{n-2}$ be smooth. Then there exists a function*

$$f \in C^\infty(\mathcal{O}, \mathbb{R}^{n-2}) \cap C(\overline{\mathcal{O}}, \mathbb{R}^{n-2}), \tag{6.55}$$

whose graph is a minimal surface, and whose boundary is the curve $\gamma \subset \mathbb{R}^n$ that is the graph of g, so

$$f = g \quad on\ \partial\mathcal{O}. \tag{6.56}$$

Proof. Let $\varphi : \overline{D} \to \mathbb{R}^n$ be the function constructed in Theorem 6.8. Set $F(x) = \big(\varphi_1(x), \varphi_2(x)\big)$. Then $F : \overline{D} \to \mathbb{R}^2$ is harmonic on D and F maps $S^1 = \partial D$ homeomorphically onto $\partial\mathcal{O}$. It follows from the convexity of $\mathcal{O}$ and the maximum principle for harmonic functions that $F : \overline{D} \to \overline{\mathcal{O}}$.

We claim that $DF(x)$ is invertible for each $x \in D$. Indeed, if $x_0 \in D$ and $DF(x_0)$ is singular, we can choose nonzero $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that, at $x = x_0$,

$$\alpha_1 \frac{\partial \varphi_1}{\partial x_j} + \alpha_2 \frac{\partial \varphi_2}{\partial x_j} = 0, \quad j = 1, 2.$$

Then the function $h(x) = \alpha_1\varphi_1(x) + \alpha_2\varphi_2(x)$ has the property (6.54), so $h(x)$ must take the value $h(x_0)$ at four distinct points of ∂D. Since $F : \partial D \to \partial\mathcal{O}$ is a homeomorphism, this forces the *linear* function $\alpha_1 x_1 + \alpha_2 x_2$ to take the same value at four distinct points of $\partial\mathcal{O}$, which contradicts the convexity of $\mathcal{O}$.

Thus $F : D \to \mathcal{O}$ is a local diffeomorphism. Since F gives a homeomorphism of the boundaries of these regions, degree theory implies that F is a diffeomorphism of D onto $\mathcal{O}$ and a homeomorphism of $\overline{D}$ onto $\overline{\mathcal{O}}$. Consequently, the desired function in (6.55) is $f = \widetilde{\varphi} \circ F^{-1}$, where $\widetilde{\varphi}(x) = \big(\varphi_2(x), \dots, \varphi_n(x)\big)$.

Functions whose graphs are minimal surfaces satisfy a certain nonlinear PDE, called the *minimal surface equation*, which we will derive and study in § 7.

Let us mention that while one ingredient in the solution to the Plateau problem presented above is a version of the Riemann mapping theorem, Proposition 6.4, there are presentations for which the Riemann mapping theorem is a consequence of the argument, rather than an ingredient (see, e.g., [Nit2]).

It is also of interest to consider the analogue of the Plateau problem when, instead of immersing the disk in $\mathbb{R}^n$ as a minimal surface with given boundary, one takes a surface of higher genus, and perhaps several boundary components. An extra complication is that Proposition 6.4 must be replaced by something more elaborate, since two compact surfaces with boundary which are diffeomorphic to each other but not to the disk may not be conformally equivalent. One needs to consider spaces of "moduli" of such surfaces; Theorem 4.2 of Chap. 5 deals with the easiest case after the disk. This problem was tackled by Douglas [Dou2] and by Courant [Cou2], but their work has been criticized by [ToT] and [Jos], who present alternative solutions. The paper [Jos] also treats the Plateau problem for surfaces in Riemannian manifolds, extending results of [Mor1].

There have been successful attacks on problems in the theory of minimal submanifolds, particularly in higher dimension, using very different techniques, involving geometric measure theory, currents, and varifolds. Material on these important developments can be found in [Alm, Fed, Morg].

So far in this section, we have devoted all our attention to minimal submanifolds of Euclidean space. It is also interesting to consider minimal submanifolds of other Riemannian manifolds. We make a few brief comments on this topic. A great deal more can be found in [Cher, Law, Law2, Mor1, Pi] and in survey articles in [Bom].

Let Y be a smooth, compact Riemannian manifold. Assume Y is isometrically imbedded in $\mathbb{R}^n$, which can always be arranged, by Nash's theorem. Let M be a compact, k-dimensional submanifold of Y. We say M is a minimal submanifold of Y if its k-dimensional volume is a critical point with respect to small variations of M, within Y. The computations in (6.1)–(6.13) extend to this case. We need to take $X = X(s,u)$ with $\partial_s X(s,u) = \xi(s,u)$, tangent to Y, rather than $X(s,u) = X_0(u) + s\xi(u)$. Then these computations show that M is a minimal submanifold of Y if and only if, for each $x \in M$,

$$\mathfrak{H}(x) \perp T_x Y, \tag{6.57}$$

where $\mathfrak{H}(x)$ is the mean curvature vector of M (as a submanifold of $\mathbb{R}^n$), defined by (6.13).

There is also a well-defined mean curvature vector $\mathfrak{H}_Y(x) \in T_x Y$, orthogonal to $T_x M$, obtained from the second fundamental form of M as a submanifold of Y. One sees that $\mathfrak{H}_Y(x)$ is the orthogonal projection of $\mathfrak{H}(x)$ onto $T_x Y$, so the condition that M be a minimal submanifold of Y is that $\mathfrak{H}_Y = 0$ on M.

The formula (6.10) continues to hold for the isometric imbedding $X : M \to \mathbb{R}^n$. Thus M is a minimal submanifold of Y if and only if, for each $x \in M$,

$$\Delta X(x) \perp T_x Y. \tag{6.58}$$

If $\dim M = 2$, the formula (6.15) holds, so if M is given a new metric, conformally scaled by a factor e^{2u}, the new Laplace operator Δ_1 has the property that $\Delta_1 X = e^{-2u} \Delta X$, hence is parallel to ΔX. Thus the property (6.58) is unaffected by such a conformal change of metric; we have the following extension of Proposition 6.2:

Proposition 6.13. *If M is a Riemannian manifold of dimension 2 and $X : M \to \mathbb{R}^n$ is a smooth imbedding, with image $M_1 \subset Y$, then M_1 is a minimal submanifold of Y provided $X : M \to M_1$ is conformal and, for each $x \in M$,*

$$\Delta X(x) \perp T_{X(x)} Y. \tag{6.59}$$

We note that (6.59) alone specifies that X is a *harmonic* map from M into Y. Harmonic maps will be considered further in §§ 11 and 12B; they will also be studied, via parabolic PDE, in Chap. 15, § 2.

Exercises

1. Consider the Gauss map $N : M \to S^2$, for a smooth, oriented surface $M \subset \mathbb{R}^3$. Show that N is *antiholomorphic* if and only if M is a minimal surface.
(*Hint*: If $N(p) = q$, $DN(p) : T_pM \to T_qS^2 \approx T_pM$ is identified with $-A_N$. Compare (4.67) in Appendix C. Check when $A_N J = -JA_N$, where J is counterclockwise rotation by $90°$, on T_pM.) Thus, if we define the antipodal Gauss map $\widetilde{N} : M \to S^2$ by $\widetilde{N}(p) = -N(p)$, this map is holomorphic precisely when M is a minimal surface.
2. If $x \in S^2 \subset \mathbb{R}^3$, pick $v \in T_xS^2 \subset \mathbb{R}^3$, set $w = Jv \in T_xS^2 \subset \mathbb{R}^3$, and take $\xi = v + iw \in \mathbb{C}^3$. Show that the one-dimensional, complex span of ξ is independent of the choice of v, and that we hence have a holomorphic map
$$\Xi : S^2 \longrightarrow \mathbb{CP}^3.$$
Show that the image $\Xi(S^2) \subset \mathbb{CP}^3$ is contained in the image of $\{\zeta \in \mathbb{C}^3 \setminus 0 : \zeta_1^2 + \zeta_2^2 + \zeta_3^3 = 0\}$ under the natural map $\mathbb{C}^3 \setminus 0 \to \mathbb{CP}^3$.
3. Suppose that $M \subset \mathbb{R}^3$ is a minimal surface constructed by the method of Proposition 6.3, via $X : \Omega \to M \subset \mathbb{R}^3$. Define $\Psi : \Omega \to \mathbb{C}^3 \setminus 0$ by $\Psi = (\psi_1, \psi_2, \psi_3)$, and define $\mathfrak{X} : \Omega \to \mathbb{CP}^3$ by composing Ψ with the natural map $\mathbb{C}^3 \setminus 0 \to \mathbb{CP}^3$. Show that, for $u \in \Omega$,
$$\mathfrak{X}(u) = \Xi \circ \widetilde{N}\big(X(u)\big).$$
For the relation between ψ_j and the Gauss map for minimal surfaces in $\mathbb{R}^n$, $n > 3$, see [Law].
4. Give a detailed demonstration of (6.38).
5. In analogy with Proposition 6.4, extend Theorem 4.3 of Chap. 5 to the following result:

Proposition. *If $\overline{M}$ is a compact Riemannian manifold of dimension 2 which is homeomorphic to an annulus, then there exists a conformal diffeomorphism*
$$\Psi : \overline{M} \longrightarrow \overline{\mathfrak{A}}_\rho,$$
for a unique $\rho \in (0, 1)$, where $\overline{\mathfrak{A}}_\rho = \{z \in \mathbb{C} : \rho \le |z| \le 1\}$.

6. If $\widetilde{II}$ is the second fundamental form of a minimal hypersurface $M \subset \mathbb{R}^n$, show that $\widetilde{II}$ has divergence zero. As in Chap. 2, § 3, we define the divergence of a second-order tensor field T by $T^{jk}{}_{;k}$. (*Hint*: Use the Codazzi equation (cf. Appendix C, § 4, especially (4.18)) plus the zero trace condition.)
7. Similarly, if $\widetilde{II}$ is the second fundamental form of a minimal submanifold M of codimension 1 in S^n (with its standard metric), show that $\widetilde{II}$ has divergence zero.
(*Hint*: The Codazzi equation, from (4.16) of Appendix C, is
$$(\nabla_Y \widetilde{II})(X, Z) - (\nabla_Y \widetilde{II})(Y, Z) = \langle R(X, Y)Z, N\rangle,$$
where ∇ is the Levi–Civita connection on M; X, Y, Z are tangent to M; Z is normal to M (but tangent to S^n); and R is the curvature tensor of S^n. In such a case, the right side vanishes. (See Exercise 6 in § 4 of Appendix C.) Thus the argument needed for Exercise 6 above extends.)
8. Extend the result of Exercises 6–7 to the case where M is a codimension-1 minimal submanifold in any Riemannian manifold Ω with constant sectional curvature.

9. Let M be a two-dimensional minimal submanifold of S^3, with its standard metric. Assume M is diffeomorphic to S^2. Show that M must be a "great sphere" in S^3. (*Hint*: By Exercise 7, $\widetilde{II}$ is a symmetric trace free tensor of divergence zero; that is, $\widetilde{II}$ belongs to
$$\mathcal{V} = \{u \in C^\infty(M, S_0^2 T^*) : \operatorname{div} u = 0\},$$
a space introduced in (10.47) of Chap. 10. As noted there, when M is a Riemann surface, $\mathcal{V} \approx \mathcal{O}(\kappa \otimes \kappa)$. By Corollary 9.4 of Chap. 10, $\mathcal{O}(\kappa \otimes \kappa) = 0$ when M has genus $g = 0$.)
10. Prove Lemma 6.11.

6B. Second variation of area

In this appendix to § 6, we take up a computation of the second variation of the area integral, and some implications, for a family of manifolds of dimension k, immersed in a Riemannian manifold Y. First, we take $Y = \mathbb{R}^n$ and suppose the family is given by $X(s,u) = X_0(u) + s\xi(u)$, as in (6.1)–(6.5).

Suppose, as in the computation (6.2)–(6.5), that $\|\partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\| = 1$ on M, while $E_j = \partial_j X_0$ form an orthonormal basis of $T_x M$, for a given point $x \in M$. Then, extending (6.3), we have

$$A'(s) = \sum_{j=1}^k \int \frac{\langle \partial_1 X \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X, \partial_1 X \wedge \cdots \wedge \partial_k X\rangle}{\|\partial_1 X \wedge \cdots \wedge \partial_k X\|}\, du_1 \cdots du_k. \tag{6b.1}$$

Consequently, $A''(0)$ will be the integral with respect to $du_1 \cdots du_k$ of a sum of three terms:

$$\begin{aligned}
&-\sum_{i,j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\rangle \\
&\qquad\qquad \times \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\rangle \\
&+ 2\sum_{i<j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0\rangle \\
&+ \sum_{i,j} \langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \xi \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_i \xi \wedge \cdots \wedge \partial_k X_0\rangle.
\end{aligned} \tag{6b.2}$$

Let us write

$$A_\xi E_i = \sum_\ell a_\xi^{i\ell} E_\ell, \tag{6b.3}$$

with $E_j = \partial_j X_0$ as before. Then, as in (6.4), the first sum in (6b.2) is equal to

$$-\sum_{i,j} a_\xi^{ii} a_\xi^{jj}. \tag{6b.4}$$

Let us move to the last sum in (6b.2). We use the Weingarten formula $\partial_j \xi = \nabla_j^1 \xi - A_\xi E_j$, to write this sum as

$$\sum_{i,j} a_\xi^{jj} a_\xi^{ii} + \sum_{i,j} \langle \nabla_j^1 \xi, \nabla_i^1 \xi \rangle, \tag{6b.5}$$

at x. Note that the first sum in (6b.5) cancels (6b.4), while the last sum in (6b.5) can be written as $\|\nabla^1 \xi\|^2$. Here, ∇^1 is the connection induced on the normal bundle of M.

Now we look at the middle term in (6b.2), namely,

$$2 \sum_{i<j} \sum_{\ell,m} a_\xi^{i\ell} a_\xi^{jm} \langle E_1 \wedge \cdots \wedge E_\ell \wedge \cdots \wedge E_m \wedge \cdots \wedge E_k, E_1 \wedge \cdots \wedge E_k \rangle, \tag{6b.6}$$

at x, where E_ℓ appears in the ith slot and E_m appears in the jth slot in the k-fold wedge product. This is equal to

$$2 \sum_{i<j} \left(a_\xi^{ii} a_\xi^{jj} - a_\xi^{ij} a_\xi^{ji} \right) = 2 \operatorname{Tr} \Lambda^2 A_\xi, \tag{6b.7}$$

at x. Thus we have

$$A''(0) = \int_M \left[\|\nabla^1 \xi\|^2 + 2 \operatorname{Tr} \Lambda^2 A_\xi \right] dA(x). \tag{6b.8}$$

If M is a hypersurface of $\mathbb{R}^n$, and we take $\xi = fN$, where N is a unit normal field, then $\|\nabla^1 \xi\|^2 = \|\nabla f\|^2$ and (6b.7) is equal to

$$2 \sum_{i<j} \langle R(E_j, E_i) E_i, E_j \rangle f^2 = S f^2, \tag{6b.9}$$

by the *Theorema Egregium*, where S is the scalar curvature of M. Consequently, if $M \subset \mathbb{R}^n$ is a hypersurface (with boundary), and the hypersurfaces M_s are given by (6.6), with area integral (6.2), then

$$A''(0) = \int_M \left[\|\nabla f\|^2 + S(x) f^2 \right] dA(x). \tag{6b.10}$$

Recall that when $\dim M = 2$, so $M \subset \mathbb{R}^3$,

$$S = 2K, \tag{6b.11}$$

where K is the Gauss curvature, which is ≤ 0 whenever M is a minimal surface in $\mathbb{R}^3$.

If M has general codimension in $\mathbb{R}^n$, we can rewrite (6b.8) using the identity

$$2 \operatorname{Tr} \Lambda^2 A_\xi = (\operatorname{Tr} A_\xi)^2 - \|A_\xi\|^2, \tag{6b.12}$$

where $\|A_\xi\|$ denotes the Hilbert–Schmidt norm of A_ξ, that is,

$$\|A_\xi\|^2 = \operatorname{Tr}(A_\xi^* A_\xi).$$

Recalling (6.13), if $k = \dim M$, we get

$$A''(0) = \int_M \Big[\|\nabla^1 \xi\|^2 - \|A_\xi\|^2 + k^2 \langle \mathfrak{H}(x), \xi\rangle^2 \Big]\, dA(x). \tag{6b.13}$$

Of course, the last term in the integrand vanishes for all compactly supported fields ξ normal to M when M is a minimal submanifold of $\mathbb{R}^n$.

We next suppose the family of manifolds M_s is contained in a manifold $Y \subset \mathbb{R}^n$. Hence, as before, instead of $X(s,u) = X_0(u) + s\xi(u)$, we require $\partial_s X(s,u) = \xi(s,u)$ to be tangent to Y. We take $X(0,u) = X_0(u)$. Then (6b.1) holds, and we need to add to (6b.2) the following term, in order to compute $A''(0)$:

$$\begin{aligned} \Phi &= \sum_{j=1}^k \big\langle \partial_1 X_0 \wedge \cdots \wedge \partial_j \kappa \wedge \cdots \wedge \partial_k X_0, \partial_1 X_0 \wedge \cdots \wedge \partial_k X_0 \big\rangle, \\ \kappa &= \partial_s \xi = \partial_s^2 X. \end{aligned} \tag{6b.14}$$

If, as before, $\partial_j X_0 = E_j$ form an orthonormal basis of $T_x M$, for a given $x \in M$, then

$$\Phi = \sum_{j=1}^k \langle \partial_j \kappa, E_j \rangle, \quad \text{at } x. \tag{6b.15}$$

Now, given the compactly supported field $\xi(0,u)$, tangent to Y and normal to M, let us suppose that, for each u, $\gamma_u(s) = X(s,u)$ is a constant-speed geodesic in Y, such that $\gamma_u'(0) = \xi(0,u)$. Thus $\kappa = \gamma_u''(0)$ is normal to Y, and, by the Weingarten formula for $M \subset \mathbb{R}^n$,

$$\partial_j \kappa = \nabla^1_{E_j} \kappa - A_\kappa E_j, \tag{6b.16}$$

at x, where ∇^1 is the connection on the normal bundle to $M \subset \mathbb{R}^n$ and A is as before the Weingarten map for $M \subset \mathbb{R}^n$. Thus

$$\Phi = -\sum_j \langle A_\kappa E_j, E_j \rangle = -\mathrm{Tr}\, A_\kappa = -k\langle \mathfrak{H}(x), \kappa \rangle, \tag{6b.17}$$

where $k = \dim M$.

If we suppose M is a minimal submanifold of Y, then $\mathfrak{H}(x)$ is normal to Y, so, for any compactly supported field ξ, normal to M and tangent to Y, the computationss (6b.13) supplemented by (6b.14)–(6b.17) gives

$$A''(0) = \int_M \Big[\|\nabla^1 \xi\|^2 - \|A_\xi\|^2 - k\langle \mathfrak{H}(x), \kappa \rangle \Big]\, dA(x). \tag{6b.18}$$

Recall that A_ξ is the Weingarten map of $M \subset \mathbb{R}^n$.

We prefer to use B_ξ, the Weingarten map of $M \subset Y$. It is readily verified that

$$A_\xi = B_\xi \in \ \mathrm{End}\, T_x M \tag{6b.19}$$

if $\xi \in T_x Y$ and $\xi \perp T_x M$; see Exercise 13 in § 4 of Appendix C. Thus in (6b.18) we can simply replace $\|A_\xi\|^2$ by $\|B_\xi\|^2$. Also recall that ∇^1 in (6b.18) is the connection on the normal bundle to $M \subset \mathbb{R}^n$. We prefer to use the connection on the normal bundle to $M \subset Y$, which we denote by $\nabla^\#$. To relate these two objects, we use the identities

$$\begin{gathered} \partial_j \xi = \nabla^1_j \xi - A_\xi E_j, \quad \partial_j \xi = \widetilde{\nabla}_j \xi + II^Y(E_j, \xi), \\ \widetilde{\nabla}_j \xi = \nabla^\#_j \xi - B_\xi E_j, \end{gathered} \tag{6b.20}$$

where $\widetilde{\nabla}$ denotes the covariant derivative on Y, and II^Y is the second fundamental form of $Y \subset \mathbb{R}^n$. In view of (6b.19), we obtain

$$\nabla^1_j \xi = \nabla^\#_j \xi + II^Y(E_j, \xi), \tag{6b.21}$$

a sum of terms tangent to Y and normal to Y, respectively. Hence

$$\|\nabla^1 \xi\|^2 = \|\nabla^\# \xi\|^2 + \sum_j \|II^Y(E_j, \xi)\|^2. \tag{6b.22}$$

Thus we can rewrite (6b.18) as

$$A''(0) = \int_M \Big[\|\nabla^\# \xi\|^2 - \|B_\xi\|^2 + \sum_j \|II^Y(E_j, \xi)\|^2 - \ \mathrm{Tr}\, A_\kappa \Big]\, dA(x). \tag{6b.23}$$

We want to replace the last two terms in this integrand by a quantity defined intrinsically by $M_s \subset Y$, not by the way Y is imbedded in $\mathbb{R}^n$. Now $\operatorname{Tr} A_\kappa = \sum \langle II^M(E_j, E_j), \kappa\rangle$, where II^M is the second fundamental form of $M \subset \mathbb{R}^n$. On the other hand, it is easily verified that

$$\kappa = \gamma_u''(0) = II^Y(\xi, \xi). \tag{6b.24}$$

Thus the last two terms in the integrand sum to

$$\Psi = \sum_j \Big[\| II^Y(E_j, \xi)\|^2 - \langle II^Y(\xi, \xi), II^M(E_j, E_j)\rangle \Big]. \tag{6b.25}$$

We can replace $II^M(E_j, E_j)$ by $II^Y(E_j, E_j)$ here, since these two objects have the same component normal to Y. Then Gauss' formula implies

$$\Psi = \sum_j \langle R^Y(\xi, E_j)\xi, E_j\rangle, \tag{6b.26}$$

where R^Y is the Riemann curvature tensor of Y. We define $\overline{\mathfrak{R}} \in \operatorname{End} N_x M$, where $N(M)$ is the normal bundle of $N \subset Y$, by

$$\langle \overline{\mathfrak{R}}(\xi), \eta\rangle = \sum_j \langle R^Y(\xi, E_j)\eta, E_j\rangle, \tag{6b.27}$$

at x, where $\{E_j\}$ is an orthonormal basis of $T_x M$. It follows easily that this is independent of the choice of such an orthonormal basis.

Our calculation of $A''(0)$ becomes

$$A''(0) = \int_M \Big[\|\nabla^\# \xi\|^2 - \|B_\xi\|^2 + \big\langle \overline{\mathfrak{R}}(\xi), \xi\big\rangle \Big]\, dA(x) \tag{6b.28}$$

when M is a minimal submanifold of Y, where $\nabla^\#$ is the connection on the normal bundle to $M \subset Y$, B is the Weingarten map for $M \subset Y$, and $\overline{\mathfrak{R}}$ is defined by (6b.27). If we define a second-order differential operator $\mathfrak{L}_0$ and a zero-order operator $\mathfrak{B}$ on $C_0^\infty\big(M, N(M)\big)$ by

$$\mathfrak{L}_0 \xi = (\nabla^\#)^* \nabla^\# \xi, \quad \langle \mathfrak{B}(\xi), \eta\rangle = \operatorname{Tr}(B_\eta^* B_\xi), \tag{6b.29}$$

respectively, we can write this as

$$A''(0) = (\mathfrak{L}\xi, \xi)_{L^2(M)}, \quad \mathfrak{L}\xi = \mathfrak{L}_0 \xi - \mathfrak{B}(\xi) + \overline{\mathfrak{R}}(\xi). \tag{6b.30}$$

We emphasize that these formulas, and the ones below, for $A''(0)$ are valid for immersed minimal submanifolds of Y as well as for imbedded submanifolds.

Suppose that M has codimension 1 in Y and that Y and M are orientable. Complete the basis $\{E_j\}$ of T_xM to an orthonormal basis

$$\{E_j : 1 \le j \le k+1\}$$

of T_xY. In this case, $E_{k+1}(x)$ and $\xi(x)$ are parallel, so

$$\langle R^Y(\xi, E_{k+1})\eta, E_{k+1}\rangle = 0.$$

Thus (6b.27) becomes

$$\overline{\mathfrak{R}}(\xi) = -\mathrm{Ric}^Y \xi \quad \text{if } \dim Y = \dim M + 1, \tag{6b.31}$$

where Ric^Y denotes the Ricci tensor of Y. In such a case, taking $\xi = fE_{k+1} = f\nu$, where ν is a unit normal field to M, tangent to Y, we obtain

$$\begin{aligned} A''(0) &= \int_M \Big[\|\nabla f\|^2 - \big(\|B_\nu\|^2 + \langle \mathrm{Ric}^Y \nu, \nu\rangle\big)|f|^2\Big]\, dA(x) \\ &= (Lf, f)_{L^2(M)}, \end{aligned} \tag{6b.32}$$

where

$$Lf = -\Delta f + \varphi f, \quad \varphi = -\|B_\nu\|^2 - \langle \mathrm{Ric}^Y \nu, \nu\rangle. \tag{6b.33}$$

We can express φ in a different form, noting that

$$\langle \mathrm{Ric}^Y \nu, \nu\rangle = S^Y - \sum_{j=1}^k \langle \mathrm{Ric}^Y E_j, E_j\rangle, \tag{6b.34}$$

where S^Y is the scalar curvature of Y. From Gauss' formula we readily obtain, for general $M \subset Y$ of any codimension,

$$\begin{aligned} \langle \mathrm{Ric}^Y E_j, E_j\rangle &= \langle R^Y(E_j, \nu)\nu, E_j\rangle + \langle \mathrm{Ric}^M E_j, E_j\rangle \\ &\qquad + \sum_\ell \|II(E_j, E_\ell)\|^2 - k\big\langle \mathfrak{H}_Y, II(E_j, E_j)\big\rangle, \end{aligned} \tag{6b.35}$$

where II denotes the second fundamental form of $M \subset Y$. Summing over $1 \le j \le k$, when M has codimension 1 in Y, and ν is a unit normal to M, we get

$$2\langle \mathrm{Ric}^Y \nu, \nu\rangle = S^Y - S^M - \|B_\nu\|^2 + \|\mathfrak{H}_Y\|^2. \tag{6b.36}$$

If M is a minimal submanifold of Y of codimension 1, this implies that

$$\begin{aligned} \varphi &= \frac{1}{2}(S^M - S^Y) - \frac{1}{2}\|B_\nu\|^2 \\ &= \frac{1}{2}(S^M - S^Y) + \operatorname{Tr} \Lambda^2 B_\nu. \end{aligned} \tag{6b.37}$$

We also note that when $\dim M = 2$ and $\dim Y = 3$, then, for $x \in M$,

$$\operatorname{Tr} \Lambda^2 B_\nu(x) = K^M(x) - K^Y(T_x M), \tag{6b.38}$$

where $K^M = (1/2)S^M$ is the Gauss curvature of M and $K^Y(T_x M)$ is the sectional curvature of Y, along the plane $T_x M$.

We consider another special case, where $\dim M = 1$. We have $\langle \overline{\mathfrak{R}}(\xi), \xi\rangle = -|\xi|^2 K^Y(\Pi_{M\xi})$, where $K^Y(\Pi_{M\xi})$ is the sectional curvature of Y along the plane in $T_x Y$ spanned by $T_x M$ and ξ. In this case, to say M is minimal is to say it is a geodesic; hence $B_\xi = 0$ and $\nabla^\# \xi = \widetilde{\nabla}_T \xi$, where $\widetilde{\nabla}$ is the covariant derivative on Y, and T is a unit tangent vector to M. Thus (6b.28) becomes the familiar formula for the second variation of arc length for a geodesic:

$$\ell''(0) = \int_\gamma \Big[\|\widetilde{\nabla}_T \xi\|^2 - |\xi|^2 K^Y(\Pi_{\gamma\xi}) \Big]\, ds, \tag{6b.39}$$

where we have used γ instead of M to denote the curve, and also ℓ instead of A and ds instead of dA, to denote arc length.

The operators $\mathfrak{L}$ and L are second-order elliptic operators that are self-adjoint, with domain $H^2(M)$, if M is compact and without boundary, and with domain $H^2(M) \cap H_0^1(M)$, if $\overline{M}$ is compact with boundary. In such cases, the spectra of these operators consist of eigenvalues $\lambda_j \nearrow +\infty$. If M is not compact, but B and $\overline{\mathfrak{R}}$ are bounded, we can use the Friedrichs method to define self-adjoint extensions $\mathfrak{L}$ and L, which might have continuous spectrum.

We say a minimal submanifold $M \subset Y$ is *stable* if $A''(0) \geq 0$ for all smooth, compactly supported variations ξ, normal to M (and vanishing on ∂M). Thus the condition that M be stable is that the spectrum of $\mathfrak{L}$ (equivalently, of L, if codim $M = 1$) be contained in $[0, \infty)$. In particular, if M is actually area minimizing with respect to small perturbations, leaving ∂M fixed (which we will just call "area minimizing"), then it must be stable, so

$$M \text{ area minimizing} \Longrightarrow \operatorname{spec} \mathfrak{L} \subset [0, \infty). \tag{6b.40}$$

The second variational formulas above provide necessary conditions for a minimal immersed submanifold to be stable. For example, suppose M is a boundaryless, codimension-1 minimal submanifold of Y, and both are orientable. Then we can take $f = 1$ in (6b.32), to get

$$\text{(6b.41)} \qquad M \text{ stable } \Longrightarrow \int_M \left(\|B_\nu\|^2 + \langle \mathrm{Ric}^Y \nu, \nu \rangle \right) dA \le 0.$$

If $\dim M = 2$ and $\dim Y = 3$, then, by (6b.37), we have

$$\text{(6b.42)} \qquad M \text{ stable } \Longrightarrow \int_M \left(\|B_\nu\|^2 + S^Y - 2K^M \right) dA \le 0.$$

In this case, if M has genus g, the Gauss–Bonnet theorem implies that $\int K^M \, dA = 4\pi(1-g)$, so

$$\text{(6b.43)} \qquad M \text{ stable } \Longrightarrow \int_M \left(\|B_\nu\|^2 + S^Y \right) dA \le 8\pi(1-g).$$

This implies some nonexistence results.

Proposition 6b.1. *Assume that Y is a compact, oriented Riemannian manifold and that Y and M have no boundary.*

If the Ricci tensor Ric^Y is positive-definite, then Y cannot contain any compact, oriented, area-minimizing immersed hypersurface M. If Ric^Y is positive-semidefinite, then any such M would have to be totally geodesic in Y.

Now assume $\dim Y = 3$. If Y has scalar curvature $S^Y > 0$ everywhere, then Y cannot contain any compact, oriented, area-minimizing immersed surface M of genus $g \ge 1$.

More generally, if $S^Y \ge 0$ everywhere, and if M is a compact, oriented, immersed hypersurface of genus $g \ge 1$, then for M to be area minimizing it is necessary that $g = 1$ and that M be totally geodesic in Y.

R. Schoen and S.-T. Yau [SY] obtained topological consequences for a compact, oriented 3-manifold Y from this together with the following existence theorem. Suppose M is a compact, oriented surface of genus $g \ge 1$, and suppose the fundamental group $\pi_1(Y)$ contains a subgroup isomorphic to $\pi_1(M)$. Then, given any Riemannian metric on Y, there is a smooth immersion of M into Y which is area minimizing with respect to small perturbations, as shown in [SY]. It follows that if Y is a compact, oriented Riemannian 3-manifold, whose scalar curvature S^Y is everywhere positive, then $\pi_1(Y)$ cannot have a subgroup isomorphic to $\pi_1(M)$, for any compact Riemann surface M of genus $g \ge 1$.

We will not prove the result of [SY] on the existence of such minimal immersions. Instead, we demonstrate a topological result, due to Synge, of a similar flavor but simpler to prove. It makes use of the second variational formula (6b.39) for arc length.

Proposition 6b.2. *If Y is a compact, oriented Riemannian manifold of even dimension, with positive sectional curvature everywhere, then Y is simply connected.*

Proof. It is a simple consequence of Ascoli's theorem that there is a length-minimizing, closed geodesic in each homotopy class of maps from S^1 to Y. Thus, if $\pi_1(Y) \neq 0$, there is a nontrivial stable geodesic, γ. Pick $p \in \gamma$, ξ_p normal to γ at p (i.e., $\xi_p \in N_p(\gamma)$), and parallel translate ξ about γ, obtaining $\overline{\xi}_p \in N_p(\gamma)$ after one circuit. This defines an orientation-preserving, orthogonal, linear transformation $\tau : N_p\gamma \to N_p\gamma$. If Y has dimension $2k$, then $N_p\gamma$ has dimension $2k-1$, so $\tau \in \mathrm{SO}(2k-1)$. It follows that τ must have an eigenvector in $N_p\gamma$, with eigenvalue 1. Thus we get a nontrivial, smooth section ξ of $N(\gamma)$ which is parallel over γ, so (6b.39) implies

$$\int_\gamma K^Y(\Pi_{\gamma\xi})\, ds \leq 0. \tag{6b.44}$$

If $K^Y(\Pi) > 0$ everywhere, this is impossible.

One might compare these results with Proposition 4.7 of Chap. 10, which states that if Y is a compact Riemannian manifold and $\mathrm{Ric}^Y > 0$, then the first cohomology group $\mathcal{H}^1(Y) = 0$.

7. The minimal surface equation

We now study a nonlinear PDE for functions whose graphs are minimal surfaces. We begin with a formula for the mean curvature of a hypersurface $M \subset \mathbb{R}^{n+1}$ defined by $u(x) = c$, where $\nabla u \neq 0$ on M. If $N = \nabla u/|\nabla u|$, we have the formula

$$\langle A_N X, Y\rangle = -|\nabla u|^{-1}(D^2u)(X,Y), \tag{7.1}$$

for $X, Y \in T_xM$, as shown in (4.26) of Appendix C. To take the trace of the restriction of D^2u to T_xM, we merely take $\mathrm{Tr}(D^2u) - D^2u(N,N)$. Of course, $\mathrm{Tr}(D^2u) = \Delta u$. Thus, for $x \in M$,

$$\mathrm{Tr}\, A_N(x) = -|\nabla u(x)|^{-1}\Big[\Delta u - |\nabla u|^{-2} D^2u(\nabla u, \nabla u)\Big]. \tag{7.2}$$

Suppose now that M is given by the equation

$$x_{n+1} = f(x'), \quad x' = (x_1, \dots, x_n).$$

Thus we take $u(x) = x_{n+1} - f(x')$, with $\nabla u = (-\nabla f, 1)$. We obtain for the mean curvature the formula

$$nH(x) = -\frac{1}{\langle \nabla f\rangle^3}\Big[\langle \nabla f\rangle^2 \Delta f - D^2 f(\nabla f, \nabla f)\Big] = \mathcal{M}(f), \tag{7.3}$$

where $\langle \nabla f \rangle^2 = 1 + |\nabla f(x')|^2$. Written out more fully, the quantity in brackets above is

$$(1 + |\nabla f|^2)\Delta f - \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = \widetilde{\mathcal{M}}(f). \tag{7.4}$$

Thus the equation stating that a hypersurface $x_{n+1} = f(x')$ be a minimal submanifold of $\mathbb{R}^{n+1}$ is

$$\widetilde{\mathcal{M}}(f) = 0. \tag{7.5}$$

In case $n = 2$, we have the minimal surface equation, which can also be written as

$$\big(1 + |\partial_2 f|^2\big)\, \partial_1^2 f - 2\big(\partial_1 f \cdot \partial_2 f\big)\, \partial_1 \partial_2 f + \big(1 + |\partial_1 f|^2\big)\, \partial_2^2 f = 0. \tag{7.6}$$

It can be verified that this PDE also holds for a minimal surface in $\mathbb{R}^n$ described by $x'' = f(x')$, where $x'' = (x_3, \dots, x_n)$, if (7.6) is regarded as a system of k equations in k unknowns, $k = n - 2$, and $(\partial_1 f \cdot \partial_2 f)$ is the dot product of $\mathbb{R}^k$-valued functions. We continue to denote the left side of (7.6) by $\widetilde{\mathcal{M}}(f)$.

Proposition 6.12 can be translated immediately into the following existence theorem for the minimal surface equation:

Proposition 7.1. *Let $\mathcal{O}$ be a bounded, convex domain in $\mathbb{R}^2$ with smooth boundary. Let $g \in C^\infty(\partial\mathcal{O}, \mathbb{R}^k)$ be given. Then there is a solution*

$$u \in C^\infty(\mathcal{O}, \mathbb{R}^k) \cap C(\overline{\mathcal{O}}, \mathbb{R}^k) \tag{7.7}$$

to the boundary problem

$$\widetilde{\mathcal{M}}(u) = 0, \quad u\big|_{\partial\mathcal{O}} = g. \tag{7.8}$$

When $k = 1$, we also have uniqueness, as a consequence of the following:

Proposition 7.2. *Let $\mathcal{O}$ be any bounded domain in $\mathbb{R}^n$. Let $u_j \in C^\infty(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ be real-valued solutions to*

$$\widetilde{\mathcal{M}}(u_j) = 0, \quad u_j = g_j \text{ on } \partial\mathcal{O}, \tag{7.9}$$

for $j = 1, 2$. Then

$$g_1 \le g_2 \text{ on } \partial\mathcal{O} \Longrightarrow u_1 \le u_2 \text{ on } \overline{\mathcal{O}}. \tag{7.10}$$

Proof. We prove this by deriving a linear PDE for the difference $v = u_2 - u_1$ and applying the maximum principle. In general,

$$\Phi(u_2) - \Phi(u_1) = Lv, \quad L = \int_0^1 D\Phi\big(\tau u_2 + (1 - \tau)u_1\big)\, d\tau. \tag{7.11}$$

Suppose Φ is a second-order differential operator:

$$\Phi(u) = F(u, \partial u, \partial^2 u), \quad F = F(u, p, \zeta). \tag{7.12}$$

Then, as in (3.4),

$$D\Phi(u) = F_\zeta(u, \partial u, \partial^2 u)\, \partial^2 v + F_p(u, \partial u, \partial^2 u)\, \partial v + F_u(u, \partial u, \partial^2 u) v. \tag{7.13}$$

When $\Phi(u) = \widetilde{\mathcal{M}}(u)$ is given by (7.4), $F_u(u, \xi, \zeta) = 0$, and we have

$$D\widetilde{\mathcal{M}}(u)v = A(u)v + B(u)v, \tag{7.14}$$

where

$$A(u)v = \bigl(1 + |\nabla u|^2\bigr)\Delta v - \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \tag{7.15}$$

is strongly elliptic, and $B(u)$ is a first-order differential operator. Consequently, we have

$$\widetilde{\mathcal{M}}(u_2) - \widetilde{\mathcal{M}}(u_1) = Av + Bv, \tag{7.16}$$

where $A = \int_0^1 A\bigl(\tau u_2 + (1-\tau)u_2\bigr)\, d\tau$ is strongly elliptic of order 2 at each point of $\mathcal{O}$, and B is a first-order differential operator, which annihilates constants. If (7.9) holds, then $Av + Bv = 0$. Now (7.10) follows from the maximum principle, Proposition 2.1 of Chap. 5.

We have as of yet no estimates on $|\nabla u_j(x)|$ as $x \to \partial\mathcal{O}$, so A, which is elliptic in $\mathcal{O}$, could conceivably degenerate at $\partial\mathcal{O}$. To achieve a situation where the results of Chap. 5, § 2, apply, we could note that the hypotheses of Proposition 7.2 imply that, for any $\varepsilon > 0$, $u_1 \le u_2 + \varepsilon$ on a neighborhood of $\partial\mathcal{O}$. Alternatively, one can check that the *proof* of Proposition 2.1 in Chap. 5 works even if the elliptic operator is allowed to degenerate at the boundary. Either way, the maximum principle then applies to yield (7.10).

While Proposition 7.2 is a sort of result that holds for a large class of second-order, scalar, elliptic PDE, the next result is much more special and has interesting consequences. It implies that the size of a solution to the minimal surface equation (7.8) can sometimes be controlled by the behavior of g on *part* of the boundary.

Proposition 7.3. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a domain contained in the annulus $r_1 < |x| < r_2$, and let $u \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ solve $\widetilde{\mathcal{M}}(u) = 0$. Set*

$$G(x; r) = r \cosh^{-1}\left(\frac{|x|}{r}\right), \textit{ for } |x| > r, \quad G(x; r) \le 0. \tag{7.17}$$

If

$$u(x) \le G(x; r_1) + M \quad \textit{on}\ \{x \in \partial\mathcal{O} : |x| > r_1\}, \tag{7.18}$$

for some $M \in \mathbb{R}$, *then*

$$u(x) \le G(x; r_1) + M \quad \textit{on}\ \mathcal{O}. \tag{7.19}$$

Here, $z = G(x; r_1)$ defines the lower half of a catenoid, over $\{x \in \mathbb{R}^2 : |x| \ge r_1\}$. This function solves the minimal surface equation on $|x| > r_1$ and vanishes on $|x| = r_1$.

Proof. Given $s \in (r_1, r_2)$, let

$$\varepsilon(s) = \max_{s \le |x| \le r_2} \left| G(x; r_1) - G(x; s) \right|. \tag{7.20}$$

The hypothesis (7.18) implies that

$$u(x) \le G(x; s) + M + \varepsilon(s) \tag{7.21}$$

on $\{x \in \partial\mathcal{O} : |x| \ge s\}$. We claim that (7.21) holds for x in

$$\mathcal{O}(s) = \mathcal{O} \cap \{x : s < |x| < r_2\}. \tag{7.22}$$

Once this is established, (7.19) follows by taking $s \searrow r_1$. To prove this, it suffices by Proposition 7.2 to show that (7.21) holds on $\partial\mathcal{O}(s)$. Since it holds on $\partial\mathcal{O}$, it remains to show that (7.21) holds for x in

$$\mathcal{C}(s) = \mathcal{O} \cap \{x : |x| = s\}, \tag{7.23}$$

illustrated by a broken arc in Fig. 7.1. If not, then $u(x) - G(x; s)$ would have a maximum $M_1 > M + \varepsilon(s)$ at some point $p \in \mathcal{C}(s)$. By Proposition 7.1, we have $u(x) - G(x; s) \le M_1$ on $\mathcal{O}(s)$. However, $\nabla u(x)$ is bounded on a neighborhood of p, while

$$\frac{\partial}{\partial r} G(x; s) = -\infty \quad \text{on } |x| = s. \tag{7.24}$$

This implies that $u(x) - G(x; s) > M_1$, for all points in $\mathcal{O}(s)$ sufficiently near p. This contradiction shows that (7.21) must hold on $\mathcal{C}(s)$, and the proposition is proved.

One implication is that if $\mathcal{O} \subset \mathbb{R}^2$ is as illustrated in Fig. 7.1, it is not possible to solve the boundary problem (7.8) with g prescribed arbitrarily on all of $\partial\mathcal{O}$. A more precise statement about domains $\mathcal{O} \subset \mathbb{R}^2$ for which (7.8) is always solvable is the following:

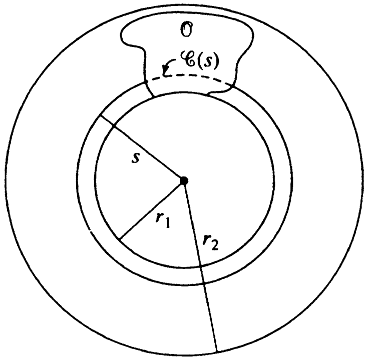

FIGURE 7.1 Nonconvex Region $\mathcal{O}$

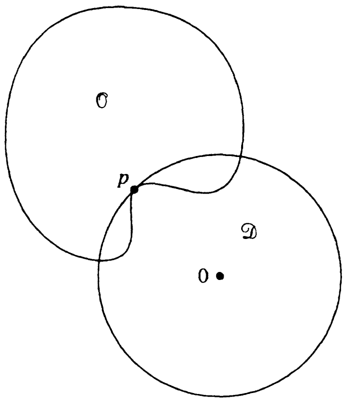

FIGURE 7.2 Another Nonconvex Region $\mathcal{O}$

Proposition 7.4. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded, connected domain with smooth boundary. Then (7.8) has a solution for all $g \in C^\infty(\partial\mathcal{O})$ if and only if $\mathcal{O}$ is convex.*

Proof. The positive result is given in Proposition 7.1. Now, if $\mathcal{O}$ is not convex, let $p \in \partial\mathcal{O}$ be a point where $\mathcal{O}$ is concave, as illustrated in Fig. 7.2. Pick a disk $\mathcal{D}$ whose boundary C is tangent to $\partial\mathcal{O}$ at p and such that, near p, C intersects the complement $\mathcal{O}^c$ only at p. Then apply Proposition 7.3 to the domain $\widetilde{\mathcal{O}} = \mathcal{O} \setminus \overline{\mathcal{D}}$, taking the origin to be the center of $\mathcal{D}$ and r_1 to be the radius of $\mathcal{D}$. We deduce that if u solves $\widetilde{\mathcal{M}}(u) = 0$ on $\mathcal{O}$, then

$$u(x) \leq M + G(x; r_1) \text{ on } \partial\mathcal{O} \setminus \mathcal{D} \Longrightarrow u(p) \leq M, \tag{7.25}$$

which certainly restricts the class of functions g for which (7.8) can be solved.

Note that the function $v(x) = G(x; r)$ defined by (7.17) also provides an example of a solution to the minimal surface equation (7.8) on an annular region

$$\mathcal{O} = \{x \in \mathbb{R}^2 : r < |x| < s\},$$

with smooth (in fact, locally constant) boundary values

$$v = 0 \text{ on } |x| = r, \quad v = -r \cosh^{-1} \frac{s}{r} \text{ on } |x| = s,$$

which is not a smooth function, or even a Lipschitz function, on $\overline{\mathcal{O}}$. This is another phenomenon that is different when $\mathcal{O}$ is convex. We will establish the following:

Proposition 7.5. *If $\mathcal{O} \subset \mathbb{R}^2$ is a bounded region with smooth boundary which is strictly convex (i.e., $\partial\mathcal{O}$ has positive curvature), and $g \in C^\infty(\partial\mathcal{O})$ is real-valued, then the solution to (7.8) is Lipschitz at each point $x_0 \in \partial\mathcal{O}$.*

Proof. Given $x_0 \in \partial\mathcal{O}$, we have $z_0 = \big(x_0, g(x_0)\big) \in \gamma \subset \mathbb{R}^3$, where γ is the boundary of the minimal surface M which is the graph of $z = u(x)$. The strict convexity hypothesis on $\mathcal{O}$ implies that there are two planes Π_j in $\mathbb{R}^3$ through z_0, such that Π_1 lies below γ and Π_2 above γ, and Π_j are given by $z = \alpha_j \cdot (x - x_0) + g(x_0) = w_{jx_0}(x)$, $\alpha_j = \alpha_j(x_0) \in \mathbb{R}^3$. There is an estimate of the form

$$|\alpha_j(x_0)| \le K(x_0) \|g \circ \rho_{x_0}\|_{C^2}, \tag{7.26}$$

where ρ_{x_0} is the radial projection (from the center of $\mathcal{O}$) of $\partial\mathcal{O}$ onto a circle $\mathcal{C}(x_0)$ containing $\mathcal{O}$ and tangent to $\partial\mathcal{O}$ at x_0, and $K(x_0)$ depends on the curvature of $\mathcal{C}(x_0)$. Now Proposition 7.2 applies to give

$$w_{1x_0}(x) \le u(x) \le w_{2x_0}(x), \quad x \in \overline{\mathcal{O}}, \tag{7.27}$$

since linear functions solve the minimal surface equation. This establishes the Lipschitz continuity, with the quantitative estimate

$$|u(x_0) - u(x)| \le A|x - x_0|, \quad x_0 \in \partial\mathcal{O}, \ x \in \overline{\mathcal{O}}, \tag{7.28}$$

where

$$A = \sup_{x_0 \in \partial\mathcal{O}} |\alpha_1(x_0)| + |\alpha_2(x_0)|. \tag{7.29}$$

This result points toward an estimate on $|\nabla u(x)|$, $x \in \overline{\mathcal{O}}$, for a solution to (7.8). We begin the line of reasoning that leads to such an estimate, a line that applies to other situations. First, let's rederive the minimal surface equation, as the stationary condition for

$$I(u) = \int_{\mathcal{O}} F\big(\nabla u(x)\big)\, dx, \tag{7.30}$$

where

$$F(p) = \Big(1 + |p|^2\Big)^{1/2}, \tag{7.31}$$

so (7.30) gives the area of the graph of $z = u(x)$. The method used in Chapter 2, § 1, yields the PDE

$$\sum A^{ij}(\nabla u)\, \partial_i \partial_j u = 0, \tag{7.32}$$

where

$$A^{ij}(p) = \frac{\partial^2 F}{\partial p_i \partial p_j}. \tag{7.33}$$

Compare this with (1.68) and (1.36) of Chap. 2. When $F(p)$ is given by (7.31), we have

$$A^{ij}(p) = \langle p \rangle^{-3} \Big(\delta_{ij} \langle p \rangle^2 - p_i p_j \Big), \tag{7.34}$$

so in this case (7.32) is equal to $-\mathcal{M}(u)$, defined by (7.3). Now, when u is a sufficiently smooth solution to (7.32), we can apply $\partial_\ell = \partial/\partial x_\ell$ to this equation and obtain the PDE

$$\sum \partial_i A^{ij}(\nabla u)\, \partial_j w_\ell = 0, \tag{7.35}$$

for $w_\ell = \partial_\ell u$, not for all PDE of the form (7.32), but whenever $A^{ij}(p)$ is symmetric in (i, j) and satisfies

$$\frac{\partial A^{ij}}{\partial p_m} = \frac{\partial A^{im}}{\partial p_j}, \tag{7.36}$$

which happens when $A^{ij}(p)$ has the form (7.33). If (7.35) satisfies the ellipticity condition

$$\sum A^{ij}\big(\nabla u(x)\big)\, \xi_i \xi_j \geq C(x) |\xi|^2, \quad C(x) > 0, \tag{7.37}$$

for $x \in \mathcal{O}$, then we can apply the maximum principle, to obtain the following:

Proposition 7.6. *Assume $u \in C^1(\overline{\mathcal{O}})$ is real-valued and satisfies the PDE (7.32), with coefficients given by (7.33). If the ellipticity condition (7.37) holds, then $\partial_\ell u(x)$ assumes its maximum and minimum values on $\partial\mathcal{O}$; hence*

$$\sup_{x\in\overline{\mathcal{O}}} |\nabla u(x)| = \sup_{x\in\partial\mathcal{O}} |\nabla u(x)|. \tag{7.38}$$

Combining this result with Proposition 7.5, we have the following:

Proposition 7.7. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded region with smooth boundary which is strictly convex, $g \in C^\infty(\partial\Omega)$ real-valued. If $u \in C^2(\mathcal{O}) \cap C^1(\overline{\mathcal{O}})$ is a solution to (7.8), then there is an estimate*

$$\|u\|_{C^1(\overline{\mathcal{O}})} \le C(\mathcal{O})\, \|g\|_{C^2(\partial\mathcal{O})}. \tag{7.39}$$

Note that the existence result of Proposition 7.1 does not provide us with the knowledge that u belongs to $C^1(\overline{\mathcal{O}})$, and thus it will take further work to demonstrate that the estimate (7.39) actually holds for an arbitrary real-valued solution to (7.8) when $\mathcal{O} \subset \mathbb{R}^2$ is strictly convex and g is smooth. We will be in a position to establish this result, and further regularity, after sufficient theory is developed in the next two sections. See in particular Theorem 10.4. For now, we can regard this as motivation to develop the tools in the following sections, on the regularity of solutions to elliptic boundary problems.

We next look at the Gauss curvature of a minimal surface M, given by $z = u(x),\ x \in \mathcal{O} \subset \mathbb{R}^2$. For a general u, the curvature is given by

$$K = \bigl(1 + |\nabla u|^2\bigr)^{-2} \det\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right). \tag{7.40}$$

See (4.29) in Appendix C. When u satisfies the minimal surface equation, there are some other formulas for K, in terms of operations on

$$\Phi(x) = F(\nabla u)^{-1} = \bigl(1 + |\nabla u|^2\bigr)^{-1/2}, \tag{7.41}$$

which we will list, leaving their verification as an exercise:

$$K = -\frac{|\nabla\Phi|^2}{1 - \Phi^2}, \tag{7.42}$$

$$K = \frac{1}{2\Phi}\,\Delta\Phi, \tag{7.43}$$

$$K = \Delta\, \log(1 + \Phi). \tag{7.44}$$

Now if we alter the metric g induced on M via its imbedding in $\mathbb{R}^3$ by a conformal factor:

$$g' = (1 + \Phi)^2 g = e^{2v} g, \quad v = \log(1 + \Phi), \tag{7.45}$$

then, as in formula (1.30), we see that the Gauss curvature k of M in the new metric is

$$k = (-\Delta v + K)e^{-2v} = 0; \tag{7.46}$$

in other words, the metric $g' = (1+\Phi)^2 g$ is flat! Using this observation, we can establish the following remarkable theorem of S. Bernstein:

Theorem 7.8. *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an everywhere-defined C^2-solution to the minimal surface equation, then u is a linear function.*

Proof. Consider the minimal surface M given by $z = u(x)$, $x \in \mathbb{R}^2$, in the metric $g' = (1+\Phi)^2 g$, which, as we have seen, is flat. Now $g' \geq g$, so this is a complete metric on M. Thus (M, g') is isometrically equivalent to $\mathbb{R}^2$. Hence (M, g) is conformally equivalent to $\mathbb{C}$.

On the other hand, the antipodal Gauss map

$$\widetilde{N} : M \longrightarrow S^2, \quad \widetilde{N} = \langle \nabla u \rangle^{-1}(\nabla u, -1), \tag{7.47}$$

is holomorphic; see Exercise 1 of § 6. But the range of $\widetilde{N}$ is contained in the lower hemisphere of S^2, so if we take $S^2 = \mathbb{C} \cup \{\infty\}$ with the point at infinity identified with the "north pole" $(0, 0, 1)$, we see that $\widetilde{N}$ yields a bounded holomorphic function on $M \approx \mathbb{C}$. By Liouville's theorem, $\widetilde{N}$ must be constant. Thus M is a flat plane in $\mathbb{R}^3$.

It turns out that Bernstein's theorem extends to $u : \mathbb{R}^n \to \mathbb{R}$, for $n \leq 7$, by work of E. DeGiorgi, F. Almgren, and J. Simons, but not to $n \geq 8$.

Exercises

1. If $D\widetilde{\mathcal{M}}(u)$ is the differential operator given by (7.14)–(7.15), show that its principal symbol satisfies

$$-\sigma_{D\widetilde{\mathcal{M}}(u)}(x, \xi) = \big(1 + |p|^2\big)|\xi|^2 - (p \cdot \xi)^2 \geq |\xi|^2, \tag{7.48}$$

   where $p = \nabla u(x)$.
2. Show that the formula (7.3) for $\mathcal{M}(f)$ is equivalent to

$$\mathcal{M}(f) = \sum_j \partial_j \big(\langle \nabla f \rangle^{-1} \, \partial_j f\big) = \operatorname{div}\big(\langle \nabla f \rangle^{-1} \nabla f\big). \tag{7.49}$$

3. Give a detailed demonstration of the estimate (7.26) on the slope of planes that can lie above and below the graph of g over $\partial\mathcal{O}$ (assumed to have positive curvature), needed for the proof of Proposition 7.5. (*Hint*: In case $\partial\mathcal{O}$ is the unit circle S^1, consider the cases $g(\theta) = \cos^k \theta$.)
4. Establish the formulas (7.42)–(7.44) for the Gauss curvature of a minimal surface.

8. Elliptic regularity II (boundary estimates)

We establish estimates and regularity for solutions to nonlinear elliptic boundary problems. We treat completely nonlinear, second-order equations, obtaining L^2-Sobolev estimates for solutions assumed a priori to belong to $C^{2+r}(\overline{M}),\ r > 0$. We make note of improved estimates for solutions to quasi-linear, second-order equations. In § 10 we will show how such results, when supplemented by the DeGiorgi–Nash–Moser theory, apply to the solvability of the Dirichlet problem for certain quasi-linear elliptic PDE.

Though we restrict attention to second-order equations, the analysis in this section extends readily to higher-order elliptic systems, such as we treated in § 11 of Chap. 5. The exposition here is taken from [T].

Having looked at interior regularity in § 4, we restrict attention to a collar neighborhood of the boundary $\partial M = X$, so we look at a PDE of the form

$$\partial_y^2 u = F(y, x, D_x^2 u, D_x^1 \partial_y u), \tag{8.1}$$

with $y \in [0, 1], x \in X$. We set

$$v_1 = \Lambda u, \quad v_2 = \partial_y u, \tag{8.2}$$

and produce a first-order system for $v = (v_1, v_2)$,

$$\begin{aligned} \frac{\partial v_1}{\partial y} &= \Lambda v_2, \\ \frac{\partial v_2}{\partial y} &= F(y, x, D_x^2 \Lambda^{-1} v_1, D_x^1 v_2). \end{aligned} \tag{8.3}$$

An operator like $T = \Lambda$ or $T = D_x^2 \Lambda^{-1}$ does not map $C^{k+1+r}(I \times X)$ to $C^{k+r}(I \times X)$, but if we set

$$C^{k+r+}(I \times X) = \bigcup_{\varepsilon > 0} C^{k+r+\varepsilon}(I \times X), \tag{8.4}$$

then

$$T : C^{k+1+r+}(I \times X) \longrightarrow C^{k+r+}(I \times X). \tag{8.5}$$

Thus we will assume $u \in C^{2+r+}$. This implies $v \in C^{1+r+}$, and the arguments $D_x^2 \Lambda^{-1} v_1$ and $D_x^1 v_2$ appearing in (8.3) belong to C^{r+}. We will be able to drop the "+" in the statement of the main result.

Now if we treat y as a parameter and apply the paradifferential operator construction developed in § 10 of Chap. 13 to the family of operators on functions of x, we obtain

$$\begin{aligned} F(y, x, D_x^2 \Lambda^{-1} v_1, D_x^1 v_2) = A_1(v; y, x, D_x)v_1 \\ + A_2(v; y, x, D_x)v_2 + R(v), \end{aligned} \tag{8.6}$$

with (for fixed y) $R(v) \in C^\infty(X)$,

$$A_j(v; y, x, \xi) \in \mathcal{A}_0^r S^1_{1,1} \subset C^r S^1_{1,0} \cap S^1_{1,1} \tag{8.7}$$

and

$$D_x^\beta A_j \in S^1_{1,1}, \text{ for } |\beta| \le r, \quad S^{1+(|\beta|-r)}_{1,1}, \text{ for } |\beta| > r, \tag{8.8}$$

provided $u \in C^{2+r+}$.

Note that if we write $F = F(y, x, \zeta, \eta)$, $\zeta_\alpha = D_x^\alpha u$ $(|\alpha| \le 2)$, $\eta_\alpha = D_x^\alpha \partial_y u$ $(|\alpha| \le 1)$, then we can set

$$B_1(v; y, x, \xi) = \sum_{|\alpha| \le 2} \frac{\partial F}{\partial \zeta_\alpha}(D_x^2 \Lambda^{-1} v_1, D_x^1 v_2) \xi^\alpha \langle \xi \rangle^{-1} \tag{8.9}$$

(suppressing the y- and x-arguments of F) and

$$B_2(v; y, x, \xi) = \sum_{|\alpha| \le 1} \frac{\partial F}{\partial \eta_\alpha}(D_x^2 \Lambda^{-1} v_1, D_x^1 v_2) \xi^\alpha. \tag{8.10}$$

Thus

$$v \in C^{1+r+} \Longrightarrow A_j - B_j \in C^r S^{1-r}_{1,1}. \tag{8.11}$$

Using (8.4), we can rewrite the system (8.3) as

$$\begin{aligned} \frac{\partial v_1}{\partial y} &= \Lambda v_2, \\ \frac{\partial v_2}{\partial y} &= A_1(x, D) v_1 + A_2(x, D) v_2 + R(v). \end{aligned} \tag{8.12}$$

We also write this as

$$\frac{\partial v}{\partial y} = K(v; y, x, D_x) v + R \quad (R \in C^\infty), \tag{8.13}$$

where $K(v; y, x, D_x)$ is a 2×2 matrix of first-order pseudodifferential operators. Let us denote the symbol obtained by replacing A_j by B_j as $\tilde{K}$, so

$$K - \tilde{K} \in C^r S^{1-r}_{1,1}. \tag{8.14}$$

The ellipticity condition can be expressed as

$$\operatorname{spec} \tilde{K}(v; y, x, \xi) \subset \{z \in \mathbb{C} : |\operatorname{Re} z| \ge C|\xi|\}, \tag{8.15}$$

for $|\xi|$ large. Hence we can make the same statement about the spectrum of the symbol K, for $|\xi|$ large, provided $v \in C^{1+r+}$ with $r > 0$.

In order to derive L^2-Sobolev estimates, we will construct a symmetrizer, in a fashion similar to § 11 in Chap. 5. In particular, we will make use of Lemma 11.4 of Chap. 5. Let $\tilde{E} = \tilde{E}(v; y, x, \xi)$ denote the projection onto the $\{\mathrm{Re}\, z > 0\}$ spectral space of $\tilde{K}$, defined by

$$\tilde{E}(y, x, \xi) = \frac{1}{2\pi i} \int_\gamma \big(z - \tilde{K}(y, x, \xi)\big)^{-1}\, dz, \tag{8.16}$$

where γ is a curve enclosing that part of the spectrum of $\tilde{K}(y, x, \xi)$ contained in $\{\mathrm{Re}\, z > 0\}$. Then the symbol

$$\tilde{A} = (2\tilde{E} - 1)\tilde{K} \in C^r S^1_{cl} \tag{8.17}$$

has spectrum in $\{\mathrm{Re}\, z > 0\}$. (The symbol class $C^r S^m_{cl}$ is defined as in (9.46) of Chap. 13.) Let $\tilde{P} \in C^r S^0_{cl}$ be a symmetrizer for the symbol $\tilde{A}$, constructed via Lemma 11.4 of Chap. 5, namely,

$$\tilde{P}(y, x, \xi) = \Phi\big(\tilde{A}(y, x, \xi)\big),$$

where Φ is as in (11.54)–(11.55) in Chap. 5. Thus $\tilde{P}$ and $(\tilde{P}\tilde{A} + \tilde{A}^*\tilde{P})$ are positive-definite symbols, for $|\xi| \ge 1$.

We now want to apply symbol smoothing to $\tilde{P}$, $\tilde{A}$, and $\tilde{E}$. It will be convenient to modify the construction slightly, and smooth in both x and y. Thus we obtain various symbols in $S^m_{1,\delta}$, with the understanding that the symbol classes reflect estimates on $D_{y,x}$-derivatives. For example, we obtain (with $0 < \delta < 1$)

$$P(y, x, \xi) \in S^0_{1,\delta}; \quad P - \tilde{P} \in C^r S^{-r\delta}_{1,\delta} \tag{8.18}$$

by smoothing $\tilde{P}$, in (y, x). We set

$$Q = \frac{1}{2}\big(P(y, x, D_x) + P(y, x, D_x)^*\big) + K\Lambda^{-1}, \tag{8.19}$$

with $K > 0$ picked to make the operator Q positive-definite on $L^2(X)$. Similarly, define A and E by smoothing $\tilde{A}$ and $\tilde{E}$ in (y, x), so

$$\begin{aligned} A(y, x, \xi) &\in S^1_{1,\delta}, \quad A - \tilde{A} \in C^r S^{1-r\delta}_{1,\delta}, \\ E(y, x, \xi) &\in S^0_{1,\delta}, \quad E - \tilde{E} \in C^r S^{-r\delta}_{1,\delta}, \end{aligned} \tag{8.20}$$

and we smooth K, writing

$$K = K_0 + K^b; \quad K_0 \in S^1_{1,\delta}, \ \ K^b \in C^r S^{1-r\delta}_{1,\delta} \cap S^{1-r\delta}_{1,1}. \tag{8.21}$$

Consequently, on the symbol level,

$$\begin{aligned} A &= (2E-1)K_0 + A^b, \quad A^b \in S^{1-r\delta}_{1,\delta}, \\ PA + A^* P &\geq C|\xi|, \quad \text{for } |\xi| \text{ large}. \end{aligned} \tag{8.22}$$

Let us note that the homogeneous symbols $\tilde{K}$, $\tilde{E}$, and $\tilde{A}$ commute, for each (y, x, ξ); hence the commutators of the various symbols K, E, A have order $\leq r\delta$ units less than the sum of the orders of these symbols; for example,

$$[E(y,x,\xi), K_0(y,x,\xi)] \in S^{1-r\delta}_{1,\delta}. \tag{8.23}$$

Using this symmetrizer construction, we will look for estimates for solutions to a system of the form (8.3) in the spaces $H_{k,s}(M) = H_{k,s}(I \times X)$, with norms

$$\|v\|^2_{k,s} = \sum_{j=0}^{k} \|\partial_y^j \Lambda^{k-j+s} v(y)\|^2_{L^2(I\times X)}. \tag{8.24}$$

We shall differentiate $(Q\Lambda^s Ev, \Lambda^s Ev)$ and $(Q\Lambda^s(1-E)v, \Lambda^s(1-E)v)$ with respect to y (these expressions being $L^2(X)$-inner products) and sum the two resulting expressions, to obtain the desired a priori estimates, parallel to the treatment in § 11 of Chap. 5.

Using (8.13), we have

$$\begin{aligned} \frac{d}{dy}(Q\Lambda^s Ev, \Lambda^s Ev) = 2\,\mathrm{Re}(Q\Lambda^s E(Kv+R), \Lambda^s Ev) & \\ + (Q'\Lambda^s Ev, \Lambda^s Ev) & \\ + 2\,\mathrm{Re}(Q\Lambda^s E'v, \Lambda^s Ev). & \end{aligned} \tag{8.25}$$

Note that given $v \in C^{1+r+}, r > 0$, Q' and E' belong to $OPS^{\delta}_{1,\delta}$. Hence, for fixed y, each of the last two terms is bounded by

$$C\|v(y)\|^2_{H^{s+\delta/2}}. \tag{8.26}$$

Here and below, we will adopt the convention that $C = C(\|v\|_{C^{1+r+}})$, with a slight abuse of notation. Namely, $v \in C^{1+r+}$ belongs to $C^{1+r+\varepsilon}$ for some $\varepsilon > 0$, and we loosely use $\|v\|_{C^{1+r+}}$ instead of $\|v\|_{C^{1+r+\varepsilon}}$.

To analyze the first term on the right side of (8.25), we write

$$\begin{aligned} (Q\Lambda^s E(Kv+R), \Lambda^s Ev) = (Q\Lambda^s EK_0 v, \Lambda^s Ev) & \\ + (Q\Lambda^s K^b v, \Lambda^s Ev) & \\ + (Q\Lambda^s ER, \Lambda^s Ev), & \end{aligned} \tag{8.27}$$

where the last term is harmless and, for fixed y,

$$|(Q\Lambda^s E K^b v, \Lambda^s E v)| \le C\|v(y)\|^2_{H^{s+(1-r\delta)/2}}, \tag{8.28}$$

provided $s + (1 - r\delta)/2 - (1 - r\delta) > -(1 - \delta)r$, that is,

$$s > \frac{1}{2} - r + \frac{1}{2}r\delta, \tag{8.29}$$

in view of (8.21).

Since $\tilde{E}(y, x, \xi)$ is a projection, we have $E(y, x, \xi)^2 - E(y, x, \xi) \in S^{-r\delta}_{1,\delta}$ and

$$\begin{aligned} E(y, x, D) - E(y, x, D)^2 &= F(y, x, D) \in OPS^{-\sigma}_{1,\delta}, \\ \sigma &= \min\,(r\delta, 1 - \delta). \end{aligned} \tag{8.30}$$

Thus

$$QEK_0 = QAE + G; \quad G(y) \in OPS^{1-\sigma}_{1,\delta}. \tag{8.31}$$

Consequently, we can write the first term on the right side of (8.27) as

$$(QAE\Lambda^s v, \Lambda^s E v) - (G\Lambda^s v, \Lambda^s E v) + (Q[\Lambda^s, EK_0]v, \Lambda^s E v). \tag{8.32}$$

The last two terms in (8.32) are bounded (for each y) by

$$C\|v(y)\|^2_{H^{s+(1-\sigma)/2}}. \tag{8.33}$$

As for the contribution of the first term in (8.32) to the estimation of (8.25), we have, for each y,

$$(QAE\Lambda^s v, \Lambda^s E v) = (QA\Lambda^s E v, \Lambda^s E v) + (QA[E, \Lambda^s]v, \Lambda^s v), \tag{8.34}$$

the last term estimable by (8.33), and

$$2\,\mathrm{Re}(QA\Lambda^s E v, \Lambda^s E v) \ge C_1\|Ev(y)\|^2_{H^{s+1/2}} - C_2\|Ev(y)\|^2_{H^s}, \tag{8.35}$$

by (8.22) and Gårding's inequality. Keeping track of the various ingredients in the analysis of (8.25), we see that

$$\begin{aligned} \frac{d}{dy}(Q\Lambda^s E v, \Lambda^s E v) \ge C_1\|Ev(y)\|^2_{H^{s+1/2}} \\ - C_2\|v(y)\|^2_{H^{s+(1-\sigma)/2}} - C_3\|R(y)\|^2_{H^s}, \end{aligned} \tag{8.36}$$

where $C_j = C_j(\|v\|_{C^{1+r+}}) > 0$.

A similar analysis gives

$$
\begin{aligned}
(8.37)\quad &\frac{d}{dy}(Q\Lambda^s(1-E)v, \Lambda^s(1-E)v) \\
&\le -C_1\|(1-E)v(y)\|^2_{H^{s+1/2}} + C_2\|v(y)\|^2_{H^{s+(1-\sigma/2)}} + C_3\|R(y)\|^2_{H^s}.
\end{aligned}
$$

Putting together these two estimates yields

$$
\begin{aligned}
(8.38)\quad &\frac{1}{2}C_1\|v(y)\|^2_{H^{s+1/2}} \le C_1\|Ev(y)\|^2_{H^{s+1/2}} + C_1\|(1-E)v(y)\|^2_{H^{s+1/2}} \\
&\le \frac{d}{dy}(Q\Lambda^s Ev, \Lambda^s Ev) - \frac{d}{dy}(Q\Lambda^s(1-E)v, \Lambda^s(1-E)v) \\
&\qquad + C_2\|v(y)\|^2_{H^{s+(1-\sigma)/2}} + C_3\|R(y)\|^2_{H^s}.
\end{aligned}
$$

Now standard arguments allow us to replace $H^{s+(1-\sigma)/2}$ by H^t, with $t << s$. Then integration over $y \in [0,1]$ gives

$$
\begin{aligned}
(8.39)\quad C_1\|v\|^2_{0,s+1/2} &\le \|\Lambda^s Ev(1)\|^2_{L^2} + \|\Lambda^s(1-E)v(0)\|^2_{L^2} \\
&\quad + C_2\|v\|^2_{0,t} + C_3\|R\|^2_{0,s}.
\end{aligned}
$$

Recalling that

$$
(8.40)\quad \|v\|^2_{1,s} = \|\Lambda^{1+s}v\|^2_{L^2(M)} + \|\Lambda^s\partial_y v\|^2_{L^2(M)}
$$

and using (8.13) to estimate $\partial_y v$, we have

$$
(8.41)\quad \|v\|^2_{1,s-1/2} \le C\Big[\|Ev(1)\|^2_{H^s} + \|(1-E)v(0)\|^2_{H^s} + \|v\|^2_{0,t} + \|R\|^2_{0,s}\Big],
$$

with $C = C(\|v\|_{C^{1+r+}})$, provided that $v \in C^{1+r+}$ with $r > 0$ and that s satisfies the lower bound (8.29). Let us note that

$$
C_1\Big[\|\Lambda^s(1-E)v(1)\|^2_{L^2} + \|\Lambda^s Ev(0)\|^2_{L^2}\Big]
$$

could have been included on the left side of (8.39), so we also have the estimate

$$
(8.42)\quad \|(1-E)v(1)\|^2_{H^s} + \|Ev(0)\|^2_{H^s} \le \text{ right side of (8.41)}.
$$

Having completed a first round of a priori estimates, we bring in a consideration of boundary conditions that might be imposed. Of course, the boundary conditions $Ev(1) = f_1, (1-E)v(0) = f_0$ are a possibility, but these are really a tool with which to analyze other, more naturally occurring boundary conditions. The “real” boundary conditions of interest include the Dirichlet condition on (8.1):

$$
(8.43)\quad u(0) = f_0, \quad u(1) = f_1,
$$

various sorts of (possibly nonlinear) conditions involving first-order derivatives:

$$G_j(x, D^1 u) = f_j, \quad \text{at } y = j \quad (j = 0, 1), \tag{8.44}$$

and when (8.1) is itself a $K \times K$ system, other possibilities, which can be analyzed in the same spirit. Now if we write $D^1 u = (u, \partial_x u, \partial_y u) = (\Lambda^{-1} v_1, \partial_x \Lambda^{-1} v_1, v_2)$, and use the paradifferential operator construction of Chap. 13, § 10, we can write (8.44) as

$$H_j(v; x, D)v = g_j, \quad \text{at } y = j, \tag{8.45}$$

where, given $v \in C^{1+r+}$,

$$H_j(v; x, \xi) \in \mathcal{A}_0^{1+r} S^0_{1,1} \subset C^{1+r} S^0_{1,0} \cap S^0_{1,1}. \tag{8.46}$$

Of course, (8.43) can be written in the same form, with $H_j v = v_1$.

Now the following is the natural regularity hypothesis to make on (8.45); namely, that we have an estimate of the form

$$\begin{aligned} \sum_j \|v(j)\|^2_{H^s} \le C\Big[\|Ev(0)\|^2_{H^s} + \|(1-E)v(1)\|^2_{H^s}\Big] \\ + C\sum_j \Big[\|H_j(v; x, D)v(j)\|^2_{H^s} + \|v(j)\|^2_{H^{s-1}}\Big]. \end{aligned} \tag{8.47}$$

We then say the boundary condition is *regular*. If we combine this with (8.41) and (8.42), we obtain the following fundamental estimate:

Proposition 8.1. *If v satisfies the elliptic system (8.3), together with the boundary condition (8.45), assumed to be regular, then*

$$\|v\|^2_{1,s-1/2} \le C\Big[\sum_j \|g_j\|^2_{H^s} + \|v\|^2_{0,t} + \|R\|^2_{0,s}\Big], \tag{8.48}$$

provided $v \in H_{1,s-1/2} \cap C^{1+r}$, $r > 0$, and s satisfies (8.29). We can take $t << s$. In case (8.44) holds, we can replace $\|g_j\|_{H^s}$ by $\|f_j\|_{H^s}$, and in case the Dirichlet condition (8.43) holds and is regular, we can replace $\|g_j\|_{H^s}$ by $\|f_j\|_{H^{s+1}}$ in (8.48).

Here, we have taken the opportunity to drop the "+" from C^{1+r+}; to justify this, we need only shift r slightly. For the same reason, we can assume that, in (8.1), $u \in C^{2+r}$, for some $r > 0$. In the rest of this section, we assume for simplicity that $s - 1/2 \in \mathbb{Z}^+ \cup \{0\}$.

We can now easily obtain higher-order estimates, of the form

$$\|v\|^2_{k,s-1/2} \le C\Big[\sum_j \|g_j\|^2_{H^{s+k-1}} + \|v\|^2_{0,t} + \|R\|^2_{k-1,s}\Big], \tag{8.49}$$

for $t << s - 1/2$, by induction from

$$\|v\|^2_{k,s-1/2} = \|v\|^2_{k-1,s+1/2} + \|\partial_y v\|^2_{k-1,s-1/2},$$

plus substituting the right side of (8.3) for $\partial_y v$. This follows from the existence of Moser-type estimates:

$$\begin{aligned} &\|F(\cdot,\cdot,w_1,w_2)\|_{k,s-1/2} \\ &\quad \le C\big(\|w_1\|_{L^\infty}, \|w_2\|_{L^\infty}\big)\big[\|w_1\|_{k,s-1/2} + \|w_2\|_{k,s-1/2}\big], \end{aligned} \tag{8.50}$$

for $k, k+s-1/2 > 0$. If $s - 1/2 \in \mathbb{Z}^+ \cup \{0\}$, such an estimate can be established by methods used in § 3 of Chap. 13.

We also obtain a corresponding regularity theorem, via inclusion of Friedrich mollifiers in the standard fashion. Thus replace Λ^s by $\Lambda^s_\varepsilon = \Lambda^s J_\varepsilon$ in (8.25) and repeat the analysis. One must keep in mind that K^b must be applicable to $v(y)$ for the analogue of (8.28) to work. Given (8.21), we need $v(y) \in H^\sigma$ with $\sigma > 1-r$. However, $v \in C^{1+r}$ already implies this. We thus have the following result.

Theorem 8.2. *Let v be a solution to the elliptic system (8.3), satisfying the boundary conditions (8.45), assumed to be regular. Assume*

$$v \in C^{1+r}, \quad r > 0, \tag{8.51}$$

and

$$g_j \in H^{s+k-1}(X), \tag{8.52}$$

with $s - 1/2 \in \mathbb{Z}^+ \cup \{0\}$. Then

$$v \in H_{k,s-1/2}(I \times X). \tag{8.53}$$

In particular, taking $s = 1/2$, and noting that

$$H_{k,0}(M) = H^k(M), \tag{8.54}$$

we can specialize this implication to

$$g_j \in H^{k-1/2}(X) \Longrightarrow v \in H^k(I \times X), \tag{8.55}$$

for $k = 1, 2, 3, \dots$, granted (8.51) (which makes the $k = 1$ case trivial).

Note that, in (8.36)–(8.38), one could replace the term $\|R(y)\|^2_{H^s}$ by the product $\|R(y)\|_{H^{s-1/2}} \cdot \|v(y)\|_{H^{s+1/2}}$; then an absorption can be performed in (8.38), and hence in (8.39)–(8.41) we can substitute $\|R\|^2_{0,s-1/2}$, and use $\|R\|^2_{k-1,s-1/2}$ in (8.49).

We note that Theorem 8.2 is also valid for solutions to a nonhomogeneous elliptic system, where R in (8.13) can contain an extra term, belonging to

$H_{k-1,s-1/2}$, and then the estimate (8.49), strengthened as indicated above, and consequent regularity theorem are still valid. If (8.1) is generalized to

$$\partial_y^2 u = F(D_x^2 u, D_x^1 \partial_y u) + f, \tag{8.56}$$

then a term of the form $(0, f)^t$ is added to (8.13).

In view of the estimate (8.11) comparing the symbol of K with that obtained from the linearization of the original PDE (8.1), and the analogous result that holds for H_j, derived from G_j, we deduce the following:

Proposition 8.3. *Suppose that, at each point on ∂M, the linearization of the boundary condition of (8.44) is regular for the linearization of the PDE (8.1). Assume $u \in C^{2+r}$, $r > 0$. Then the regularity estimate (8.49) holds. In particular, this holds for the Dirichlet problem, for any scalar (real) elliptic PDE of the form (8.1).*

We next establish a strengthened version of Theorem 8.2 when u solves a quasi-linear, second-order elliptic PDE, with a regular boundary condition. Thus we are looking at the special case of (8.1) in which

$$\begin{aligned} F(y, x, D_x^2 u, D_x^1 \partial_y u) = & -\sum_j B^j(x, y, D^1 u)\, \partial_j \partial_y u \\ & -\sum_{j,k} A^{jk}(x, y, D^1 u)\, \partial_j \partial_k u \\ & + F_1(x, y, D^1 u). \end{aligned} \tag{8.57}$$

All the calculations done above apply, but some of the estimates are better. This is because when we derive the equation (8.13), namely,

$$\frac{\partial v}{\partial y} = K(v; y, x, D_x)v + R \quad (R \in C^\infty) \tag{8.58}$$

for $v = (v_1, v_2) = (\Lambda u, \partial_y u)$, (8.7) is improved to

$$u \in C^{1+r+} \Longrightarrow K \in \mathcal{A}_0^r S_{1,1}^1 + S_{1,1}^{1-r} \quad (r > 0). \tag{8.59}$$

Compare with (4.62). Under the hypothesis $u \in C^{1+r+}$, one has the result (8.17), $\tilde{A} \in C^r S_{cl}^1$, which before required $u \in C^{2+r+}$. Also (8.20)–(8.22) now hold for $u \in C^{1+r+}$. Thus all the a priori estimates, down through (8.49), hold, with $C = C(\|u\|_{C^{1+r+}})$. As before, we can delete the "+." One point that must be taken into consideration is that, for the estimates to work, one needs $v(y) \in H^\sigma$ with $\sigma > 1 - r$, and now this does not necessarily follow from the hypothesis $u \in C^{1+r}$. Hence we have the following regularity result. Compare the interior regularity established in Theorem 4.5.

Theorem 8.4. *Let u satisfy a second-order, quasi-linear elliptic PDE with a regular boundary condition, of the form (8.45), for $v = (\Lambda u, \partial_y u)$. Assume that*

$$u \in C^{1+r} \cap H_{1,\sigma}, \quad r > 0, \quad r + \sigma > 1. \tag{8.60}$$

Then, for $k = 0, 1, 2, \dots,$

$$g_j \in H^{k-1/2}(X) \Longrightarrow v \in H^k(I \times X). \tag{8.61}$$

The Dirichlet boundary condition is regular (if the PDE is real and scalar), and

$$u(j) = f_j \in H^{k+s}(X) \Longrightarrow v \in H_{k,s-\frac{1}{2}}(I \times X) \tag{8.62}$$

if $s > (1 - r)/2$. In particular,

$$\begin{aligned} u(j) = f_j \in H^{k+1/2}(X) &\Longrightarrow v \in H^k(I \times X) \\ &\Longrightarrow u \in H^{k+1}(I \times X). \end{aligned} \tag{8.63}$$

We consider now the further special case

$$\begin{aligned} F(y, x, D_x^2 u, D_x^1 \partial_y u) = &-\sum_j B^j(x, y, u)\, \partial_j \partial_y u \\ &- \sum_{j,k} A^{jk}(x, y, u)\, \partial_j \partial_k u + F_1(x, y, D^1 u). \end{aligned} \tag{8.64}$$

In this case, when we derive the system (8.58), we have the implication

$$u \in C^{r+}(\overline{M}) \Longrightarrow K \in \mathcal{A}_0^r S_{1,1}^1 + S_{1,1}^{1-r} \quad (r > 0). \tag{8.65}$$

Similarly, under this hypothesis, we have $\tilde{A} \in C^r S_{cl}^1$, and so forth. Therefore we have the following:

Proposition 8.5. *If u satisfies the PDE (8.1) with F given by (8.64), then the conclusions of Theorem 8.4 hold when the hypothesis (8.60) is weakened to*

$$u \in C^r \cap H_{1,\sigma}, \quad r + \sigma > 1. \tag{8.66}$$

Note that associated to this regularity is an estimate. For example, if u satisfies the Dirichlet boundary condition, we have, for $k \geq 2$,

$$\|u\|_{H^k(M)} \leq C_k(\|u\|_{C^r(\overline{M})}) \big[\|u|_{\partial M}\|_{H^{k-1/2}(\partial M)} + \|u\|_{L^2(M)} \big], \tag{8.67}$$

where we have used Poincaré's inequality to replace the $H_{1,\sigma}$-norm of u by the L^2-norm on the right.

Let us see to what extent the results obtained here apply to solutions to the minimal surface equation produced in § 7. Recall the boundary problem (7.8):

$$\langle \nabla u \rangle^2 \Delta u - \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad u = g \text{ on } \partial \mathcal{O}, \tag{8.68}$$

where $\mathcal{O}$ is a strictly convex region in $\mathbb{R}^2$, with smooth boundary. For this boundary problem, Theorem 8.4 applies, to yield the implication

$$g \in H^{k+1/2}(\partial \mathcal{O}) \Longrightarrow u \in H^{k+1}(\mathcal{O}), \quad k = 0, 1, 2, \dots, \tag{8.69}$$

provided we know that

$$u \in C^{1+r}(\overline{\mathcal{O}}) \cap H_{1,\sigma}(\mathcal{A}), \quad r > 0, \ \ r + \sigma > 1, \tag{8.70}$$

where $\mathcal{A}$ is a collar neighborhood of $\partial \mathcal{O}$ in $\overline{\mathcal{O}}$. Now, while we know that solutions to the minimal surface equation are smooth inside $\mathcal{O}$ (having proved that minimal surfaces are real analytic), we so far have only *continuity* of a solution u on $\overline{O}$, plus a Lipschitz bound on $u\big|_{\partial \mathcal{O}}$ and a hope of obtaining a bound in $C^1(\overline{\mathcal{O}})$. We therefore have a gap to close to be able to apply the results of this section to solutions of (8.68).

The material of the next two sections will close this gap. As we'll see, we will be able to treat (8.68), not only for $\dim \mathcal{O} = 2$, but also for $\dim \mathcal{O} = n > 2$. Also, the gap will be closed on a number of other quasi-linear elliptic PDE.

Exercises

1. Suppose u is a solution to a quasi-linear elliptic PDE of the form

$$\sum a_{jk}(x, u) \partial_j \partial_k u + b(x, u, \nabla u) = 0 \ \text{ on } \overline{M},$$

satisfying boundary conditions

$$B_0(x, u) u = g_0, \quad B_1(x, u, D) u = g_1, \quad \text{on } \partial M,$$

assumed to be regular. The operators B_j have order j. Generalizing (8.67), show that, for any $r > 0,\ k \geq 2$, there is an estimate

$$\|u\|_{H^k(M)} \leq C_k\big(\|u\|_{C^r(\overline{M})}\big)\Big(\|g_0\|_{H^{k-1/2}(\partial M)} + \|g_1\|_{H^{k-3/2}(\partial M)} + \|u\|_{L^2(M)}\Big). \tag{8.71}$$

2. Extend Theorem 8.4 to nonhomogeneous, quasi-linear equations,

$$\sum a_{jk}(x, D^1 u)\, \partial_j \partial_k u + b(x, D^1 u) = h(x), \tag{8.72}$$

satisfying regular boundary conditions. If one uses the Dirichlet boundary condition, $u\big|_{\partial M} = g$, show that

(8.73)
$$\|u\|_{H^k(M)} \le C_k\big(\|u\|_{C^{1+r}(\overline{M})}\big)\Big(\|g\|_{H^{k-1/2}(\partial M)} + \|h\|_{H^{k-2}(M)} + \|u\|_{L^2(M)}\Big).$$

3. Give a proof of the mapping property (8.5).
4. Prove the Moser-type estimate (8.50), when $s - 1/2 = \ell \in \mathbb{Z}^+ \cup \{0\}$. (*Hint*. Rework Propositions 3.2–3.9 of Chap. 13, with H^k replaced by $H_{k,\ell}$.)

9. Elliptic regularity III (DeGiorgi–Nash–Moser theory)

As noted at the end of § 8, there is a gap between conditions needed on the solution of boundary problems for many nonlinear elliptic PDEs, in order to obtain higher-order regularity, and conditions that solutions constructed by methods used so far in this chapter have been shown to satisfy. One method of closing this gap, that has proved useful in many cases, involves the study of second-order, scalar, linear elliptic PDE, in divergence form, whose coefficients have no regularity beyond being bounded and measurable.

In this section we establish regularity for a class of PDE $Lu = f$, for second-order operators of the form (using the summation convention)

$$Lu = b^{-1}\partial_j\big(a^{jk}b\, \partial_k u\big), \tag{9.1}$$

where $(a^{jk}(x))$ is a positive-definite, bounded matrix and $0 < b_0 \le b(x) \le b_1, b$ scalar, and a^{jk}, b are merely measurable. The breakthroughs on this were first achieved by DeGiorgi [DeG] and Nash [Na2]. We will present Moser's derivation of interior bounds and Hölder continuity of solutions to $Lu = 0$, from [Mo2], and then Morrey's analysis of the nonhomogeneous equation $Lu = f$ and proof of boundary regularity, from [Mor2]. Other proofs can be found in [GT] and [KS].

We make a few preliminary remarks on (9.1). We will use a^{jk} to define an inner product of vectors:

$$\langle V, W\rangle = V_j a^{jk} W_k, \tag{9.2}$$

and use $b\, dx = dV$ as the volume element. In case $g_{jk}(x)$ is a metric tensor, if one takes $a^{jk} = g^{jk}$ and $b = g^{1/2}$, then (9.1) defines the Laplace operator. For a compactly supported function w,

$$(Lu, w) = -\int \langle \nabla u, \nabla w\rangle\, dV. \tag{9.3}$$

The behavior of L on a nonlinear function of u, $v = f(u)$, plays an important role in estimates; we have

$$v = f(u) \Longrightarrow Lv = f'(u)Lu + f''(u)|\nabla u|^2, \tag{9.4}$$

where we set $|V|^2 = \langle V, V\rangle$. Also, taking $w = \psi^2 u$ in (9.3) gives the following important identity. If $Lu = g$ on an open set Ω and $\psi \in C_0^1(\Omega)$, then

$$\int \psi^2 |\nabla u|^2 \, dV = -2 \int \langle \psi \nabla u, u \nabla \psi\rangle \, dV - \int \psi^2 g u \, dV. \tag{9.5}$$

Applying Cauchy's inequality to the first term on the right yields the useful estimate

$$\frac{1}{2} \int \psi^2 |\nabla u|^2 \, dV \le 2 \int |u|^2 |\nabla \psi|^2 \, dV - \int \psi^2 g u \, dV. \tag{9.6}$$

Given these preliminaries, we are ready to present an approach to sup norm estimates known as "Moser iteration." Once this is done (in Theorem 9.3 below), we will then tackle Hölder estimates.

To implement Moser iteration, consider a nested sequence of open sets with smooth boundary

$$\Omega_0 \supset \cdots \supset \Omega_j \supset \Omega_{j+1} \supset \cdots \tag{9.7}$$

with intersection $\mathcal{O}$, as illustrated in Fig. 9.1. We will make the geometrical hypothesisthat the distance of any point on $\partial\Omega_{j+1}$ to $\partial\Omega_j$ is $\sim Cj^{-2}$. We want to estimate the sup norm of a function v on $\mathcal{O}$ in terms of its L^2-norm on Ω_0, assuming

$$v > 0 \text{ is a subsolution of } L \quad (\text{i.e., } Lv \ge 0). \tag{9.8}$$

In view of (9.4), an example is

$$v = (1 + u^2)^{1/2}, \quad Lu = 0. \tag{9.9}$$

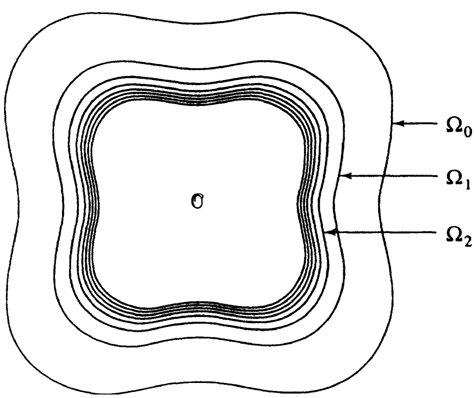

FIGURE 9.1 Setup for Moser Iteration

We will obtain such an estimate in terms of the Sobolev constants $\gamma(\Omega_j)$ and C_j, defined below. Ingredients for the analysis include the following two lemmas, the first being a standard Sobolev inequality.

Lemma 9.1. *For* $v \in H^1(\Omega_j)$, $\kappa \le n/(n-2)$,

$$\|v^\kappa\|^2_{L^2(\Omega_j)} \le \gamma(\Omega_j)\big[\|\nabla v\|^{2\kappa}_{L^2(\Omega_j)} + \|v\|^{2\kappa}_{L^2(\Omega_j)}\big]. \tag{9.10}$$

The next lemma follows from (9.6) if we take $\psi = 1$ on Ω_{j+1}, tending roughly linearly to 0 on $\partial\Omega_j$.

Lemma 9.2. *If* $v > 0$ *is a subsolution of* L*, then, with* $C_j = C(\Omega_j, \Omega_{j+1})$*,*

$$\|\nabla v\|_{L^2(\Omega_{j+1})} \le C_j \|v\|_{L^2(\Omega_j)}. \tag{9.11}$$

Under the geometrical conditions indicated above on Ω_j, we can assume

$$\gamma(\Omega_j) \le \gamma_0, \quad C_j \le C(j^2+1). \tag{9.12}$$

Putting together the two lemmas, we see that when v satisfies (9.8),

$$\begin{aligned} \|v^\kappa\|^2_{L^2(\Omega_{j+1})} &\le \gamma(\Omega_{j+1})\Big[C_j^{2\kappa}\|v\|^{2\kappa}_{L^2(\Omega_j)} + \|v\|^{2\kappa}_{L^2(\Omega_{j+1})}\Big] \\ &\le \gamma_0(C_j^{2\kappa}+1)\|v\|^{2\kappa}_{L^2(\Omega_j)}. \end{aligned} \tag{9.13}$$

Fix $\kappa \in (1, n/(n-2)]$. Now, if v satisfies (9.8), so does

$$v_j = v^{\kappa^j}, \tag{9.14}$$

by (9.4). Note that $v_{j+1} = v_j^\kappa$. Now let

$$N_j = \|v\|_{L^{2\kappa^j}(\Omega_j)} = \|v_j\|^{1/\kappa^j}_{L^2(\Omega_j)}, \tag{9.15}$$

so

$$\|v\|_{L^\infty(\mathcal{O})} \le \limsup_{j\to\infty} N_j. \tag{9.16}$$

If we apply (9.13) to v_j, we have

$$\|v_{j+1}\|^2_{L^2(\Omega_{j+1})} \le \gamma_0(C_j^{2\kappa}+1)\|v_j\|^{2\kappa}_{L^2(\Omega_j)}. \tag{9.17}$$

Note that the left side is equal to $N_{j+1}^{2\kappa^{j+1}}$, and the norm on the right is equal to $N_j^{2\kappa^{j+1}}$. Thus (9.17) is equivalent to

$$N_{j+1}^2 \leq \Big[\gamma_0(C_j^{2\kappa}+1)\Big]^{1/\kappa^{j+1}} N_j^2. \tag{9.18}$$

By (9.12), $C_j^{2\kappa}+1 \leq C_0(j^{4\kappa}+1)$, so

$$\begin{aligned}\limsup_{j\to\infty} N_j^2 &\leq \prod_{j=0}^{\infty}\Big[\gamma_0 C_0(j^{4\kappa}+1)\Big]^{1/\kappa^{j+1}} N_0^2 \\ &\leq (\gamma_0 C_0)^{1/(\kappa-1)}\left[\exp\sum_{j=0}^{\infty}\kappa^{-j-1}\ \log(j^{4\kappa}+1)\right] N_0^2 \\ &\leq K^2 N_0^2,\end{aligned} \tag{9.19}$$

for finite K. This gives Moser's sup-norm estimate:

Theorem 9.3. *If $v > 0$ is a subsolution of L, then*

$$\|v\|_{L^\infty(\mathcal{O})} \leq K\|v\|_{L^2(\Omega_0)}, \tag{9.20}$$

where $K = K(\gamma_0, C_0, n)$.

Hölder continuity of a solution to $Lu = 0$ will be obtained as a consequence of the following "Harnack inequality." Let $B_\rho = \{x : |x| < \rho\}$.

Proposition 9.4. *Let $u \geq 0$ be a solution of $Lu = 0$ in B_{2r}. Pick $c_0 \in (0,\infty)$. Suppose*

$$\mathit{meas}\{x \in B_r : u(x) \geq 1\} > c_0^{-1} r^n. \tag{9.21}$$

Then there is a constant $c > 0$ such that

$$u(x) > c^{-1} \ \text{ in } B_{r/2}. \tag{9.22}$$

This will be established by examining $v = f(u)$ with

$$f(u) = \max\{-\log(u+\varepsilon), 0\}, \tag{9.23}$$

where ε is chosen in $(0,1)$. Note that f is convex, so v is a subsolution. Our first goal will be to estimate the $L^2(B_r)$-norm of ∇v. Once this is done, Theorem 9.3 will be applied to estimate v from above (hence u from below) on $B_{r/2}$.

We begin with a variant of (9.5), obtained by taking $w = \psi^2 f'(u)$ in (9.3). The identity (for smooth f) is

$$\int \psi^2 f''|\nabla u|^2\, dV + 2\int \langle \psi f'\nabla u, \nabla\psi\rangle\, dV = -(Lu, \psi^2 f'). \tag{9.24}$$

This vanishes if $Lu = 0$. Applying Cauchy's inequality to the second integral, we obtain

$$\int \psi^2 \Big[f''(u) - \delta^2 f'(u)^2 \Big] |\nabla u|^2 \, dV \le \frac{1}{\delta^2} \int |\nabla \psi|^2 \, dV. \tag{9.25}$$

Now the function $f(u)$ in (9.23) has the property that

$$h = -e^{-f} \text{ is a convex function;} \tag{9.26}$$

indeed, in this case $h(u) = \max\{-(u+\varepsilon), -1\}$. Thus

$$f'' - (f')^2 = e^f h'' \ge 0. \tag{9.27}$$

Thus $f''(u)|\nabla u|^2 \ge f'(u)^2 |\nabla u|^2 = |\nabla v|^2$ if $v = f(u)$. Taking $\delta^2 = 1/2$ in (9.25), we obtain

$$\int \psi^2 |\nabla v|^2 \, dV \le 4 \int |\nabla \psi|^2 \, dV, \tag{9.28}$$

after one overcomes the minor problem that f' has a jump discontinuity. If we pick ψ to $= 1$ on B_r and go linearly to 0 on ∂B_{2r}, we obtain the estimate

$$\int_{B_r} |\nabla v|^2 \, dV \le C r^{n-2}, \tag{9.29}$$

for $v = f(u)$, given that $Lu = 0$ and that (9.26) holds.

Now the hypothesis (9.21) implies that v vanishes on a subset of B_r of measure $> c_0^{-1} r^n$. Hence there is an elementary estimate of the form

$$r^{-n} \int_{B_r} v^2 \, dV \le C r^{2-n} \int_{B_r} |\nabla v|^2 \, dV, \tag{9.30}$$

which is bounded from above by (9.29). Now Theorem 9.3, together with a simple scaling argument, gives

$$v(x)^2 \le C r^{-n} \int_{B_r} v^2 \, dV \le C_1^2, \qquad x \in B_{r/2}, \tag{9.31}$$

so

$$u + \varepsilon \ge e^{-C_1}, \quad \text{for } x \in B_{r/2}, \tag{9.32}$$

for all $\varepsilon \in (0,1)$. Taking $\varepsilon \to 0$, we have the proof of Proposition 9.4.

We remark that Moser obtained a stronger Harnack inequality in [Mo3], by a more elaborate argument. In that work, the hypothesis (9.21) is weakened to

$$\sup_{B_r} u(x) \geq 1. \tag{9.21a}$$

To deduce the Hölder continuity of a solution to $Lu = 0$ given Proposition 9.4 is fairly simple. Following [Mo2], who followed DeGiorgi, we have from (9.20) a bound

$$|u(x)| \leq K \tag{9.33}$$

on any compact subset $\mathcal{O}$ of Ω_0, given $u \in H^1(\Omega_0),\ Lu = 0$. Fix $x_0 \in \mathcal{O}$, such that $B_\rho(x_0) \subset \mathcal{O}$, and, for $r \leq \rho$, let

$$\omega(r) = \sup_{B_r} u(x) - \inf_{B_r} u(x), \tag{9.34}$$

where $B_r = B_r(x_0)$. Clearly, $\omega(\rho) \leq 2K$. Adding a constant to u, we can assume

$$\sup_{B_\rho} u(x) = -\inf_{B_\rho} u(x) = \frac{1}{2}\omega(\rho) = M. \tag{9.35}$$

Then $u_+ = 1 + u/M$ and $u_- = 1 - u/M$ are also annihilated by L. They are both ≥ 0 and at least one of them satisfies the hypothesis (9.21), with $r = \rho/2$. If, for example, u_+ does, then Proposition 9.4 implies

$$u_+(x) > c^{-1} \quad \text{in } B_{\rho/4}, \tag{9.36}$$

so

$$-M\Big(1 - \frac{1}{c}\Big) \leq u(x) \leq M \quad \text{in } B_{\rho/4}. \tag{9.37}$$

Hence

$$\omega(\rho/4) \leq \Big(1 - \frac{1}{2c}\Big)\omega(\rho), \tag{9.38}$$

which gives Hölder continuity:

$$\omega(r) \leq \omega(\rho)\Big(\frac{r}{\rho}\Big)^\alpha, \quad \alpha = -\log_4\Big(1 - \frac{1}{2c}\Big). \tag{9.39}$$

We state the result formally.

Theorem 9.5. *If $u \in H^1(\Omega_0)$ solves $Lu = 0$, then for every compact $\mathcal{O}$ in Ω_0, there is an estimate*

$$\|u\|_{C^\alpha(\mathcal{O})} \le C\,\|u\|_{L^2(\Omega_0)}. \tag{9.40}$$

It will be convenient to replace (9.40) by an estimate involving Morrey spaces, which are discussed in Appendix A at the end of this chapter. We claim that under the hypotheses of Theorem 9.5,

$$\nabla u\big|_{\mathcal{O}} \in M_2^p, \quad p = \frac{n}{1-\alpha}, \tag{9.41}$$

where the Morrey space M_2^p consists of functions f satisfying the $q = 2$ case of (A.2). The property (9.41) is stronger than (9.40), by Morrey's lemma (Lemma A.1). To see (9.41), if B_R is a ball of radius R centered at y, $B_{2R} \subset \Omega$, then let $c = u(y)$ and replace u by $u(x) - c$ in (9.6), to get

$$\frac{1}{2}\int \psi^2 |\nabla u|^2 \, dV \le 2\int |u(x) - c|^2 |\nabla\psi|^2 \, dV.$$

Taking $\psi = 1$ on B_R, going linearly to 0 on ∂B_{2R}, gives

$$\int_{B_R} |\nabla u|^2 \, dV \le C\, R^{n-2+2\alpha}, \tag{9.42}$$

as needed to have (9.41).

So far we have dealt with the homogeneous equation, $Lu = 0$. We now turn to regularity for solutions to a nonhomogeneous equation. We will follow a method of Morrey, and Morrey spaces will play a very important role in this analysis. We take L as in (9.1), with a^{jk} measurable, satisfying

$$0 < \lambda_0 |\xi|^2 \le \sum a^{jk}(x)\xi_j\xi_k \le \lambda_1 |\xi|^2, \tag{9.43}$$

while for simplicity we assume $b, b^{-1} \in \operatorname{Lip}(\overline{\Omega})$. We consider a PDE

$$Lu = f. \tag{9.44}$$

It is clear that, for $u \in H_0^1(\Omega)$,

$$(Lu, u) \ge C \sum \|\partial_j u\|_{L^2}^2, \tag{9.45}$$

so we have an isomorphism

$$L : H_0^1(\Omega) \xrightarrow{\approx} H^{-1}(\Omega). \tag{9.46}$$

Thus, for any $f \in H^{-1}(\Omega)$, (9.44) has a unique solution $u \in H_0^1(\Omega)$. One can write such f as

$$f = \sum \partial_j g_j, \quad g_j \in L^2(\Omega). \tag{9.47}$$

The solution $u \in H_0^1(\Omega)$ then satisfies

$$\|u\|^2_{H^1(\Omega)} \le C \sum \|g_j\|^2_{L^2}. \tag{9.48}$$

Here C depends on $\Omega, \lambda_0, \lambda_1$, and $b \in \operatorname{Lip}(\overline{\Omega})$.

One can also consider the boundary problem

$$Lv = 0 \text{ on } \Omega, \qquad v = w \text{ on } \partial\Omega, \tag{9.49}$$

given $w \in H^1(\Omega)$, where the latter condition means $v - w \in H_0^1(\Omega)$. Indeed, setting $v = u + w$, the equation for u is $Lu = -Lw$, $u \in H_0^1(\Omega)$. Thus (9.49) is uniquely solvable, with an estimate

$$\|\nabla v\|_{L^2(\Omega)} \le C\|\nabla w\|_{L^2(\Omega)}, \tag{9.50}$$

where C has a dependence as in (9.48).

Our present goal is to give Morrey's proof of the following local regularity result.

Theorem 9.6. *Suppose $u \in H^1(\Omega)$ solves (9.44), with $f = \sum \partial_j g_j$, $g_j \in M_2^q(\Omega)$, $q > n$, that is,*

$$\int_{B_r} |g_j|^2 \, dV \le K_1^2 \Big(\frac{r}{R}\Big)^{n-2+2\mu}, \quad \mu = 1 - \frac{n}{q} \in (0,1). \tag{9.51}$$

Assume L is of the form (9.1), where the coefficients a^{jk} satisfy (9.43) and $b, b^{-1} \in \operatorname{Lip}(\overline{\Omega})$. Let $\mathcal{O} \subset\subset \Omega$, and assume $\mu < \mu_0 = \alpha$, for which Theorem 9.5 holds. Then $u \in C^\mu(\mathcal{O})$; more precisely, $\nabla u \in M_2^q(\mathcal{O})$, that is,

$$\int_{B_r} |\nabla u|^2 \, dV \le K_2^2 \Big(\frac{r}{R}\Big)^{n-2+2\mu}. \tag{9.52}$$

Morrey established this by using (9.48), (9.50), and an elegant dilation argument, in concert with Theorem 9.5. For this, suppose $B_R = B_R(y) \subset \Omega$ for each $y \in \mathcal{O}$. We can write $u = U + H$ on B_R, where

$$\begin{aligned} LU &= \sum \partial_j g_j \text{ on } B_R, \quad U \in H_0^1(B_R), \\ LH &= 0 \text{ on } B_R, \quad H - u \in H_0^1(B_R), \end{aligned} \tag{9.53}$$

and we have

$$\|\nabla U\|_{L^2(B_R)} \le C_1 \|g\|_{L^2(B_R)}, \quad \|\nabla H\|_{L^2(B_R)} \le C_2 \|\nabla u\|_{L^2(B_R)}, \tag{9.54}$$

where $\|g\|_{L^2}^2 = \sum \|g_j\|_{L^2}^2$. Let us set

$$\|F\|_r = \|F\|_{L^2(B_r)}. \tag{9.55}$$

Also let $\kappa(g_j, R)$ be the best constant K_1 for which (9.51) is valid for $0 < r \le R$. If $g_\tau(x) = g(\tau x)$, note that

$$\kappa(g_\tau, \tau^{-1} S) = \tau^{n/2} \kappa(g, S).$$

Now define

$$\begin{aligned} \varphi(r) = \sup\Bigl\{\|\nabla U\|_{rS} : U \in H_0^1(B_S),\ LU = \sum \partial_j g_j, \text{ on } B_S,& \\ \kappa(g_j, S) \le 1, 0 < S \le R\Bigr\}.& \end{aligned} \tag{9.56}$$

Let us denote by $\varphi_S(r)$ the sup in (9.56) with S fixed, in $(0, R]$. Then $\varphi_S(r)$ coincides with $\varphi_R(r)$, with L replaced by the dilated operator, coming from the dilation taking B_S to B_R. More precisely, the dilated operator is

$$L_S = b_S\, \partial_j\, a_S^{jk}\, b_S^{-1}\, \partial_k, \tag{9.57}$$

with

$$a_S^{jk}(x) = a^{jk}(SR^{-1}x), \quad b_S(x) = b(SR^{-1}x),$$

assuming 0 has been arranged to be the center of B_R. To see this, note that if $\tau = S/R$, $U_\tau(x) = \tau^{-1} U(\tau x)$, and $g_{j\tau}(x) = g_j(\tau x)$, then

$$LU = \sum \partial_j g_j \Longleftrightarrow L_S U_\tau = \sum \partial_j g_{j\tau}. \tag{9.58}$$

Also, $\nabla U_\tau(x) = (\nabla U)(\tau x)$, so $\|\nabla U_\tau\|_{S/\tau} = \tau^{n/2} \|\nabla U\|_S$.

Now for this family L_S, one has a *uniform* bound on C in (9.48); hence $\varphi(r)$ is *finite* for $r \in (0, 1]$. We also note that the bounds in (9.40) and (9.42) are uniformly valid for this family of operators. Theorem 9.6 will be proved when we show that

$$\varphi(r) \le A\, r^{n/2-1+\mu}. \tag{9.59}$$

In fact, this will give the estimate (9.52) with u replaced by U; meanwhile such an estimate with u replaced by H is a consequence of (9.42). Let H satisfy (9.42) with $\alpha = \mu_0$. We take $\mu < \mu_0$.

Pick $S \in (0, R]$ and pick g_j satisfying (9.51), with R replaced by S and K_1 by K. Write the U of (9.53) as $U = U_S + H_S$ on B_S, where $U_S \in H_0^1(B_S)$, $LU_S = LU = \sum \partial_j g_j$ on B_S. Clearly, (9.51) implies

$$\int_{B_r} |g_j|^2 \, dV \le K^2 \Big(\frac{S}{R}\Big)^{n-2+2\mu} \Big(\frac{r}{S}\Big)^{n-2+2\mu}. \tag{9.60}$$

Thus, as in (9.54) (and recalling the definition of φ), we have

$$\begin{aligned} \|\nabla U_S\|_S &\le A_1 K \Big(\frac{S}{R}\Big)^{n/2-1+\mu}, \\ \|\nabla H_S\|_S &\le A_2 \|\nabla U\|_S \le A_2 K \varphi\Big(\frac{S}{R}\Big). \end{aligned} \tag{9.61}$$

Now, suppose $0 < r < S < R$. Then, applying (9.42) to H_S, we have

$$\begin{aligned} \|\nabla U\|_r &\le \|\nabla U_S\|_r + \|\nabla H_S\|_r \\ &\le K\Big(\frac{S}{R}\Big)^{n/2-1+\mu} \varphi\Big(\frac{r}{S}\Big) + A_3 K \varphi\Big(\frac{S}{R}\Big)\Big(\frac{r}{S}\Big)^{n/2-1+\mu_0}. \end{aligned} \tag{9.62}$$

Therefore, setting $s = r/R,\ t = S/R$, we have the inequality

$$\varphi(s) \le t^{n/2-1+\mu} \varphi\Big(\frac{s}{t}\Big) + A_3 \varphi(t) \Big(\frac{s}{t}\Big)^{n/2-1+\mu_0}, \tag{9.63}$$

valid for $0 < s < t \le 1$. Since it is clear that $\varphi(r)$ is monotone and finite on $(0,1]$, it is an elementary exercise to deduce from (9.63) that $\varphi(r)$ satisfies an estimate of the form (9.59), as long as $\mu < \mu_0$. This proves Theorem 9.6.

Now that we have interior regularity estimates for the nonhomogeneous problem, we will be able to use a few simple tricks to establish regularity up to the boundary for solutions to the Dirichlet problem

$$Lu = \sum \partial_j g_j, \quad u = f \text{ on } \partial\Omega, \tag{9.64}$$

where L has the form (9.1), $\overline{\Omega}$ is compact with smooth boundary, $f \in \operatorname{Lip}(\partial\Omega)$, and $g_j \in L^q(\Omega)$, with $q > n$. First, extend f to $f \in \operatorname{Lip}(\overline{\Omega})$. Then $u = v + f$, where v solves

$$Lv = \sum \partial_j h_j, \quad v = 0 \text{ on } \partial\Omega, \tag{9.65}$$

where

$$\partial_j h_j = \partial_j g_j - b^{-1} \partial_j \big(a^{jk} b \, \partial_k f\big). \tag{9.66}$$

We will assume $b \in \operatorname{Lip}(\overline{\Omega})$; then h_j can be chosen in L^q also.

The class of equations (9.65) is invariant under smooth changes of variables (indeed, invariant under Lipschitz homeomorphisms with Lipschitz inverses, having the further property of preserving volume up to a factor in $\operatorname{Lip}(\overline{\Omega})$). Thus make a change of variables to flatten out the boundary (locally), so we consider a solution $v \in H^1$ to (9.65) in $x_n > 0,\ |x| \le R$. We can even arrange that $b = 1$. Now extend v to negative x_n, to be *odd* under the reflection $x_n \mapsto -x_n$. Also extend $a^{jk}(x)$ to be even when $j, k < n$ or $j = k = n$, and odd when j or $k = n$ (but not both). Extend h_j to be odd for $j < n$ and even for $j = n$. With these extensions, we continue to have (9.65) holding, this time in the ball $|x| \le R$. Thus interior regularity applies to this extension of v, yielding Hölder continuity. The following is hence proved.

Theorem 9.7. *Let $u \in H^1(\Omega)$ solve the PDE*

$$\sum b^{-1}\partial_j\big(a^{jk} b\, \partial_k u\big) = \sum \partial_j g_j \ \text{on } \Omega, \quad u = f \ \text{on } \partial\Omega. \tag{9.67}$$

Assume $g_j \in L^q(\Omega)$ with $q > n = \dim \Omega$, and $f \in Lip(\partial\Omega)$. Assume that $b, b^{-1} \in Lip(\overline{\Omega})$ and that (a^{jk}) is measurable and satisfies the uniform ellipticity condition (9.43). Then u has a Hölder estimate

$$\|u\|_{C^\mu(\overline{\Omega})} \le C_1\Big(\sum \|g_j\|_{L^q(\Omega)} + \|f\|_{Lip(\partial\Omega)}\Big). \tag{9.68}$$

More precisely, if $\mu = 1 - n/q \in (0,1)$ is sufficiently small, then ∇u belongs to the Morrey space $M_2^q(\Omega)$, and

$$\|\nabla u\|_{M_2^q(\Omega)} \le C_2\Big(\sum \|g_j\|_{L^q(\Omega)} + \|f\|_{Lip(\partial\Omega)}\Big). \tag{9.69}$$

In these estimates, $C_j = C_j(\Omega, \lambda_1, \lambda_2, b)$.

So far in this section we have looked at differential operators of the form (9.1) in which (a^{jk}) is symmetric, but unlike the nondivergence case, where $a^{jk}(x)\,\partial_j\partial_k u = a^{kj}(x)\,\partial_j\partial_k u$, nonsymmetric cases do arise; we will see an example in § 15. Thus we briefly describe the extension of the analysis of (9.1) to

$$Lu = b^{-1}\,\partial_j\big([a^{jk} + \omega^{jk}]b\,\partial_k u\big). \tag{9.70}$$

We make the same hypotheses on $a^{jk}(x)$ and $b(x)$ as before, and we assume (ω^{jk}) is antisymmetric and bounded:

$$\omega^{jk}(x) = -\omega^{kj}(x), \quad \omega^{jk} \in L^\infty(\Omega). \tag{9.71}$$

We thus have both a positive symmetric form and an antisymmetric form defined at almost all $x \in \Omega$:

$$\langle V, W\rangle = V_j a^{jk}(x) W_k, \quad [V, W] = V_j \omega^{jk}(x) W_k. \tag{9.72}$$

We use the subscript L^2 to indicate the integrated quantities:

$$\langle v, w\rangle_{L^2} = \int \langle v, w\rangle \, dV, \quad [v, w]_{L^2} = \int [v, w] \, dV. \tag{9.73}$$

Then, in place of (9.3), we have

$$(Lu, w) = -\langle \nabla u, \nabla w\rangle_{L^2} - [\nabla u, \nabla w]_{L^2}. \tag{9.74}$$

The formula (9.4) remains valid, with $|\nabla u|^2 = \langle \nabla u, \nabla u\rangle$, as before. Instead of (9.5), we have

$$\int \psi^2 |\nabla u|^2 \, dV = -2\langle \psi \nabla u, u \nabla \psi\rangle_{L^2} - 2[\psi \nabla u, u\nabla \psi]_{L^2} - \int \psi^2 g u \, dV, \tag{9.75}$$

when $Lu = g$ on Ω and $\psi \in C_0^1(\Omega)$. This leads to a minor change in (9.6):

$$\frac{1}{2}\int \psi^2 |\nabla u|^2 \, dV \le (2 + C_0) \int |u|^2 |\nabla \psi|^2 \, dV - \int \psi^2 g u \, dV, \tag{9.76}$$

where C_0 is determined by the operator norm of (ω^{jk}), relative to the inner product $\langle \, , \, \rangle$.

From here, the proofs of Lemmas 9.1 and 9.2, and that of Theorem 9.3, go through without essential change, so we have the sup-norm estimate (9.20). In the proof of the Harnack inequality, (9.24) is replaced by

$$\begin{aligned} \int \psi^2 f'' |\nabla u|^2 \, dV + 2\langle \psi f' \nabla u, \nabla \psi\rangle_{L^2} + 2[\psi f' \nabla u, \nabla \psi]_{L^2} \\ = -(Lu, \psi^2 f'). \end{aligned} \tag{9.77}$$

Hence (9.25) still works if you replace the factor $1/\delta^2$ by $(1 + C_1)/\delta^2$, where again C_1 is estimated by the size of (ω^{jk}). Thus Proposition 9.4 extends to our present case, and hence so does the key regularity result, Theorem 9.5. Let us record what has been noted so far:

Proposition 9.8. *Assume Lu has the form (9.70), where (a^{jk}) and b satisfy the hypotheses of Theorem 9.5, and (ω^{jk}) satisfies (9.71). If $u \in H^1(\Omega_0)$ solves $Lu = 0$, then, for every compact $\mathcal{O} \subset \Omega_0$, there is an estimate*

$$\|u\|_{C^\alpha(\mathcal{O})} \le C \|u\|_{L^2(\Omega_0)}. \tag{9.78}$$

The Morrey space estimates go through as before, and the analysis of (9.64) is also easily modified to incorporate the change in L. Thus we have the following:

Proposition 9.9. *The boundary regularity of Theorem 9.7 extends to the operators L of the form (9.70), under the hypothesis (9.71) on (ω^{jk}).*

Exercises

1. Given the strengthened form of the Harnack inequality, in which the hypothesis (9.21) is replaced by (9.21a), produce a shorter form of the argument in (9.33)–(9.40) for Hölder continuity of solutions to $Lu = 0$.
2. Show that in the statement of Theorem 9.7, $\sum \partial_j g_j$ in (9.67) can be replaced by
$$h + \sum \partial_j g_j, \quad g_j \in L^q(\Omega), \quad h \in L^p(\Omega), \quad q > n,\ p > \frac{n}{2}.$$
(*Hint*: Write $h = \sum \partial_j h_j$ for some $h_j \in L^q(\Omega)$.)
3. With L given by (9.1), consider
$$L_1 = L + X, \quad X = \sum A_j(x)\,\partial_j.$$
Show that in place of (9.4) and (9.6), we have
$$v = f(u) \Longrightarrow L_1 v = f'(u) L_1 u + f''(u)|\nabla u|^2$$
and
$$\frac{1}{2}\int \psi^2 |\nabla u|^2\, dV \le \int \Bigl(4|\nabla\psi|^2 + 2A\psi^2\Bigr)|u|^2\, dV - \int \psi^2 u(L_1 u)\, dV,$$
where $A(x)^2 = \sum A_j(x)^2$.
Extend the sup-norm estimate of Theorem 9.3 to this case, given $A_j \in L^\infty(\Omega)$.
4. With L given by (9.1), suppose u solves
$$Lu + \sum \partial_j\bigl(A_j(x)u\bigr) + C(x)u = g \ \text{ on } \Omega \in \mathbb{R}^n.$$
Supppose we have
$$A_j \in L^q(\Omega),\ C \in L^p(\Omega),\ g \in L^p(\Omega), \quad p > \frac{n}{2},\ q > n,$$
and suppose we also have
$$\|u\|_{H^1(\Omega)} + \|u\|_{L^\infty(\Omega)} \le K, \quad u\big|_{\partial\Omega} = f \in \operatorname{Lip}(\partial\Omega).$$
Show that, for some $\mu > 0$, $u \in C^\mu(\overline{\Omega})$. (*Hint*: Apply Theorem 9.7, together with Exercise 2.)

10. The Dirichlet problem for quasi-linear elliptic equations

The primary goal in this section is to establish the existence of smooth solutions to the Dirichlet problem for a quasi-linear elliptic PDE of the form

$$\sum F_{p_j p_k}(\nabla u)\partial_j \partial_k u = 0 \text{ on } \Omega, \quad u = \varphi \text{ on } \partial\Omega. \tag{10.1}$$

More general equations will also be considered. As noted in (7.32), this is the PDE satisfied by a critical point of the function

$$I(u) = \int_\Omega F(\nabla u)\, dx \tag{10.2}$$

defined on the space

$$V^1_\varphi = \{u \in H^1(\Omega) : u = \varphi \text{ on } \partial\Omega\}.$$

Assume $\varphi \in C^\infty(\overline{\Omega})$. We assume F is smooth and satisfies

$$A_1(p)|\xi|^2 \le \sum F_{p_j p_k}(p)\xi_j \xi_k \le A_2(p)|\xi|^2, \tag{10.3}$$

with $A_j : \mathbb{R}^n \to (0,\infty)$, continuous.

We use the method of continuity, showing that, for each $\tau \in [0,1]$, there is a smooth solution to

$$\Phi_\tau(D^2u) = 0 \text{ on } \Omega, \quad u = \varphi_\tau \text{ on } \partial\Omega, \tag{10.4}$$

where $\Phi_1(D^2u) = \Phi(D^2u)$ is the left side of (10.1) and $\varphi_1 = \varphi$. We arrange a situation where (10.4) is clearly solvable for $\tau = 0$. For example, we might take $\varphi_\tau \equiv \varphi$ and

$$\Phi_\tau(D^2u) = \tau\Phi(D^2u) + (1-\tau)\Delta u = \sum A^{jk}_\tau(\nabla u)\,\partial_j\partial_k u, \tag{10.5}$$

with

$$A^{jk}_\tau(p) = \partial_{p_j}\partial_{p_k}\Big[\tau F(p) + \frac{1}{2}(1-\tau)|p|^2\Big]. \tag{10.6}$$

Another possibility is to take

$$\Phi_\tau(D^2u) = \Phi(D^2u), \quad \varphi_\tau(x) = \tau\varphi(x), \tag{10.7}$$

since at $\tau = 0$ we have the solution $u = 0$ in this case.

Let J be the largest interval containing $\{0\}$ such that (10.7) has a solution $u = u_\tau \in C^\infty(\overline{\Omega})$ for each $\tau \in J$. We will show that J is all of $[0,1]$ by showing it is both open and closed in $[0,1]$. We will deal specifically with the method (10.5)–(10.6), but a similar argument can be applied to the method (10.7).

Demonstrating the openness of J is the relatively easy part.

Lemma 10.1. *If $\tau_0 \in J$, then, for some $\varepsilon > 0$, $[\tau_0, \tau_0 + \varepsilon) \subset J$.*

Proof. Fix k large and define

$$\Psi : [0,1] \times V_\varphi^k \longrightarrow H^{k-2}(\Omega) \tag{10.8}$$

by $\Psi(\tau, u) = \Phi_\tau(D^2 u)$, where

$$V_\varphi^k = \{u \in H^k(\Omega) : u = \varphi \text{ on } \partial\Omega\}. \tag{10.9}$$

This map is C^1, and its derivative with respect to the second argument is

$$D_2\Psi(\tau_0, u)v = Lv, \tag{10.10}$$

where

$$L : V_0^k = H^k \cap H_0^1 \longrightarrow H^{k-2}(\Omega) \tag{10.11}$$

is given by

$$Lv = \sum \partial_j A_{\tau_0}^{jk}(\nabla u(x))\, \partial_k v. \tag{10.12}$$

L is an elliptic operator with coefficients in $C^\infty(\overline{\Omega})$ when $u = u_{\tau_0}$, clearly an isomorphism in (10.11). Thus, by the inverse function theorem, for τ close enough to τ_0, there will be u_τ, close to u_{τ_0}, such that $\Psi(\tau, u_\tau) = 0$. Since $u_\tau \in H^k(\Omega)$ solves the regular elliptic boundary problem (10.4), if we pick k large enough, we can apply the regularity result of Theorem 8.4 to deduce $u_\tau \in C^\infty(\overline{\Omega})$.

The next task is to show that J is closed. This will follow from a sufficient a priori bound on solutions $u = u_\tau,\ \tau \in J$. We start with fairly weak bounds. First, the maximum principle implies

$$\|u\|_{L^\infty(M)} = \|\varphi\|_{L^\infty(\partial M)}, \tag{10.13}$$

for each $u = u_\tau,\ \tau \in J$.

Next we estimate derivatives. Each $w_\ell = \partial_\ell u$ satisfies

$$\sum \partial_j A^{jk}(\nabla u)\partial_k w_\ell = 0, \tag{10.14}$$

where $A^{jk}(\nabla u)$ is given by (10.6); we drop the subscript τ.

The next ingredient is a "boundary gradient estimate," of the form

$$|\nabla u(x)| \le K, \quad \text{for } x \in \partial\Omega, \tag{10.15}$$

As we have seen in the discussion of the minimal surface equation in § 7, whether this holds depends on the nature of the PDE and the region M. For now, we will make (10.15) a hypothesis. Then the maximum principle applied to (10.14) yields a uniform bound

$$\|\nabla u\|_{L^\infty(\Omega)} \le K. \tag{10.16}$$

For the next step of the argument, we will suppose for simplicity that $\overline{\Omega} = \mathbb{T}^{n-1} \times [0,1]$, for the present, and discuss the modification of the argument for the general case later. Under this assumption, in addition to (10.14), we also have

$$w_\ell = \partial_\ell \varphi \text{ on } \partial\Omega, \quad \text{for } 1 \le \ell \le n-1, \tag{10.17}$$

since ∂_ℓ is tangent to $\partial\Omega$ for $1 \le \ell \le n-1$.

Now we can say that Theorem 9.7 applies to $u_\ell = \partial_\ell u$, for $1 \le \ell \le n-1$. Thus there is an $r > 0$ for which we have bounds

$$\|w_\ell\|_{C^r(\overline{\Omega})} \le K, \quad 1 \le \ell \le n-1. \tag{10.18}$$

Let us note that Theorem 9.7 yields the bounds

$$\|\nabla w_\ell\|_{M_2^p(\Omega)} \le K', \quad 1 \le \ell \le n-1, \tag{10.19}$$

which are more precise than (10.18); here $1 - r = n/p$. Away from the boundary, such a property on *all* first derivatives of a solution to (10.1) leads to the applicability of Schauder estimates to establish interior regularity.

In the case of examining regularity at the boundary, more work is required since (10.18) does not include a derivative ∂_n transverse to the boundary. Now, using (10.4), we can solve for $\partial_n^2 u$ in terms of $\partial_j \partial_k u$, for $1 \le j \le n,\ 1 \le k \le n-1$. This will lead to the estimate

$$\|u\|_{C^{r+1}(\overline{\Omega})} \le K, \tag{10.20}$$

as we will now show.

In order to prove (10.20), note that, by (10.19),

$$\partial_k \partial_\ell u \in M_2^p(\Omega), \quad \text{for } 1 \le \ell \le n-1,\ 1 \le k \le n, \tag{10.21}$$

where $p \in (n, \infty)$ and $r \in (0,1)$ are related by $1 - r = n/p$. Now the PDE (10.4) enables us to write $\partial_n^2 u$ as a linear combination of the terms in (10.21), with $L^\infty(\Omega)$-coefficients. Hence

$$\partial_n^2 u \in M_2^p(\Omega), \tag{10.22}$$

so

$$\nabla(\partial_n u) \in M_2^p(\Omega) \subset M^p(\Omega). \tag{10.23}$$

Morrey's lemma (Lemma A.1) states that

$$\nabla v \in M^p(\Omega) \Longrightarrow v \in C^r(\overline{\Omega}) \quad \text{if } r = 1 - \frac{n}{p} \in (0,1). \tag{10.24}$$

Thus

$$\partial_n u \in C^r(\overline{\Omega}), \tag{10.25}$$

and this together with (10.18) yields (10.20). From this, plus the Morrey space inclusions (10.21)–(10.22), we have the hypothesis (8.60) of Theorem 8.4, with $r > 0$ and $\sigma = 1$. Thus, by Theorem 8.4, and the associated estimate (8.73), we deduce estimates

$$\|u\|_{H^k(\Omega)} \le K_k, \tag{10.26}$$

for $k = 2, 3, \dots$. Therefore, if $[0, \tau_1) \subset J$, as $\tau_\nu \nearrow \tau_1$, we can pick a subsequence of u_{τ_ν} converging weakly in $H^{k+1}(\Omega)$, hence strongly in $H^k(\Omega)$. If k is picked large enough, the limit u_1 is an element of $H^{k+1}(\Omega)$, solving (10.4) for $\tau = \tau_1$, and furthermore the regularity result Theorem 8.4 is applicable; hence $u_1 \in C^\infty(\overline{\Omega})$. This implies that J is closed.

Hence we have a proof of the solvability of the boundary problem (10.1), for the special case $\overline{\Omega} = \mathbb{T}^{n-1} \times [0,1]$, granted the validity of the boundary gradient estimate (10.15).

As noted, to have ∂_ℓ, $1 \le \ell \le n-1$, tangent to ∂M, we required $\overline{\Omega} = \mathbb{T}^{n-1} \times [0,1]$. For $\overline{\Omega} \subset \mathbb{R}^n$, if $X = \sum b_\ell \partial_\ell$ is a smooth vector field tangent to $\partial\Omega$, then $u_X = Xu$ solves, in place of (10.14),

$$\sum \partial_j A^{jk}(\nabla u)\, \partial_k u_X = \sum \partial_j F_j, \tag{10.27}$$

with $F_j \in L^\infty$ calculable in terms of ∇u. Thus Theorem 9.7 still applies, and the rest of the argument above extends easily. We have the following result.

Theorem 10.2. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying (10.3). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $\varphi \in C^\infty(\partial\Omega)$. Then the Dirichlet problem (10.1) has a unique solution $u \in C^\infty(\overline{\Omega})$, provided the boundary gradient estimate (10.15) is valid for all solutions $u = u_\tau$ to (10.4), for $\tau \in [0,1]$.*

Proof. Existence follows from the fact that J is open and closed in $[0,1]$, and nonempty, as $0 \in J$. Uniqueness follows from the maximum principle argument used to establish Proposition 7.2.

Let us record a result that implies uniqueness.

Proposition 10.3. *Let Ω be any bounded domain in $\mathbb{R}^n$. Assume that $u_\nu \in C^\infty(\Omega) \cap C(\overline{\Omega})$ are real-valued solutions to*

$$G(\nabla u_\nu, \partial^2 u_\nu) = 0 \text{ on } \Omega, \quad u_\nu = g_\nu \text{ on } \partial\Omega, \tag{10.28}$$

for $\nu = 1, 2$, where $G = G(p, \zeta)$, $\zeta = (\zeta_{jk})$. Then, under the ellipticity hypothesis

$$\sum \frac{\partial G}{\partial \zeta_{jk}}(p, \zeta)\, \xi_j \xi_k \geq A(p)|\xi|^2 > 0, \tag{10.29}$$

we have

$$g_1 \leq g_2 \text{ on } \partial\Omega \Longrightarrow u_1 \leq u_2 \text{ on } \overline{\Omega}. \tag{10.30}$$

Proof. Same as Proposition 7.2. As shown there, $v = u_2 - u_1$ satisfies the identity $Lv = G(\nabla u_2, \partial^2 u_2) - G(\nabla u_1, \partial^2 u_1)$, and L satisfies the conditions for the maximum principle, in the form of Proposition 2.1 of Chap. 5, given (10.29).

It is also useful to note that we can replace the first part of (10.28) by

$$G(\nabla u_2, \partial^2 u_2) \leq G(\nabla u_1, \partial^2 u_1), \tag{10.31}$$

and the maximum principle still yields the conclusion (10.30).

Since the boundary gradient estimate was verified in Proposition 7.5 for the minimal surface equation whenever $\Omega \subset \mathbb{R}^2$ has strictly convex boundary, we have existence of smooth solutions in that case. In fact, the proof of Proposition 7.5 works when $\Omega \subset \mathbb{R}^n$ is strictly convex, so that $\partial\Omega$ has positive Gauss curvature everywhere. We hence have the following result.

Theorem 10.4. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary that is strictly convex, then the Dirichlet problem*

$$\langle \nabla u \rangle^2 \Delta u - \sum_{j,k} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_j \partial x_k} = 0, \quad u = g \text{ on } \partial\Omega, \tag{10.32}$$

for a minimal hypersurface, has a unique solution $u \in C^\infty(\overline{\Omega})$, given $g \in C^\infty(\partial\Omega)$.

In Proposition 7.1, it was shown that when $n = 2$, the equation (10.32) has a solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$, and Proposition 7.2 showed that such a solution must be unique. Hence in the case $n = 2$, Theorem 10.4 implies the regularity at $\partial\Omega$ for this solution, given $\varphi \in C^\infty(\partial\Omega)$.

We now look at other cases where the boundary gradient estimate can be verified, by extending the argument used in Proposition 7.5. Some terminology is useful. Let us be given a nonlinear operator $F(D^2u)$, and $g \in C^\infty(\partial\Omega)$. We say a function $B_+ \in C^2(\Omega)$ is an *upper barrier* at $y \in \partial\Omega$ (for g), provided

$$\begin{aligned} F(D^2B_+) \le 0 \text{ on } \Omega, \qquad & B_+ \in C^1(\overline{\Omega}), \\ B_+ \ge g \text{ on } \partial\Omega, \qquad & B_+(y) = g(y). \end{aligned} \tag{10.33}$$

Similarly, we say $B_- \in C^2(\Omega)$ is a *lower barrier* at y (for g), provided

$$\begin{aligned} F(D^2B_-) \ge 0 \text{ on } \Omega, \qquad & B_- \in C^1(\overline{\Omega}), \\ B_- \le g \text{ on } \partial\Omega, \qquad & B_-(y) = g(y). \end{aligned} \tag{10.34}$$

An alternative expression is that g has an upper (or lower) barrier at y. Note well the requirement that $B_\pm$ belong to $C^1(\overline{\Omega})$. We say g has upper (resp., lower) barriers along $\partial\Omega$ if there are upper (resp., lower) barriers for g at each $y \in \partial\Omega$, with uniformly bounded $C^1(\overline{\Omega})$-norms. The following result parallels Proposition 7.5.

Proposition 10.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary. Consider a nonlinear differential operator of the form $F(D^2u) = G(\nabla u, \partial^2 u)$, satisfying the ellipticity hypothesis (10.29). Assume that g has upper and lower barriers along $\partial\Omega$, whose gradients are everywhere bounded by K. Then a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to $F(D^2u) = 0$, $u = g$ on $\partial\Omega$, satisfies*

$$|u(y) - u(x)| \le 2K|y - x|, \quad y \in \partial\Omega, \ x \in \overline{\Omega}. \tag{10.35}$$

If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then

$$|\nabla u(x)| \le 2K, \quad x \in \overline{\Omega}. \tag{10.36}$$

Proof. Same as Proposition 7.5. If $B_{\pm y}$ are the barriers for g at $y \in \partial\Omega$, then

$$B_{-y}(x) \le u(x) \le B_{+y}(x), \quad x \in \overline{\Omega},$$

which readily yields (10.35). Note that $w_\ell = \partial_\ell u$ satisfies the PDE

$$\sum \frac{\partial G}{\partial \zeta_{jk}} \partial_j \partial_k w_\ell + \sum \frac{\partial G}{\partial p_j} \partial_j w_\ell = 0 \quad \text{on } \Omega, \tag{10.37}$$

so the maximum principle yields (10.36).

Now, behind the specific implementation of Proposition 7.5 is the fact that when $\partial\Omega$ is strictly convex and $g \in C^\infty(\partial\Omega)$, there are *linear* functions $B_{\pm y}$, satisfying $B_{-y} \le g \le B_{+y}$ on $\partial\Omega$, $B_{-y}(y) = g(y) = B_{+y}(y)$, with bounded gradients. Such functions $B_{\pm y}$ are annihilated by operators of the form (10.1). Therefore, we have the following extension of Theorem 10.4.

Theorem 10.6. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary that is strictly convex, then the Dirichlet problem (10.1) has a unique solution $u \in C^\infty(\overline{\Omega})$, given $\varphi \in C^\infty(\partial\Omega)$, provided the ellipticity hypothesis (10.3) holds.*

We next consider the construction of upper and lower barriers when $F(D^2u) = \sum A^{jk}(\nabla u)\,\partial_j\partial_k u$ satisfies the *uniform* ellipticity condition

$$\lambda_0|\xi|^2 \le \sum A^{jk}(p)\xi_j\xi_k \le \lambda_1|\xi|^2, \tag{10.38}$$

for some $\lambda_j \in (0,\infty)$, independent of p. Given $z \in \mathbb{R}^n$, $R = |y-z|$, $\alpha \in (0,\infty)$, set

$$E_{y,z}(x) = e^{-\alpha r^2} - e^{-\alpha R^2}, \quad r^2 = |x-z|^2. \tag{10.39}$$

A calculation, used already in the derivation of maximum principles in §2 of Chap. 5, gives

$$\begin{aligned} &\sum A^{jk}(p)\,\partial_j\partial_k E_{y,z}(x) \\ &= e^{-\alpha r^2}\Big[4\alpha^2 A^{jk}(p)(x_j-z_j)(x_k-z_k) - 2\alpha A^j{}_j(p)\Big]. \end{aligned} \tag{10.40}$$

Under the hypothesis (10.38), we have

$$\sum A^{jk}(p)\,\partial_j\partial_k E_{y,z}(x) \ge 2\alpha e^{-\alpha r^2}\left[2\alpha\lambda_0|x-z|^2 - n\lambda_1\right]. \tag{10.41}$$

To make use of these functions, we proceed as follows. Given $y \in \partial\Omega$, pick $z = z(y) \in \mathbb{R}^n \setminus \overline{\Omega}$ such that y is the closest point to z on $\overline{\Omega}$. Given that $\overline{\Omega}$ is compact and $\partial\Omega$ is smooth, we can arrange that $|y-z| = R$, a positive constant, with the property that R^{-1} is greater than twice the absolute value of any principal curvature of $\partial\Omega$ at any point. Note that, for any choice of $\alpha > 0$, $E_{y,z}(y) = 0$ and $E_{y,z}(x) < 0$ for $x \in \overline{\Omega} \setminus \{y\}$. From (10.41) we see that if α is picked sufficiently large (namely, $\alpha > n\lambda_1/2R^2\lambda_0$), then

$$\sum A^{jk}(p)\,\partial_j\partial_k E_{y,z}(x) > 0, \quad x \in \overline{\Omega}, \tag{10.42}$$

for all p, since $|x-z| \ge R$. Now, given $g \in C^\infty(\partial\Omega)$, we can find $K \in (0,\infty)$ such that, for all $x \in \partial\Omega$,

$$B_{\pm y}(x) = g(y) \mp KE_{y,z}(x) \Longrightarrow B_{-y}(x) \le g(x) \le B_{+y}(x). \tag{10.43}$$

Consequently, we have upper and lower barriers for g along $\partial\Omega$. Therefore, we have the following existence theorem.

Theorem 10.7. *Let $\Omega \subset \mathbb{R}^n$ be any bounded region with smooth boundary. If the PDE (10.1) is uniformly elliptic, then (10.1) has a unique solution $u \in C^\infty(\overline{\Omega})$ for any $\varphi \in C^\infty(\partial\Omega)$.*

Certainly the equation (10.32) for minimal hypersurfaces is not uniformly elliptic. Here is an example of a uniformly elliptic equation. Take

$$F(p) = \left(\sqrt{1+|p|^2} - a\right)^2 = |p|^2 - 2a\sqrt{1+|p|^2} + 1 + a^2, \tag{10.44}$$

with $a \in (0,1)$. This models the potential energy of a stretched membrane, say a surface $S \subset \mathbb{R}^3$, given by $z = u(x)$, with the property that each point in S is constrained to move parallel to the z-axis. Compare with (1.5) in Chap. 2.

It is also natural to look at the variational equation for a stretched membrane for which gravity also contributes to the potential energy. Thus we replace $F(p)$ in (10.44) by

$$F^\#(u,p) = F(p) + au, \tag{10.45}$$

where a is a positive constant. This is of a form not encompassed by the class considered so far in this section. The PDE for u in this case has the form

$$\operatorname{div} F_p^\#(u,\nabla u) - F_u^\#(u,\nabla u) = 0, \tag{10.46}$$

which, when $F^\#(u,p)$ has the form (10.45), becomes

$$\sum F_{p_j p_k}(\nabla u)\,\partial_j\partial_k u - a = 0. \tag{10.47}$$

We want to extend the existence argument to this case, to produce a solution $u \in C^\infty(\overline{\Omega})$, with given boundary data $\varphi \in C^\infty(\partial\Omega)$. Using the continuity method, we need estimates parallel to (10.13)–(10.20). Now, since $a > 0$, the maximum principle implies

$$\sup_{x\in\overline{\Omega}} u(x) = \sup_{y\in\partial\Omega} \varphi(y). \tag{10.48}$$

To estimate $\|u\|_{L^\infty}$, we also need control of $\inf_\Omega u(x)$. Such an estimate will follow if we obtain an estimate on $\|\nabla u\|_{L^\infty(\Omega)}$. To get this, note that the equation (10.14) for $w_\ell = \partial_\ell u$ continues to hold. Again the maximum principle applies, so the boundary gradient estimate (10.15) continues to imply (10.16). Furthermore, the construction of upper and lower barriers in (10.39)–(10.43) is easily extended, so one has such a boundary gradient estimate.

Now one needs to apply the DeGiorgi–Nash–Moser theory. Since (10.14) continues to hold, this application goes through without change, to yield (10.20), and the argument producing (10.26) also goes through as before. Thus Theorem 10.7 extends to PDE of the form (10.47).

One might consider more general force fields, replacing the potential energy function (10.45) by

$$F^{\#}(u, p) = F(p) + V(u). \tag{10.49}$$

Then the PDE for u becomes

$$\sum F_{p_j p_k}(\nabla u)\partial_j \partial_k u - V'(u) = 0. \tag{10.50}$$

In this case, $w_\ell = \partial_\ell u$ satisfies

$$\sum \partial_j A^{jk}(\nabla u)\partial_k w_\ell - V''(u) w_\ell = 0. \tag{10.51}$$

This time, we won't start with an estimate on $\|u\|_{L^\infty}$, but we will aim directly for an estimate on $\|\nabla u\|_{L^\infty}$, which will serve to bound $\|u\|_{L^\infty}$, given that $u = \varphi$ on $\partial\Omega$.

The maximum principle applies to (10.51), to yield

$$\|\nabla u\|_{L^\infty(\Omega)} = \sup_{y \in \partial\Omega} |\nabla u(y)|, \text{ provided } V''(u) \ge 0. \tag{10.52}$$

Next, we check whether the barrier construction (10.39)–(10.43) yields a boundary gradient estimate in this case. Having (10.43) (with $g = \varphi$), we want

$$H(D^2 B_{+y}) \le H(D^2 u) \le H(D^2 B_{-y}) \text{ on } \Omega, \tag{10.53}$$

in place of (10.42), where $H(D^2 u)$ is given by the left side of (10.50), and we want this sequence of inequalities together with (10.43) to yield

$$B_{-y}(x) \le u(x) \le B_{+y}(x), \quad x \in \overline{\Omega}. \tag{10.54}$$

To obtain (10.53), note that we can arrange the left side of (10.42) to exceed a large constant, and also a large multiple of $E_{y,z}(x)$. Note that the middle quentity in (10.53) is zero, so we want $H(D^2 B_{+y}) \le 0$ and $H(D^2 B_{-y}) \ge 0$, on Ω. We can certainly achieve this under the hypothesis that there is an estimate

$$|V'(u)| \le A_1 + A_2 |u|. \tag{10.55}$$

In such a case, we have (10.53). To get (10.54) from this, we use the following extension of Proposition 10.3.

Proposition 10.8. *Let $\Omega \subset \mathbb{R}^n$ be bounded. Consider a nonlinear differential operator of the form*

$$H(x, D^2 u) = G(x, u, \nabla u, \partial^2 u), \tag{10.56}$$

where $G(x, u, p, \zeta)$ satisfies the ellipticity hypothesis (10.29), and

$$\partial_u G(x, u, p, \zeta) \le 0. \tag{10.57}$$

Then, given $u_\nu \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$H(D^2 u_2) \le H(D^2 u_1) \text{ on } \Omega,\ u_1 \le u_2 \text{ on } \partial\Omega \Longrightarrow u_1 \le u_2 \text{ on } \overline{\Omega}. \tag{10.58}$$

Proof. Same as Proposition 10.3. For the relevant maximum principle, replace Proposition 2.1 of Chap. 5 by Proposition 2.6 of that chapter.

To continue our analysis of the PDE (10.50), Proposition 10.8 applies to give (10.53) $\Rightarrow$ (10.54), provided $V''(u) \ge 0$. Consequently, we achieve a bound on $\|\nabla u\|_{L^\infty(\Omega)}$, and hence also on $\|u\|_{L^\infty(\Omega)}$, provided $V(u)$ satisfies the hypotheses stated in (10.52) and (10.55).

It remains to apply the DeGiorgi–Nash–Moser theory. In the simplified case where $\overline{\Omega} = \mathbb{T}^{n-1} \times [0, 1]$, we obtain (10.18), this time by regarding (10.51) as a nonhomogeneous PDE for w_ℓ, of the form (9.67), with one term $\partial_j g_j$, namely $\partial_\ell V'(u)$. The L^∞-estimate we have on u is more than enough to apply Theorem 9.7, so we again have (10.18)–(10.19). Next, the argument (10.21)–(10.23) goes through, so we again have (10.20) and the Morrey space inclusions (10.21)–(10.22). Hence the hypothesis (8.60) of Theorem 8.4 holds, with $r > 0$ and $\sigma = 1$. Theorem 8.4 yields

$$\|u\|_{H^k(\Omega)} \le K_k, \tag{10.59}$$

and a modification of the argument parallel to the use of (10.27) works for $\Omega \subset \mathbb{R}^n$.

The estimates above work for

$$\tau \sum F_{p_j p_k}(\nabla u)\partial_j\partial_k u - \tau V'(u) + (1-\tau)\Delta u = 0, \quad u\big|_{\partial\Omega} = \varphi, \tag{10.60}$$

for all $\tau \in [0, 1]$. Also, each linearized operator is seen to be invertible, provided $V''(u) \ge 0$. Thus all the ingredients needed to use the method of continuity are in place. We have the following existence result.

Proposition 10.9. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain with smooth boundary. If the PDE*

$$\sum F_{p_j p_k}(\nabla u)\partial_j\partial_k u - V'(u) = 0, \quad u = \varphi \text{ on } \partial\Omega, \tag{10.61}$$

is uniformly elliptic, and if $V'(u)$ satisfies

$$|V'(u)| \le A_1 + A_2|u|, \quad V''(u) \ge 0, \tag{10.62}$$

then (10.61) has a unique solution $u \in C^\infty(\overline{\Omega})$, given $\varphi \in C^\infty(\partial\Omega)$.

Consider the case $V(u) = Au^2$. This satisfies (10.62) if $A \geq 0$ but not if $A < 0$. The case $A < 0$ corresponds to a repulsive force (away from $u = 0$) that increases linearly with distance. The physical basis for the failure of (10.61) to have a solution is that if $u(x)$ takes a large enough value, the repulsive force due to the potential V cannot be matched by the elastic force of the membrane. If $F_{p_j p_k}(p)$ is independent of p and $2A < 0$ is an eigenvalue of the linear operator $\sum F_{p_j p_k} \partial_j \partial_k$, then certainly (10.61) is not solvable.

On the other hand, if $V(u) = Au^2$ with $0 > A > -\ell_0$, where ℓ_0 is less than the smallest eigenvalue of all operators $\sum A^{jk} \partial_j \partial_k$ with coefficients satisfying (10.38), then one can still hope to establish solvability for (10.61), in the uniformly elliptic case. We will not pursue the details on such existence results.

We now consider more general equations, of the form

$$H(D^2 u) = \sum F_{p_j p_k}(\nabla u)\, \partial_j \partial_k u + g(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \varphi. \tag{10.63}$$

Consider the family

$$H_\tau(D^2 u) = \sum F_{p_j p_k}(\nabla u)\, \partial_j \partial_k u + \tau g(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \tau\varphi. \tag{10.64}$$

We will prove the following:

Proposition 10.10. *Assume that the equation (10.63) satisfies the ellipticity condition (10.3) and that $\partial_u g(x, u, p) \leq 0$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\varphi \in C^\infty(\partial\Omega)$ be given. Assume that, for $\tau \in [0, 1]$, any solution $u = u_\tau$ to (10.64) has an a priori bound in $C^1(\overline{\Omega})$. Then (10.63) has a solution $u \in C^\infty(\overline{\Omega})$.*

Proof. For $w_\ell = \partial_\ell u$, we have, in place of (10.14),

$$\sum \partial_j A^{jk}(\nabla u)\, \partial_k w_\ell = -\partial_\ell g(x, u, \nabla u). \tag{10.65}$$

The C^1-bound on u yields an L^∞-bound on $g(x, u, \nabla u)$, so, as in the proof of Proposition 10.9, we can use Theorem 9.7 and proceed from there to obtain high-order Sobolev estimates on solutions to (10.64).

Thus the largest interval J in $[0, 1]$ that contains $\tau = 0$ and such that (10.64) is solvable for all $\tau \in J$ is closed. The hypothesis $\partial_u g \leq 0$ implies that the linearized equation at $\tau = \tau_0$ is uniquely solvable, so, as in Lemma 10.1, J is open in $[0, 1]$, and the proposition is proved.

A simple example of (10.63) is the equation for a surface $z = u(x)$ of given constant mean curvature H:

$$\langle \nabla u \rangle^{-3} \Big[\langle \nabla u \rangle^2 \Delta u - D^2 u(\nabla u, \nabla u) \Big] + nH = 0, \quad u = \varphi \text{ on } \partial\Omega, \tag{10.66}$$

which is of the form (10.63), with $F(p) = \big(1 + |p|^2\big)^{1/2}$ and $g(x, u, p) = nH$. Note that members of the family (10.64) are all of the same type in this case, namely equations for surfaces with mean curvature τH. We see that Proposition 10.3 applies to this equation. This implies uniqueness of solutions to (10.66), provided they exist, and also gives a tool to estimate L^∞-norms, at least in some cases, by using equations of graphs of spheres of radius $1/H$ as candidates to bound u from above and below. We can also use such functions to construct barriers, replacing the linear functions used in the proof of Proposition 7.5. This change means that the class of domains and boundary data for which upper and lower barriers can be constructed is different when $H \neq 0$ than it is in the minimal surface case $H = 0$.

Note that if u solves (10.66), then $w_\ell = \partial_\ell u$ solves a PDE of the form (10.14). Thus the maximum principle yields $\|\nabla u\|_{L^\infty(\Omega)} = \sup_{\partial\Omega} |\nabla u(y)|$. Consequently, we have the solvability of (10.66) whenever we can construct barriers to prove the boundary gradient estimate.

The methods for constructing barriers described above do not exhaust the results one can obtain on boundary gradient estimates, which have been pushed quite far. We mention a result of H. Jenkins and J. Serrin. They have shown that the Dirichlet problem (10.66) for surfaces of constant mean curvature H is solvable for arbitrary $\varphi \in C^\infty(\partial\Omega)$ if and only if the mean curvature $\varkappa(y)$ of $\partial\Omega \subset \mathbb{R}^n$ satisfies

$$\varkappa(y) \geq \frac{n}{n-1}|H|, \quad \forall\, y \in \partial\Omega. \tag{10.67}$$

In the special case $n = 2, H = 0$, this implies Proposition 7.3 in this chapter. See [GT] and [Se2] for proofs of this and extensions, including variable mean curvature $H(x)$, as well as extensive general discussions of boundary gradient estimates. We will have a little more practice constructing barriers and deducing boundary gradient estimates in §§ 13 and 15 of this chapter. See the proofs of Lemma 13.12 and of the estimate (15.54).

Results discussed above extend to more general second-order, scalar, quasi-linear PDE. In particular, Proposition 10.10 can be extended to all equations of the form

$$\sum a_{jk}(x, u, \nabla u)\, \partial_j \partial_k u + b(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \varphi. \tag{10.68}$$

Let $\varphi \in C^\infty(\partial\Omega)$ be given. As long as it can be shown that, for each $\tau \in [0, 1]$, a solution to

$$\sum a_{jk}(x, u, \nabla u)\, \partial_j \partial_k u + \tau b(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \tau\varphi, \tag{10.69}$$

has an a priori bound in $C^1(\overline{\Omega})$, then (10.68) has a solution $u \in C^\infty(\overline{\Omega})$. This result, due to O. Ladyzhenskaya and N. Ural'tseva, is proved in [GT] and [LU]. These references, as well as [Se2], also discuss conditions under which one can establish a boundary gradient estimate for solutions to such PDE, and when one can pass from that to a $C^1(\overline{\Omega})$-estimate on solutions. The DeGiorgi–Nash–Moser estimates are still a major analytical tool in the proof of this general result, but further work is required beyond what was used to prove Proposition 10.10.

Exercises

1. Carry out the construction of barriers for the equation of a surface of constant mean curvature mentioned below (10.66) and thus obtain some existence results for this equation. Compare these results with the result of Jenkins and Serrin, stated in (10.67).

Exercises 2–4 deal with quasi-linear elliptic equations of the form

$$\sum \partial_j A^{jk}(x,u)\partial_k u = 0 \quad \text{on } \Omega, \quad u\big|_{\partial\Omega} = \varphi. \tag{10.70}$$

Assume there are positive functions A_j such that

$$A_1(u)|\xi|^2 \le \sum A^{jk}(x,u)\xi_j\xi_k \le A_2(u)|\xi|^2.$$

2. Fix $\varphi \in C^\infty(\partial\Omega)$. Consider the operator $\Phi(u) = v$, the solution to

$$\sum \partial_j A^{jk}(x,u)\partial_k v = 0, \quad v\big|_{\partial\Omega} = \varphi.$$

Show that, for some $r > 0$,

$$\Phi : C(\overline{\Omega}) \longrightarrow C^r(\overline{\Omega}),$$

continuously. Use the Schauder fixed-point theorem to deduce that Φ has a fixed point in $\{u \in C(\overline{\Omega}) : \sup |u| \le \sup |\varphi|\} \cap C^r(\overline{\Omega})$.
3. Show that this fixed point lies in $C^\infty(\overline{\Omega})$.
4. Examine whether solutions to (10.70) are unique.
5. Extend results on (10.1) to the case

$$\sum \partial_j F_{p_j}(x,\nabla u) = 0, \quad u\big|_{\partial\Omega} = \varphi, \tag{10.71}$$

arising from the search for critical points of $I(u) = \int_\Omega F(x,\nabla u)\,dx$, generalizing the case considered in (10.2).

In Exercises 6–9, we consider a PDE of the form

$$\sum \partial_j a^j(x,u,\nabla u) + b(x,u) = 0 \ \text{ on } \ \Omega. \tag{10.72}$$

We assume a^j and b are smooth in their arguments and

$$|a^j(x,u,p)| \le C(u)\langle p\rangle, \quad |\nabla_p a^j(x,u,p)| \le C(u).$$

We make the ellipticity hypothesis

$$\sum \frac{\partial a^j}{\partial p_k}(x,u,p)\xi_j\xi_k \ge A(u)|\xi|^2, \quad A(u) > 0.$$

6. Show that if $u \in H^1(\Omega) \cap L^\infty(\Omega)$ solves (10.72), then u solves a PDE of the form

$$\sum \partial_j A^{jk}(x)\partial_k u + \partial_j c^j(x,u) + b(x,u) = 0,$$

with

$$A^{jk} \in L^\infty, \quad \sum A^{jk}(x)\xi_j\xi_k \geq A|\xi|^2.$$

(*Hint*: Start with

$$a^j(x,u,p) = a^j(x,u,0) + \sum_k \widetilde{A}^{jk}(x,u,p)p_k,$$
$$\widetilde{A}^{jk}(x,u,p) = \int_0^1 \frac{\partial a^j}{\partial p_k}(x,u,sp)\,ds.)$$

7. Deduce that if $u \in H^1(\Omega) \cap L^\infty(\Omega)$ solves (10.72), then u is Hölder continuous on the interior of Ω.
8. If Ω is a smooth, bounded region in $\mathbb{R}^n$ and $u \in H^1(\Omega) \cap L^\infty(\Omega)$ satisfies (10.72) and $u\big|_{\partial\Omega} = \varphi \in C^1(\partial\Omega)$, show that u is Hölder continuous on $\overline{\Omega}$ and that $\nabla u \in M_2^q(\Omega)$, for some $q > n$.
9. If $u \in C^2(\overline{\Omega})$ satisfies (10.72), show that $u_\ell = \partial_\ell u$ satisfies

$$\begin{aligned}&\partial_j a^j_{p_k}(x,u,\nabla u)\,\partial_k u_\ell + \partial_j\big[a^j_u(x,u,\nabla u)u_\ell\big]\\ &+\partial_j a^j_{x_\ell}(x,u,\nabla u) + b_u(x,u)u_\ell + b_{x_\ell}(x,u) = 0.\end{aligned}$$

Discuss obtaining estimates on u in $C^{1+r}(\overline{\Omega})$, given estimates on u in $C^1(\overline{\Omega})$.

11. Direct methods in the calculus of variations

We study the existence of minima (or other stationary points) of functionals of the form

$$I(u) = \int_\Omega F(x,u,\nabla u)\,dV(x), \tag{11.1}$$

on some set of functions, such as $\{u \in B : u = g \text{ on } \partial\Omega\}$, where B is a suitable Banach space of functions on Ω, possibly taking values in $\mathbb{R}^N$, and g is a given smooth function on $\partial\Omega$. We assume $\overline{\Omega}$ is a compact Riemannian manifold with boundary and

$$F : \mathbb{R}^N \times (\mathbb{R}^N \otimes T^*\overline{\Omega}) \longrightarrow \mathbb{R} \text{ is continuous.} \tag{11.2}$$

Let us begin with a fairly direct generalization of the hypotheses (1.3)–(1.8) made in § 1. Thus, let

$$V = \{u \in H^1(\Omega, \mathbb{R}^N) : u = g \text{ on } \partial\Omega\}. \tag{11.3}$$

For now, we assume that, for each $x \in \overline{\Omega}$,

$$F(x, \cdot, \cdot) : \mathbb{R}^N \times (\mathbb{R}^N \otimes T_x^*\overline{\Omega}) \longrightarrow \mathbb{R} \text{ is convex,} \tag{11.4}$$

where the domain has its natural linear structure. We also assume

$$A_0|\xi|^2 - B_0|u| - C_0 \le F(x, u, \xi), \tag{11.5}$$

for some positive constants $A_0,\ B_0,\ C_0$, and

$$|F(x, u, \xi) - F(x, v, \zeta)| \le C\big(|u - v| + |\xi - \zeta|\big)\big(|\xi| + |\zeta| + 1\big). \tag{11.6}$$

These hypotheses will be relaxed below.

Proposition 11.1. *Assume Ω is connected, with nonempty boundary. Assume $I(u) < \infty$ for some $u \in V$. Under the hypotheses (11.2)–(11.6), I has a minimum on V.*

Proof. As in the situation dealt with in Proposition 1.2, we see that $I : V \to \mathbb{R}$ is Lipschitz continuous, bounded below, and convex. Thus, if $\alpha_0 = \inf_V\ I(u)$, then

$$K_\varepsilon = \{u \in V : \alpha_0 \le I(u) \le \alpha_0 + \varepsilon\} \tag{11.7}$$

is, for each $\varepsilon \in (0, 1]$, a nonempty, closed, convex subset of V. Hence K_ε is weakly compact in $H^1(\Omega, \mathbb{R}^N)$. Hence $\bigcap_{\varepsilon>0} K_\varepsilon = K_0 \ne \emptyset$, and inf $I(u)$ is assumed on K_0.

We will state a rather general result whose proof is given by the argument above.

Proposition 11.2. *Let V be a closed, convex subset of a reflexive Banach space W, and let $\Phi : V \to \mathbb{R}$ be a continuous map, satisfying:*

$$\inf_V\ \Phi = \alpha_0 \in (-\infty, \infty), \tag{11.8}$$

$$\exists\, b > \alpha_0 \text{ such that } \Phi^{-1}\big([\alpha_0, b]\big) \text{ is bounded in } W, \tag{11.9}$$

$$\forall\ y \in (\alpha_0, b], \quad \Phi^{-1}\big([\alpha_0, y]\big) \text{ is convex.} \tag{11.10}$$

Then there exists $v \in V$ such that $\Phi(v) = \alpha_0$.

As above, the proof comes down to the observation that, for $0 < \varepsilon \le b - \alpha_0,\ K_\varepsilon$ is a nested family of subsets of W that are compact when W has the weak topology. This result encompasses such generalizations of Proposition 11.1 as the following. Given $p \in (1, \infty),\ g \in C^\infty(\partial\Omega, \mathbb{R}^N)$, let

$$V = \{u \in H^{1,p}(\Omega, \mathbb{R}^N) : u = g \text{ on } \partial\Omega\}. \tag{11.11}$$

We continue to assume (11.4), but replace (11.5) and (11.6) by

$$A_0|\xi|^p - B_0|u| - C_0 \le F(x,u,\xi), \tag{11.12}$$

for some positive A_0, B_0, C_0, and

$$|F(x,u,\xi) - F(x,v,\zeta)| \le C\big(|u-v| + |\xi-\zeta|\big)\big(|\xi| + |\zeta| + 1\big)^{p-1}. \tag{11.13}$$

Then we have the following:

Proposition 11.3. *Assume Ω is connected, with nonempty boundary. Take $p \in (1,\infty)$, and assume $I(u) < \infty$ for some $u \in V$. Under the hypotheses (11.2), (11.4), and (11.11)–(11.13), I has a minumum on V.*

It is useful to extend Propositions 11.1 and 11.3, replacing (11.4) by a hypothesis of convexity only in the last set of variables.

Proposition 11.4. *Make the hypotheses of Proposition 11.1, or more generally of Proposition 11.3, but weaken (11.4) to the hypothesis that*

$$F(x,u,\cdot) : \mathbb{R}^N \otimes T_x^*\overline{\Omega} \longrightarrow \mathbb{R} \text{ is convex,} \tag{11.14}$$

for each $(x,u) \in \overline{\Omega} \times \mathbb{R}^N$. Then I has a minimum on V.

Proof. Let $\alpha_0 = \inf_V \; I(u)$. The hypothesis (11.12) plus Poincaré's inequality imply that $\alpha_0 > -\infty$ and that

$$B = \{u \in V : I(u) \le \alpha_0 + 1\} \text{ is bounded in } H^{1,p}(\Omega,\mathbb{R}^N). \tag{11.15}$$

Pick $u_j \in B$ so that $I(u_j) \to \alpha_0$. Passing to a subsequence, we can assume

$$u_j \to u \text{ weakly in } H^{1,p}(\Omega,\mathbb{R}^N). \tag{11.16}$$

Hence $u_j \to u$ strongly in $L^p(\Omega,\mathbb{R}^N)$. We want to show that

$$I(u) = \alpha_0. \tag{11.17}$$

To this end, set

$$\Phi(u,v) = \int_\Omega F(x,u,v)\, dV(x). \tag{11.18}$$

With $v_j = \nabla u_j$, we have

$$\Phi(u_j,v_j) \to \alpha_0. \tag{11.19}$$

Also $v_j \to v = \nabla u$ weakly in $L^p(\Omega,\mathbb{R}^N \otimes T^*)$.

We can conclude that $I(u) \le \alpha_0$, and hence (11.17) holds if we show that

$$\Phi(u, v) \le \alpha_0. \tag{11.20}$$

Now, by hypothesis (11.13) we have

$$\begin{aligned} |\Phi(u_j, v_j) - \Phi(u, v_j)| &\le C \int_\Omega |u_j - u|\big(|v_j| + 1\big)^{p-1}\, dV(x) \\ &\le C' \|u_j - u\|_{L^p(\Omega)}, \end{aligned} \tag{11.21}$$

so

$$\Phi(u, v_j) \longrightarrow \alpha_0. \tag{11.22}$$

This time, by (11.5), (11.6), and (11.14) we have that, for each $\varepsilon \in (0, 1]$,

$$\mathcal{K}_\varepsilon = \{w \in L^p(\Omega, \mathbb{R}^N \otimes T^*) : \Phi(u, w) \le \alpha_0 + \varepsilon\} \tag{11.23}$$

is a closed, convex subset of $L^p(\Omega, \mathbb{R}^N \otimes T^*)$. Hence $\mathcal{K}_\varepsilon$ is weakly compact, provided it is nonempty. Furthermore, by (11.22), $v_j \in \mathcal{K}_{\varepsilon_j}$ with $\varepsilon_j \to 0$, so we have $v \in \mathcal{K}_0$. This implies (11.20), so Proposition 11.4 is proved.

The following extension of Proposition 11.4 applies to certain constrained minimization problems.

Proposition 11.5. *Let $p \in (1, \infty)$, and let $F(x, u, \xi)$ satisfy the hypotheses of Proposition 11.4. Then, if S is any subset of V (given by (11.11)) that is closed in the weak topology of $H^{1,p}(\Omega, \mathbb{R}^N)$, it follows that $I\big|_S$ has a minimum in S.*

Proof. Let $\alpha_0 = \inf_S\ I(u)$, and take $u_j \in S$, $I(u_j) \to \alpha_0$. Since (11.15) holds, we can take a subsequence $u_j \to u$ weakly in $H^{1,p}(\Omega, \mathbb{R}^N)$, so $u \in S$. We want to show that $I(u) = \alpha_0$. Indeed, if we form $\Phi(u, v)$ as in (11.18), then the argument involving (11.19)–(11.23) continues to hold, and our assertion is proved.

For example, if $X \subset \mathbb{R}^N$ is a closed subset, we could take

$$S = \{u \in V : u(x) \in X \text{ for a.e. } x \in \Omega\}, \tag{11.24}$$

and Proposition 11.5 applies. As a specific example, X could be a compact Riemannian manifold, isometrically imbedded in $\mathbb{R}^N$, and we could take $p = 2$, $F(x, u, \nabla u) = |\nabla u|^2$. The resulting minimum of $I(u)$ is a harmonic map of $\overline{\Omega}$ into X. If $u : \Omega \to X$ is a harmonic map, it satisfies the PDE

$$\Delta u - \Gamma(u)(\nabla u, \nabla u) = 0, \tag{11.25}$$

where $\Gamma(u)(\nabla u, \nabla u)$ is a certain quadratic form in ∇u. See §2 of Chap. 15 for a derivation.

A generalization of the notion of harmonic map arises in the study of "liquid crystals." One takes

$$F(x,u,\nabla u) = a_1|\nabla u|^2 + a_2(\operatorname{div} u)^2 + a_3(u\cdot\operatorname{curl} u)^2 + a_4|u\times\operatorname{curl} u|^2, \tag{11.26}$$

where the coefficients a_j are positive constants, and then one minimizes the functional $\int_\Omega F(x,u,\nabla u)\,dV(x)$ over a set S of the form (11.24), with $X = S^2 \subset \mathbb{R}^3$, namely, over

$$S = \{u \in H^1(\Omega,\mathbb{R}^3) : |u(x)| = 1 \text{ a.e. on } \Omega,\ u = g \text{ on } \partial\Omega\}. \tag{11.27}$$

In this case, $F(x,u,\xi)$ has the form

$$F(x,u,\xi) = \sum_{j,\alpha} b_{j\alpha}(u)\xi_{j\alpha}^2, \quad b_{j,\alpha}(u) \ge a_1 > 0,$$

where each coefficient $b_{j\alpha}(u)$ is a polynomial of degree 2 in u. Clearly, this function is convex in ξ. The function $F(x,u,\xi)$ does not satisfy (11.6); hence, in going through the argument establishing Proposition 11.4, we would need to replace the $p = 2$ case of (11.22) by

$$|\Phi(u_j,v_j) - \Phi(u,v_j)| \le C\int_\Omega |u_j - u|\cdot|v_j|^2\,dV(x). \tag{11.28}$$

The following result covers integrands of the form (11.26), as well as many others. It assumes a slightly bigger lower bound on F than the previous results, but it greatly relaxes the hypotheses on how rapidly F can vary.

Theorem 11.6. *Assume Ω is connected, with nonempty boundary. Take $p \in (1,\infty)$, and set*

$$V = \{u \in H^{1,p}(\Omega,\mathbb{R}^N) : u = g \text{ on } \partial\Omega\}.$$

Assume $I(u) < \infty$ for some $u \in V$. Assume that $F(x,u,\xi)$ is smooth in its arguments and satisfies the convexity condition (11.14) in ξ and the lower bound

$$A_0|\xi|^p \le F(x,u,\xi), \tag{11.29}$$

for some $A_0 > 0$. Then I has a minimum on V.

Also, if S is a subset of V that is closed in the weak topology of $H^{1,p}(\Omega,\mathbb{R}^N)$, then $I\big|_S$ has a minimum in S.

Proof. Clearly, $\alpha_0 = \inf_S I(u) \geq 0$. With B as in (11.15), pick $u_j \in B \cap S$ so that

$$I(u_j) \to \alpha_0, \quad u_j \to u \text{ weakly in } H^{1,p}(\Omega, \mathbb{R}^N). \tag{11.30}$$

Passing to a subsequence, we can assume $u_j \to u$ a.e. on Ω. We need to show that

$$\int_\Omega F(x, u, \nabla u)\, dV \leq \alpha_0. \tag{11.31}$$

By Egorov's theorem, we can pick measurable sets $E_\nu \supset E_{\nu+1} \supset \cdots$ in Ω, of measure $< 2^{-\nu}$, such that $u_j \to u$ uniformly on $\Omega \setminus E_\nu$. We can also arrange that

$$|u(x)| + |\nabla u(x)| \leq C \cdot 2^\nu, \ \text{ for } \ x \in \Omega \setminus E_\nu. \tag{11.32}$$

Now, we have

$$\begin{aligned} \int_{\Omega \setminus E_\nu} F(x, u, \nabla u)\, dV = & \int_{\Omega \setminus E_\nu} F(x, u_j, \nabla u_j)\, dV \\ & + \int_{\Omega \setminus E_\nu} \big[F(x, u_j, \nabla u) - F(x, u_j, \nabla u_j)\big]\, dV \\ & + \int_{\Omega \setminus E_\nu} \big[F(x, u, \nabla u) - F(x, u_j, \nabla u)\big]\, dV. \end{aligned} \tag{11.33}$$

To estimate the second integral on the right side of (11.33), we use the convexity hypothesis to write

$$F(x, u_j, \nabla u) - F(x, u_j, \nabla u_j) \leq D_\xi F(x, u_j, \nabla u) \cdot (\nabla u - \nabla u_j). \tag{11.34}$$

Now, for each ν,

$$D_\xi F(x, u_j, \nabla u) \longrightarrow D_\xi F(x, u, \nabla u), \ \text{ uniformly on } \ \Omega \setminus E_\nu, \tag{11.35}$$

while $\nabla u - \nabla u_j \to 0$ weakly in $L^p(\Omega, \mathbb{R}^n)$, so

$$\lim_{j \to \infty} \int_{\Omega \setminus E_\nu} \big[F(x, u_j, \nabla u) - F(x, u_j, \nabla u_j)\big]\, dV = 0. \tag{11.36}$$

Estimating the last integral in (11.33) is easy, since

$$F(x, u, \nabla u) - F(x, u_j, \nabla u) \longrightarrow 0, \quad \text{uniformly on } \ \Omega \setminus E_\nu. \tag{11.37}$$

Thus, from our analysis of (11.33), we have

$$\int_{\Omega\setminus E_\nu} F(x,u,\nabla u)\,dV \le \limsup_{j\to\infty} \int_{\Omega\setminus E_\nu} F(x,u_j,\nabla u_j)\,dV \le \alpha_0, \tag{11.38}$$

for all ν, and taking $\nu \to \infty$ gives (11.31). The theorem is proved.

There are a number of variants of the results above. We mention one:

Proposition 11.7. *Assume that F is smooth in (x,u,ξ), that*

$$F(x,u,\xi) \ge 0, \tag{11.39}$$

and that

$$F(x,u,\cdot) : \mathbb{R}^N \otimes T_x^*\overline{\Omega} \longrightarrow \mathbb{R} \text{ is convex,} \tag{11.40}$$

for each x, u. Suppose

$$u_\nu \to u \text{ weakly in } H^{1,1}_{loc}(\Omega,\mathbb{R}^N). \tag{11.41}$$

Then

$$I(u) \le \liminf_{\nu\to\infty} I(u_\nu). \tag{11.42}$$

For a proof, and other extensions, see [Gia] or [Dac]. It is a result of J. Serrin [Se1] that, in the case where u is real-valued, the hypothesis (11.41) can be weakened to

$$u_\nu, u \in H^{1,1}_{\mathrm{loc}}(\Omega), \quad u_\nu \to u \text{ in } L^1_{\mathrm{loc}}(\Omega). \tag{11.43}$$

In [Mor2] there is an attempt to extend Serrin's result to systems, but it was shown by [Eis] that such an extension is false.

In [Dac] there is also a discussion of a replacement for convexity, due to Morrey, called "quasi-convexity." For other contexts in which the convexity hypothesis is absent, and one often looks not for a minimizer but some sort of saddle point, see [Str2] and [Gia2].

In this section we have obtained solutions to extremal problems, but these solutions lie in Sobolev spaces with rather low regularity. The problem of higher regularity for such solutions is considered in § 12.

Exercises

1. In Theorem 11.6, take $p > n = \dim \Omega = N$, and consider

$$S = \{u \in V : \det Du = 1, \text{ a.e. on } \Omega\}.$$

Show that S is closed in the weak topology of $H^{1,p}(\Omega, \mathbb{R}^n)$ and hence that Theorem 11.6 applies. (*Hint*: See (6.35)–(6.36) of Chap. 13.)

2. In Theorem 11.6, take $p \in (1, \infty)$, $\Omega \subset \mathbb{R}^n$, $N = 1$. Let $h \in C^\infty(\overline{\Omega})$, and consider

$$S = \{u \in V : u \geq h \text{ on } \Omega\}.$$

Show that S is closed in the weak topology of $H^{1,p}(\Omega)$ and hence that Theorem 11.6 applies.

Say $I\big|_S$ achieves its minimum at u, and suppose you are given that $u \in C(\Omega)$, so

$$\mathcal{O} = \{x \in \Omega : u(x) > h(x)\}$$

is open. Assume also that $\partial F/\partial \xi_j$ and $\partial F/\partial u$ satisfy convenient bounds. Show that, on $\mathcal{O}$, u satisfies the PDE

$$\sum_j \partial_j F_{\xi_j}(x, u, \nabla u) + F_u(x, u, \nabla u) = 0.$$

For more on this sort of variational problem, see [KS].

12. Quasi-linear elliptic systems

Here we (partially) extend the study of the scalar equation (10.1) to a study of an $N \times N$ system

$$A^{jk}_{\alpha\beta}(\nabla u)\partial_j \partial_k u^\beta = 0 \ \text{ on } \ \Omega, \quad u = \varphi \ \text{ on } \ \partial\Omega, \tag{12.1}$$

where $\varphi \in C^\infty(\partial\Omega, \mathbb{R}^N)$ is given. The hypothesis of strong ellipticity used previously is

$$\sum A^{jk}_{\alpha\beta}(p) v_\alpha v_\beta \xi_j \xi_k \geq C|v|^2|\xi|^2, \quad C > 0, \tag{12.2}$$

but many nonlinear results require that $A^{jk}_{\alpha\beta}(p)$ satisfy the *very strong ellipticity hypothesis*:

$$\sum A^{jk}_{\alpha\beta}(p) \zeta_{j\alpha} \zeta_{k\beta} \geq \kappa|\zeta|^2, \quad \kappa > 0. \tag{12.3}$$

We mention that, in much of the literature, (12.3) is called strong ellipticity and (12.2) is called the "Legendre–Hadamard condition."

In the case when (12.1) arises from minimizing the function

$$I(u) = \int_\Omega F(\nabla u)\, dx, \tag{12.4}$$

we have

$$A^{jk}_{\alpha\beta}(p) = \partial_{p_{j\alpha}} \partial_{p_{k\beta}} F(p). \tag{12.5}$$

In such a case, (12.3) is the statement that $F(p)$ is a uniformly strongly convex function of p. If (12.5) holds, (12.1) can be written as

$$\sum_j \partial_j G^j_\alpha(\nabla u) = 0 \ \text{ on } \ \Omega, \quad u = \varphi \ \text{ on } \ \partial\Omega; \quad G^j_\alpha(p) = \partial_{p_{j\alpha}} F(p). \tag{12.6}$$

We will assume

$$\begin{gathered} a_0|p|^2 - b_0 \le F(p) \le a_1|p|^2 + b_1, \\ |G^j_\alpha(p)| \le C_0\langle p\rangle, \qquad |A^{jk}_{\alpha\beta}(p)| \le C_1. \end{gathered} \tag{12.7}$$

These are called "controllable growth conditions."

If (12.5) holds, then

$$\begin{gathered} \partial_j G^j_\alpha(\nabla u) - \partial_j G^j_\alpha(\nabla v) = \partial_j \mathcal{A}^{jk}_{\alpha\beta}(x) \partial_k(u^\beta - v^\beta), \\ \mathcal{A}^{jk}_{\alpha\beta}(x) = \int_0^1 A^{jk}_{\alpha\beta}\big(s\nabla u + (1-s)\nabla v\big)\, ds. \end{gathered} \tag{12.8}$$

This leads to a uniqueness result:

Proposition 12.1. *Assume $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain, and assume that (12.3) and (12.7) hold. If $u, v \in H^1(\Omega, \mathbb{R}^N)$ both solve (12.6), then $u = v$ on Ω.*

Proof. By (12.8), we have

$$\int_\Omega \mathcal{A}^{jk}_{\alpha\beta}(x)\, \partial_j(u^\alpha - v^\alpha)\, \partial_k(u^\beta - v^\beta)\, dx = 0, \tag{12.9}$$

so (12.3) implies $\partial_j(u - v) = 0$, which immediately gives $u = v$.

Let $X = \sum b^\ell \partial_\ell$ be a smooth vector field on $\overline{\Omega}$, tangent to $\partial\Omega$. If we knew that $u \in H^2(\Omega)$, we could deduce that $u_X = Xu$ is the unique solution in $H^1(\Omega, \mathbb{R}^N)$ to

$$\sum \partial_j A^{jk}(\nabla u)\, \partial_k u_X = \sum \partial_j f^j + g, \quad u_X = X\varphi \ \text{ on } \ \partial\Omega, \tag{12.10}$$

where

$$\begin{aligned} f^j &= A^{jk}(\nabla u)(\partial_k b^\ell)(\partial_\ell u) + (\partial_\ell b^j) G^\ell_\alpha(\nabla u), \\ g &= -(\partial_\ell \partial_j b^\ell) G^j_\alpha(\nabla u). \end{aligned} \tag{12.11}$$

Under the growth hypothesis (12.7), $|f^j(x)| \le C|\nabla u(x)|$, so $\|f^j\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}$. Similarly, $\|g\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} + C$. Hence, we can say that (12.10) has a unique solution, satisfying

$$\|u_X\|_{H^1(\Omega)} \le C\bigl(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^2(\Omega)} + 1\bigr). \tag{12.12}$$

It is unsatisfactory to hypothesize that u belong to $H^2(\Omega)$, so we replace the differentiation of (12.6) by taking difference quotients. Let $\mathcal{F}_X^t$ denote the flow on $\overline{\Omega}$ generated by X, and set $u_h = u \circ \mathcal{F}_X^h$. Then u_h extremizes a functional

$$I_h(u_h) = \int\limits_\Omega F_h(x, \nabla u_h)\, dx, \tag{12.13}$$

where $F_h(x,p)$ depends smoothly on (h,x,p) and $F_0(x,p) = F(p)$. (In fact, (12.13) is simply (12.4), after a coordinate change.) Thus u_h satisfies the PDE

$$\partial_j(\partial_{p_{j\alpha}} F_h)(x, \nabla u_h) = 0, \quad u_h = \varphi_k \text{ on } \partial\Omega. \tag{12.14}$$

Applying the fundamental theorem of calculus to the difference of (12.14) and (12.6), we have

$$\partial_j \mathcal{A}^{jk}_{\alpha\beta h}(x)\, \partial_k\Bigl(\frac{u_h^\beta - u^\beta}{h}\Bigr) = \partial_j H^j_{\alpha h}(x, \nabla u_h), \tag{12.15}$$

where $\mathcal{A}^{jk}_{\alpha\beta h}(x)$ is as in (12.8), with $v = u_h$, and

$$H^j_{\alpha h}(x,p) = \int_0^h \frac{d}{ds}(\partial_{p_{j\alpha}} F_s)(x,p)\, ds. \tag{12.16}$$

As in the analysis of (12.10), we have

$$\|h^{-1}(u_h - u)\|_{H^1(\Omega)} \le C\bigl(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^2(\Omega)} + 1\bigr). \tag{12.17}$$

Taking $h \to 0$, we have $u_X \in H^1(\Omega, \mathbb{R}^N)$, with the estimate (12.12).

From here, a standard use of ellipticity, parallel to the argument in (10.21)–(10.25), gives an H^1-bound on a transversal derivative of u; hence $u \in H^2(\Omega, \mathbb{R}^n)$, and

$$\|u\|_{H^2(\Omega)} \le C\bigl(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^2(\Omega)} + 1\bigr). \tag{12.18}$$

As in the scalar case, one of the keys to the further analysis of a solution to (12.6) is an examination of regularity for solutions to linear elliptic systems with L^∞-coefficients. Thus we consider linear operators of the form

$$Lu = b(x)^{-1} \sum_{j,k=1}^{n} \partial_j \big(A^{jk}(x) b(x) \, \partial_k u \big), \tag{12.19}$$

Compare with (9.1). Here u takes values in $\mathbb{R}^N$ and each A^{jk} is an $N \times N$ matrix, with real-valued entries $A^{jk}_{\alpha\beta} \in L^\infty(\Omega)$. We assume $A^{jk}_{\alpha\beta} = A^{kj}_{\beta\alpha}$. As in (12.3), we make the hypothesis

$$\lambda_1 |\zeta|^2 \geq \sum A^{jk}_{\alpha\beta}(x) \zeta_{j\alpha} \zeta_{k\beta} \geq \lambda_0 |\zeta|^2, \quad \lambda_0 > 0, \tag{12.20}$$

of very strong ellipticity. Thus $A^{jk}_{\alpha\beta}$ defines a positive-definite inner product $\langle \ , \ \rangle$ on $T^* \otimes \mathbb{R}^N$. We also assume

$$0 < C_0 \leq b(x) \leq C_1. \tag{12.21}$$

Then $b(x)\, dx = dV$ defines a volume element, and, for $\varphi \in C^1_0(\Omega, \mathbb{R}^N)$,

$$(Lu, \varphi) = -\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dV. \tag{12.22}$$

We will establish the following result of [Mey].

Proposition 12.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $f_j \in L^q(\Omega, \mathbb{R}^N)$ for some $q > 2$, and let u be the unique solution in $H^{1,2}_0(\Omega)$ to*

$$Lu = \sum \partial_j f_j. \tag{12.23}$$

Assume L has the form (12.19), with coefficients $A^{jk} \in L^\infty(\Omega)$, satisfying (12.20), and $b \in C^\infty(\overline{\Omega})$, satisfying (12.21). Then $u \in H^{1,p}(\Omega)$, for some $p > 2$.

Proof. We define the affine map

$$T : H^{1,p}_0(\Omega) \longrightarrow H^{1,p}_0(\Omega) \tag{12.24}$$

as follows. Let Δ be the Laplace operator on $\overline{\Omega}$, endowed with a smooth Riemannian metric whose volume element is $dV = b(x)\, dx$, and adjust λ_0, λ_1 so (12.20) holds when $|\zeta|^2$ is computed via the inner product $(\ , \ )$ on $T^* \otimes \mathbb{R}^N$ associated with this metric, so that

$$(\Delta u, \varphi) = -\int_\Omega (\nabla u, \nabla \varphi) \, dV. \tag{12.25}$$

Then we define $Tw = v$ to be the unique solution in $H_0^{1,2}(\Omega)$ to

$$\Delta v = \Delta w - \lambda_1^{-1} L w + \lambda_1^{-1} \sum \partial_j f_j. \tag{12.26}$$

The mapping property (12.24) holds for $2 \le p \le q$, by the L^p-estimates of Chap. 13. In fact, if $\Delta v = \sum \partial_j g_j,\ v \in H_0^{1,2}(\Omega)$, then

$$\|\nabla v\|_{L^p(\Omega)} \le C(p) \|g\|_{L^p(\Omega)}. \tag{12.27}$$

If we fix $r > 2$, then, for $2 \le p \le r$, interpolation yields such an estimate, with

$$C(p) = C(r)^\theta, \quad \frac{1-\theta}{2} + \frac{\theta}{r} = \frac{1}{p}, \quad \text{i.e., } \theta = \frac{r}{p}\frac{p-2}{r-2}. \tag{12.28}$$

Hence $C(p) \searrow 1$, as $p \searrow 2$. Now we see that $Tw_1 - Tw_2 = v_1 - v_2$ satisfies

$$\Delta(v_1 - v_2) = \big(\Delta - \lambda_1^{-1} L\big)(w_1 - w_2) = \nabla g, \tag{12.29}$$

where

$$g_j^\alpha = \partial_j(w_1^\alpha - w_2^\alpha) - \lambda_1^{-1} A_{\alpha\beta}^{jk}\, \partial_k(w_1^\beta - w_2^\beta), \tag{12.30}$$

and hence, under our hypotheses,

$$\|g\|_{L^p(\Omega)} \le \Big(1 - \frac{\lambda_0}{\lambda_1}\Big) \|\nabla(w_1 - w_2)\|_{L^p(\Omega)}, \tag{12.31}$$

so

$$\|\nabla(v_1 - v_2)\|_{L^p(\Omega)} \le C(p)\Big(1 - \frac{\lambda_0}{\lambda_1}\Big) \|\nabla(w_1 - w_2)\|_{L^p(\Omega)}, \tag{12.32}$$

for $2 \le p \le q$. We see that, for some $p > 2,\ C(p)\big(1 - \lambda_0/\lambda_1\big) < 1$; hence T is a contraction on $H^{1,p}(\Omega)$ in such a case. Thus T has a unique fixed point. This fixed point is u, so we have $u \in H_0^{1,p}(\Omega)$, as claimed.

Corollary 12.3. *With hypotheses as in Proposition 12.2, given a function $\psi \in H^{1,q}(\Omega)$, the unique solution $u \in H^{1,2}(\Omega)$ satisfying (12.23) and*

$$u = \psi \ \text{ on } \ \partial\Omega \tag{12.33}$$

also belongs to $H^{1,p}(\Omega)$, for some $p > 2$.

Proof. Apply Proposition 12.2 to $u - \psi$.

Let us return to the analysis of a solution $u \in H^1(\Omega, \mathbb{R}^N)$ to the nonlinear system (12.6), under the hypotheses of Proposition 12.1. Since we have established that $u \in H^2(\Omega, \mathbb{R}^N)$, we have a bound

$$\|\nabla u\|_{L^q(\Omega)} \le A, \quad q > 2. \tag{12.34}$$

In fact, this holds with $q = 2n/(n-2)$ if $n \ge 3$, and for all $q < \infty$ if $n = 2$. As above, if $X = \sum b^\ell \partial_\ell$ is a smooth vector field on $\overline{\Omega}$, tangent to $\partial\Omega$, then $u_X = Xu$ is the unique solution in $H^1(\Omega, \mathbb{R}^N)$ to (12.10), and we can now say that $f^j \in L^q(\Omega)$. Thus Corollary 12.3 gives

$$Xu \in H^{1,p}(\Omega), \quad \text{for some } p > 2, \tag{12.35}$$

with a bound, and again a standard use of ellipticity gives an $H^{1,p}$-bound on a transversal derivative of u. We have established the following result.

Theorem 12.4. *If $u \in H^1(\Omega, \mathbb{R}^N)$ solves (12.6) on a smoothly bounded domain $\Omega \in \mathbb{R}^n$, and if the very strong ellipticity hypothesis (12.3) and the controllable growth hypothesis (12.7) hold, then $u \in H^{2,p}(\Omega, \mathbb{R}^N)$, for some $p > 2$, and*

$$\|u\|_{H^{2,p}(\Omega)} \le C\big(\|\nabla u\|_{L^2(\Omega)} + \|\varphi\|_{H^{2,q}(\Omega)} + 1\big). \tag{12.36}$$

The case $n = \dim \Omega = 2$ of this result is particularly significant, since, for $p > n$, $H^{1,p}(\Omega) \subset C^r(\overline{\Omega})$, $r > 0$. Thus, under the hypotheses of Theorem 12.4, we have $u \in C^{1+r}(\overline{\Omega})$, for some $r > 0$, if $n = 2$. Then the material of § 8 applies to (12.1), so we have the following:

Proposition 12.5. *If $u \in H^1(\Omega, \mathbb{R}^N)$ solves (12.6) on a smoothly bounded domain $\Omega \subset \mathbb{R}^2$, and the hypotheses (12.3) and (12.7) hold, then $u \in C^\infty(\overline{\Omega})$, provided $\varphi \in C^\infty(\partial\Omega)$.*

When $n = 2$, we then have existence of a unique smooth solution to (12.1), given $\varphi \in C^\infty(\partial\Omega)$. In fact, we have two routes to such existence. We could obtain a minimizer $u \in H^1(\Omega, \mathbb{R}^N)$ for (12.4), subject to the condition that $u\big|_{\partial\Omega} = \varphi$, by the results of § 11, and then apply Proposition 12.5 to deduce smoothness.

Alternatively, we could apply the continuity method, to solve

$$A^{jk}_{\alpha\beta}(\nabla u)\partial_j \partial_k u^\beta = 0 \ \text{ on } \ \Omega, \quad u = \tau\varphi \ \text{ on } \ \partial\Omega. \tag{12.37}$$

This is clearly solvable for $\tau = 0$, and the proof that the biggest τ-interval $J \subset [0,1]$, containing 0, on which (12.37) has a unique solution $u \in C^\infty(\overline{\Omega})$, is both open and closed is accomplished along lines similar to arguments in § 10. However, unlike in § 10, we do not need to establish a sup-norm bound on ∇u, or even on u; we make do with an H^1-norm bound, which can be deduced from (12.3) as follows.

If $\mathcal{A}^{jk}_{\alpha\beta}(x)$ is given by (12.8), with $v = \varphi$, we have

$$\int_\Omega \mathcal{A}^{jk}_{\alpha\beta}(x)\, \partial_k(u^\beta - \varphi^\beta)\, \partial_j(u^\alpha - \varphi^\alpha)\, dx = \int_\Omega \partial_j G^j_\alpha(\nabla\varphi)(u^\alpha - \varphi^\alpha)\, dx, \tag{12.38}$$

for a solution to (12.37) (in case $\tau = 1$). Hence

$$\kappa \|\nabla(u-\varphi)\|^2_{L^2(\Omega)} \le C\|u-\varphi\|_{L^2(\Omega)}. \tag{12.39}$$

Note the different exponents. We have $\|u-\varphi\|^2_{L^2(\Omega)} \le C_2\|\nabla(u-\varphi)\|^2_{L^2(\Omega)}$, by Poincaré's inequality, so

$$\|u-\varphi\|_{L^2(\Omega)} \le \frac{C}{\kappa C_2}. \tag{12.40}$$

Plugging this back into (12.39) gives

$$\|\nabla(u-\varphi)\|^2_{L^2(\Omega)} \le \frac{C^2}{\kappa^2 C_2}, \tag{12.41}$$

which implies the desired H^1-bound on u.

Once we have the H^1-bound on $u = u_\tau$, (12.36) gives an $H^{2,p}$-bound for some $p > 2$, hence a bound in $C^{1+r}(\overline{\Omega})$, for some $r > 0$. Then the results of § 8 give bounds in higher norms, sufficient to show that J is closed.

Proposition 12.5 does not in itself imply all the results of § 10 when $\dim \Omega = 2$, since the hypotheses (12.3) and (12.7) imply that (12.1) is uniformly elliptic. For example, the minimal surface equation is not covered by Proposition 12.5. However, it is a simple matter to prove the following result, which does (essentially) contain the $n = 2$ case of Theorem 10.2.

Proposition 12.6. *Assume $A^{jk}_{\alpha\beta}(p)$ is smooth in p and satisfies*

$$A^{jk}_{\alpha\beta}(p)\zeta_{j\alpha}\zeta_{k\beta} \ge C(p)|\zeta|^2, \quad C(p) > 0. \tag{12.42}$$

Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. Then the Dirichlet problem (12.1) has a unique solution $u \in C^\infty(\overline{\Omega})$, provided one has an a priori bound

$$\|\nabla u_\tau\|_{L^\infty(\Omega)} \le K, \tag{12.43}$$

for all smooth solutions $u = u_\tau$ to (12.37), for $\tau \in [0, 1]$.

Proof. Use the method of continuity, as above. To prove that J is closed, simply modify $F(p)$ on $\{p : |p| \geq K + 1\}$ to obtain $\widetilde{F}(p)$, satisfying (12.3) and (12.7). The solution u_τ to (12.1) for $\tau \in J$ also solves the modified equation, for which (12.36) works, so as above we have strong norm bounds on u_τ as τ approaches an endpoint of J.

Recall that, for scalar equations, (12.43) follows from a boundary gradient estimate, via the maximum principle. The maximum principle is not available for general elliptic $N \times N$ systems, even under the very strong ellipticity hypothesis, so (12.43) is then a more severe hypothesis.

Moving beyond the case $n = 2$, we need to confront the fact that solutions to elliptic PDE of the form (12.1) need not be smooth everywhere. A number of examples have been found; we give one of J. Necas [Nec], where $A^{jk}_{\alpha\beta}(p)$ in (12.1) has the form (12.5), satisfying (12.3), such that $F(p)$ satisfies $|D^\alpha F(p)| \leq C_\alpha \langle p\rangle^{-|\alpha|}|p|^2, \ \forall\, \alpha \geq 0$. Namely, take

$$
\begin{aligned}
F(\nabla u) = {} & \frac{1}{2}\frac{\partial u^{ij}}{\partial x_k}\frac{\partial u^{ij}}{\partial x_k} + \frac{\mu}{2}\frac{\partial u^{ij}}{\partial x_i}\frac{\partial u^{kk}}{\partial x_j} \\
& + \lambda \frac{\partial u^{ij}}{\partial x_i}\frac{\partial u^{ak}}{\partial x_a}\frac{\partial u^{\ell b}}{\partial x_\ell}\frac{\partial u^{jk}}{\partial x_b}\langle \nabla u\rangle^{-2},
\end{aligned}
\tag{12.44}
$$

where u takes values in $M_{n\times n} \approx \mathbb{R}^{n^2}$, and we set

$$
\lambda = 2\frac{n^3 - 1}{n(n-1)(n^3 - n + 1)}, \qquad \mu = \frac{4 + n\lambda}{n^2 - n + 1}. \tag{12.45}
$$

Since $\lambda, \mu \to 0$ as $n \to \infty$, we have ellipticity for sufficiently large n. But for any n,

$$
u^{ij}(x) = \frac{x_i x_j}{|x|} \tag{12.46}
$$

is a solution to (12.1). Thus u is Lipschitz but not C^1 on every neighborhood of $0 \in \mathbb{R}^n$. See [Gia] for other examples. Also, when one looks at more general classes of nonlinear elliptic systems, there are examples of singular solutions even in the case $n = 2$; this is discussed further in § 12B.

We now discuss some results known as *partial regularity*, to the effect that solutions $u \in H^1(\Omega, \mathbb{R}^N)$ to (12.1) can be singular only on relatively *small* subsets of Ω.

We will measure how small the singular set is via the Hausdorff s-dimensional measure $\mathcal{H}^s$, which is defined for $s \in [0, \infty)$ as follows. First, given $\rho > 0, \ S \subset \mathbb{R}^n$, set

$$
h^*_{s,\rho}(S) = \inf\Big\{ \sum_{j\geq 1} (\operatorname{diam} Y_j)^s : S \subset \bigcup_{j \geq 1} Y_j, \ \operatorname{diam} Y_j \leq \rho \Big\}. \tag{12.47}
$$

Here diam $Y_j = \sup\{|x-y| : x, y \in Y_j\}$. Each set function $h^*_{s,\rho}$ is an outer measure on $\mathbb{R}^n$. As ρ decreases, $h^*_{s,\rho}(S)$ increases. Set

$$h^*_s(S) = \lim_{\rho\to 0} h^*_{s,\rho}(S). \tag{12.48}$$

Then $h^*_s(S)$ is an outer measure. It is seen to be a *metric* outer measure, that is, if $A, B \subset \mathbb{R}^n$ and $\inf\{|x-y| : x \in A, y \in B\} > 0$, then $h^*_s(A \cup B) = h^*_s(A) + h^*_s(B)$. It follows by a fundamental theorem of Caratheodory that every Borel set in $\mathbb{R}^n$ is h^*_s-measurable. For any h^*_s-measurable set A, we set

$$\mathcal{H}^s(A) = \gamma_s h^*_s(A), \quad \gamma_s = \frac{\pi^{s/2}2^{-s}}{\Gamma(\frac{s}{2}+1)}, \tag{12.49}$$

the factor γ_s being picked so that if $k \le n$ is an integer and $S \subset \mathbb{R}^n$ is a smooth, k-dimensional surface, then $\mathcal{H}^k(S)$ is exactly the k-dimensional surface area of S. Treatments of Hausdorff measure can be found in [EG, Fed, Fol].

Our next goal will be to establish the following result. Assume $n \ge 3$.

Theorem 12.7. *If $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain and $u \in H^1(\Omega, \mathbb{R}^N)$ solves (12.1), then there exists an open $\Omega_0 \subset \Omega$ such that $u \in C^\infty(\Omega_0)$ and*

$$\mathcal{H}^r(\Omega \setminus \Omega_0) = 0, \ \textit{for some} \ r < n-2. \tag{12.50}$$

We know from Theorem 12.4 that $u \in H^{2,p}(\Omega, \mathbb{R}^N)$, for some $p > 2$. Hence (12.10) holds for derivatives of u; in particular,

$$u_\ell = \partial_\ell u \Longrightarrow u_\ell \in H^{1,p}(\Omega, \mathbb{R}^N) \tag{12.51}$$

and

$$\partial_j A^{jk}(\nabla u)\partial_k u_\ell = 0, \quad 1 \le \ell \le n. \tag{12.52}$$

Regarding this as an elliptic system for $v = (\partial_1 u, \dots, \partial_n u)$, we see that to establish Theorem 12.7, it suffices to prove the following:

Proposition 12.8. *Assume that $v \in H^{1,p}(\Omega, \mathbb{R}^M)$, for some $p > 2$, and that v solves the system*

$$\partial_j A^{jk}(x, v)\, \partial_k v = 0, \tag{12.53}$$

where $A^{jk}_{\alpha\beta}(x, v)$ is uniformly continuous in (x, v) and satisfies

$$\lambda_1|\zeta|^2 \ge A^{jk}_{\alpha\beta}(x, v)\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0|\zeta|^2, \quad \lambda_0 > 0. \tag{12.54}$$

Then there is an open $\Omega_0 \subset \Omega$ such that v is Hölder continuous on Ω_0, and (12.50) holds.

In turn, we will derive Proposition 12.8 from the following more precise result:

Proposition 12.9. *Under the hypotheses of Proposition 12.8, consider the subset* $\Sigma \subset \Omega$ *defined by*

$$x \in \Sigma \Longleftrightarrow \liminf_{R \to 0} \; R^{-n} \int\limits_{B_R(x)} |v(y) - v_{x,R}|^2 \, dy > 0, \tag{12.55}$$

where

$$v_{x,R} = \mathit{Avg}_{B_R(x)} \, v = \frac{1}{\mathit{Vol}\; B_R(x)} \int\limits_{B_R(x)} v(y) \, dy. \tag{12.56}$$

Then

$$\mathcal{H}^r(\Sigma) = 0, \;\; \text{for some} \;\; r < n - 2, \tag{12.57}$$

and Σ *contains a closed subset* $\widetilde{\Sigma}$ *of* Ω *such that* v *is Hölder continuous on* $\Omega_0 = \Omega \setminus \widetilde{\Sigma}$.

Note that every point of continuity of v belongs to $\Omega \setminus \Sigma$; it follows from Proposition 12.9 that v is Hölder continuous on a neighborhood of every point of continuity, under the hypotheses of Proposition 12.8. As Lemma 12.11 will show, for this fact we need assume only that $u \in H^{1,2}$, instead of $u \in H^{1,p}$ for some $p > 2$.

Let us first prove that Σ, defined by (12.55), has the property (12.57). First, by Poincaré's inequality,

$$\Sigma \subset \Big\{ x \in \Omega : \liminf_{R \to 0} \; R^{2-n} \int\limits_{B_R(x)} |\nabla v(y)|^2 \, dy > 0 \Big\}. \tag{12.58}$$

Since $\nabla v \in L^p(\Omega)$ for some $p > 2$, Hölder's inequality implies

$$\Sigma \subset \Big\{ x \in \Omega : \liminf_{R \to 0} \; R^{p-n} \int\limits_{B_R(x)} |\nabla v(y)|^p \, dy > 0 \Big\}. \tag{12.59}$$

Therefore, (12.57) is a consequence of the following.

Lemma 12.10. *Given* $w \in L^1(\Omega), \; 0 \le s < n$, *let*

$$E_s = \Big\{ x \in \Omega : \limsup_{r \to 0} \; r^{-s} \int\limits_{B_r(x)} |w(y)| \, dy > 0 \Big\}. \tag{12.60}$$

Then

$$\mathcal{H}^{s+\varepsilon}(E_s) = 0, \quad \forall\, \varepsilon > 0. \tag{12.61}$$

It is actually true that $\mathcal{H}^s(E_s) = 0$ (see [EG] and [Gia]), but to shorten the argument we will merely prove the weaker result (12.61), which will suffice for our purposes. In fact, we will show that

$$\mathcal{H}^s(E_{s\delta}) < \infty, \quad \forall\, \delta > 0, \tag{12.62}$$

where

$$E_{s\delta} = \Big\{x \in \Omega : \limsup_{r\to 0}\ r^{-s} \int_{B_r(x)} |w(y)|\, dy \geq \delta\Big\}.$$

This implies that $\mathcal{H}^{s+\varepsilon}(E_{s\delta}) = 0$, $\forall\ \varepsilon > 0$, and since $E_s = \bigcup_n E_{s,1/n}$, this yields (12.61).

As a tool in the argument, we use the following:

Vitali covering lemma. *Let $\mathcal{C}$ be a collection of closed balls in $\mathbb{R}^n$ (with positive radius) such that diam $B < C_0 < \infty$, for all $B \in \mathcal{C}$. Then there exists a countable family $\mathcal{F}$ of disjoint balls in $\mathcal{C}$ such that*

$$\bigcup_{B\in\mathcal{F}} \widehat{B} \supset \bigcup_{B\in\mathcal{C}} B, \tag{12.63}$$

where $\widehat{B}$ is a ball concentric with B, with five times its radius.

Sketch of proof. Take $\mathcal{C}_j = \{B \in \mathcal{C} : 2^{-j} C_0 \leq \text{ diam } B < 2^{1-j} C_0\}$. Let $\mathcal{F}_1$ be a maximal disjoint collection of balls in $\mathcal{C}_1$. Inductively, let $\mathcal{F}_k$ be a maximal disjoint set of balls in

$$\{B \in \mathcal{C}_k : B \text{ disjoint from all balls in } \mathcal{F}_1, \dots, \mathcal{F}_{k-1}\}.$$

Then set $\mathcal{F} = \bigcup \mathcal{F}_k$. One can then verify (12.63).

To begin the proof of (12.62), note that, for each $\rho > 0$, $E_{s\delta}$ is covered by a collection $\mathcal{C}$ of balls B_x of radius $r_x < \rho$, such that

$$\int_{B_x} |w(y)|\, dy \geq \delta r_x^s. \tag{12.64}$$

Thus there is a collection $\mathcal{F}$ of disjoint balls B_ν in $\mathcal{C}$ (of radius r_ν) such that (12.63) holds. In particular, $\{\widehat{B}_\nu\}$ covers $E_{s\delta}$, so

$$h^*_{s,5\rho}(E_{s\delta}) \le C_n \sum_\nu r_\nu^s \le \frac{C_n}{\delta} \int\limits_{\bigcup B_\nu} |w(y)|\, dy \le \frac{C_n}{\delta} \|w\|_{L^1(\Omega)}, \tag{12.65}$$

where C_n is independent of ρ. This proves (12.62) and hence Lemma 12.10.

Thus we have (12.57) in Proposition 12.9. To prove the other results stated in that proposition, we will establish the following:

Lemma 12.11. *Given $\tau \in (0,1)$, there exist constants*

$$\varepsilon_0 = \varepsilon_0(\tau, n, M, \lambda_0^{-1}\lambda_1), \quad R_0 = R_0(\tau, n, M, \lambda_0^{-1}\lambda_1),$$

and furthermore there exists a constant

$$A_0 = A_0(n, M, \lambda_0^{-1}\lambda_1),$$

independent of τ, such that the following holds. If $u \in H^1(\Omega, \mathbb{R}^M)$ solves (12.53) and if, for some $x_0 \in \Omega$ and some

$$R < R_0(x_0) = \min(R_0, dist(x_0, \partial\Omega)),$$

we have

$$U(x_0, R) < \varepsilon_0^2, \tag{12.66}$$

where

$$U(x_0, R) = R^{-n} \int\limits_{B_R(x_0)} |u(y) - u_{x_0,R}|^2\, dy, \tag{12.67}$$

then

$$U(x_0, \tau R) \le 2A_0 \tau^2 U(x_0, R). \tag{12.68}$$

Let us show how this result yields Proposition 12.9. Pick $\alpha \in (0,1)$, and choose $\tau \in (0,1)$ such that $2A_0\tau^{2-2\alpha} = 1$. Suppose $x_0 \in \Omega$ and $R < \min(R_0, \operatorname{dist}(x_0, \partial\Omega))$, and suppose (12.66) holds. Then (12.68) implies

$$U(x_0, \tau R) \le \tau^{2\alpha} U(x_0, R).$$

In particular, $U(x_0, \tau R) < U(x_0, R) < \varepsilon_0^2$, so inductively the implication (12.66) $\Rightarrow$ (12.68) yields

$$U(x_0, \tau^k R) \le \tau^{2\alpha k} U(x_0, R).$$

Hence, for $\rho < R$,

$$U(x_0, \rho) \le C\left(\frac{\rho}{R}\right)^{2\alpha} U(x_0, R). \tag{12.69}$$

Note that, for fixed $R > 0$, $U(x_0, R)$ is continuous in x_0, so if (12.66) holds at x_0, then we have $U(x, R) < \varepsilon_0^2$ for every x in some neighborhood $B_r(x_0)$ of x_0, and hence

$$U(x, \rho) \le C\left(\frac{\rho}{R}\right)^{2\alpha} U(x, R), \quad x \in B_r(x_0);$$

that is, we have

$$\int_{B_\rho(x)} |u(y) - u_{x,\rho}|^2 \, dy \le C\rho^{n+2\alpha} \tag{12.70}$$

uniformly for $x \in B_r(x_0)$. This implies, by Proposition A.2,

$$u \in C^\alpha\big(B_r(x_0)\big). \tag{12.71}$$

In fact, we can say more. Extending some of the preliminary results of § 9, we have, for a solution $u \in H^1(\Omega)$ of (12.53), estimates of the form

$$\|\nabla u\|^2_{L^2(B_{\rho/2}(x))} \le C\rho^{-2} \int_{B_\rho(x)} |u(y) - u_{x,\rho}|^2 \, dy; \tag{12.72}$$

see Exercise 2 below. Consequently, (12.70) implies

$$\nabla u\big|_{B_r(x_0)} \in M_2^q\big(B_r(x_0)\big), \quad q = \frac{n}{1-\alpha}. \tag{12.73}$$

which by Morrey's lemma implies (12.71). Thus, granted Lemma 12.11, Proposition 12.9 is proved, with

$$\Omega_0 = \{x_0 \in \Omega : \inf_{R < R_0(x_0)} U(x_0, R) < \varepsilon_0^2\}, \tag{12.74}$$

since clearly $\Sigma \supset \Omega \setminus \Omega_0 = \widetilde{\Sigma}$.

The proof of Lemma 12.11 (following the exposition in [Gia]) evolved from work of E. DeGiorgi [DeG2] and F. Almgren [Alm2] on regularity for minimal surfaces. It consists of blowing up small neighborhoods of x_0 and obtaining a limiting PDE for a limit of the resulting dilations of u. As a preliminary to the proof of Lemma 12.11, we first identify the constant A_0.

Lemma 12.12. *There is a constant $A_0 = A_0(n, M, \lambda_1/\lambda_0)$ such that whenever $b^{jk}_{\alpha\beta}$ are constants satisfying*

$$\lambda_1|\zeta|^2 \ge \sum b^{jk}_{\alpha\beta}\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0|\zeta|^2, \quad \lambda_0 > 0, \tag{12.75}$$

the following holds. If $u \in H^1\big(B_1(0), \mathbb{R}^M\big)$ *solves*

$$\partial_j b^{jk}_{\alpha\beta} \partial_k u^\beta = 0 \quad on \;\; B_1(0), \tag{12.76}$$

then, for all $\rho \in (0,1)$,

$$U(0,\rho) \le A_0 \rho^2 U(0,1). \tag{12.77}$$

Proof. For $\rho \in (0, 1/2]$, we have

$$U(0,\rho) \le \rho^{2-n} \int\limits_{B_\rho(0)} |\nabla u(y)|^2 \, dy \le C_n \rho^2 \|\nabla u\|^2_{L^\infty(B_{1/2}(0))}. \tag{12.78}$$

On the other hand, regularity for the constant-coefficient, elliptic PDE (12.76) readily yields an estimate

$$\|\nabla u\|^2_{L^\infty(B_{1/2}(0))} \le B_0 \|\nabla u\|^2_{L^2(B_{3/4}(0))} \le B_1 \|u - u_{0,1}\|^2_{L^2(B_1(0))}, \tag{12.79}$$

with $B_j = B_j(n, M, \lambda_1/\lambda_0)$, from which (12.77) easily follows.

We now tackle the proof of Lemma 12.11. If the conclusion (12.68) is false, then there exist $\tau \in (0,1)$ and $x_\nu \in \Omega,\ \varepsilon_\nu \to 0,\ R_\nu \to 0$, and $u_\nu \in H^1(\Omega, \mathbb{R}^M)$, solving (12.53), such that

$$U_\nu(x_\nu, R_\nu) = \varepsilon_\nu^2, \quad U_\nu(x_\nu, \tau R_\nu) > 2A_0 \tau^2 \varepsilon_\nu^2. \tag{12.80}$$

To implement the dilation argument mentioned above, we set

$$v_\nu(x) = \varepsilon_\nu^{-1}\big[u_\nu(x_\nu + R_\nu x) - u_{\nu x_\nu, R_\nu}\big]. \tag{12.81}$$

Then v_ν solves

$$\partial_j A^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big)\, \partial_k v_\nu^\beta = 0 \quad \text{on} \;\; B_1(0). \tag{12.82}$$

If we set

$$\begin{aligned} V_\nu(0,\rho) &= \rho^{-n} \int\limits_{B_\rho(0)} |v_\nu(y) - v_{\nu 0,\rho}|^2 \, dy \\ &= \varepsilon_\nu^{-2} \rho^{-n} R_\nu^{-n} \int\limits_{B_{\rho R_\nu}(x_\nu)} |u_\nu(y) - u_{\nu x_\nu, R_\nu}|^2 \, dy, \end{aligned} \tag{12.83}$$

we have (since $v_{\nu 0,1} = 0$)

$$V_\nu(0,1) = \|v_\nu\|^2_{L^2(B_1(0))} = 1, \quad V_\nu(0,\tau) > 2A_0 \tau^2. \tag{12.84}$$

Passing to a subsequence, we can assume that

$$(12.85)\qquad v_\nu \to v \text{ weakly in } L^2\big(B_1(0), \mathbb{R}^M\big), \quad \varepsilon_\nu v_\nu \to 0 \text{ a.e. in } B_1(0).$$

Also

$$(12.86)\qquad A^{jk}_{\alpha\beta}(x_\nu, u_{\nu x_\nu, R_\nu}) \longrightarrow b^{jk}_{\alpha\beta},$$

an array of constants satisfying (12.75). The uniform continuity of $A^{jk}_{\alpha\beta}$ then implies

$$(12.87)\qquad A^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big) \longrightarrow b^{jk}_{\alpha\beta} \text{ a.e. in } B_1(0).$$

Now, as in (12.72), the fact that v_ν solves (12.82) implies

$$(12.88)\qquad \|v_\nu\|_{H^1(B_\rho(0))} \le C_\rho, \quad \forall\, \rho < 1.$$

Hence, passing to a further subsequence if necessary, we have

$$(12.89)\qquad \begin{aligned} v_\nu &\longrightarrow v \text{ strongly in } L^2_{\mathrm{loc}}\big(B_1(0)\big), \\ \nabla v_\nu &\longrightarrow \nabla v \text{ weakly in } L^2_{\mathrm{loc}}\big(B_1(0)\big). \end{aligned}$$

Since the functions in (12.87) are uniformly bounded on $B_1(0)$, these results imply that we can pass to the limit in (12.82), to conclude that

$$(12.90)\qquad \partial_j b^{jk}_{\alpha\beta}\, \partial_k v^\beta = 0 \quad \text{on } B_1(0).$$

Then Lemma 12.12 implies

$$(12.91)\qquad V(0,\tau) \le A_0 \tau^2 V(0,1),$$

which is $\le A_0\tau^2$ by (12.85). On the other hand, (12.89) implies

$$(12.92)\qquad V(0,\tau) \ge 2A_0\tau^2$$

if (12.80) holds. This contradiction proves Lemma 12.11.

Hence the proof of Proposition 12.9 is complete, so we have Theorem 12.7.

Theorem 12.7 can be extended to a result on partial regularity up to the boundary (see [Gia]).

There is a condition more general than strong convexity on the integrand in (12.4), known as "quasi-convexity," under which extrema for (12.4) have been shown to possess partial regularity of the sort established in Theorem 12.7 (see [Ev3]).

There are also some results on regularity everywhere for stationary points of (12.4) when Ω has dimension ≥ 3. A notable result of [U] is that such solutions are smooth on Ω provided $F(\nabla u)$ in (12.4), in addition to being strongly convex in ∇u and satisfying the controllable growth conditions, depends only on $|\nabla u|^2$. A proof can also be found in [Gia].

Exercises

In Exercises 1–3, we consider an $N \times N$ system

$$\sum \partial_j A^{jk}_{\alpha\beta}(x)\partial_k u^\beta = \sum \partial_j f^\alpha_j \quad \text{on } B_1 = \{x \in \mathbb{R}^n : |x| < 1\}, \tag{12.93}$$

under the very strong ellipticity hypothesis (12.20). Assume $f_j \in L^2(B_1)$.

1. Show that, with $C = C(\lambda_0, \lambda_1)$,

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C\|u\|_{L^2(B_1)} + C \sum \|f_j\|_{L^2(B_1)}. \tag{12.94}$$

(*Hint*: Extend (9.6).)

2. Let $\delta_r v(x) = v(rx)$. Show that, for $r \in (0, 1]$,

$$\|\delta_r(\nabla u)\|_{L^2(B_{1/2})} \leq Cr^{-1}\|\delta_r(u - \overline{u})\|_{L^2(B_1)} + C \sum \|\delta_r f_j\|_{L^2(B_1)}, \tag{12.95}$$

where $\overline{u} = \operatorname{Avg}_{B_1} u$. (*Hint*: First apply a dilation argument to (12.94). Then apply the result to $u - \overline{u}$.) This sort of estimate is called a “Caccioppoli inequality.”

3. Deduce from Exercise 2 that if $u \in H^1(\Omega)$ solves (12.93), then

(12.96)
$$\|\delta_r(\nabla u)\|_{L^2(B_{1/2})} \leq C\|\delta_r(\nabla u)\|_{L^q(B_1)} + C \sum \|\delta_r f_j\|_{L^2(B_1)}, \quad q = \frac{2n}{n+2} < 2.$$

This sort of estimate is sometimes called a “reverse Hölder inequality.”

4. Deduce from (12.95) that if $u \in H^1(\Omega)$ solves (12.93), then, for $0 < r < 1$,

$$u \in C^r(B_1),\ f_j \in M^p_2(B_1),\ p = \frac{n}{1-r} \Longrightarrow \nabla u \in M^p_2(B_{1/2}). \tag{12.97}$$

Compare (9.41)–(9.42).

5. Let $C(p)$ be the constant in (12.27), in case $\Omega = B_1$. Show that if $C(n)\big(1 - \lambda_0/\lambda_1\big) < 1$, then a solution $u \in H^1_0(\Omega)$ to (12.93) is Hölder continuous on $\overline{B}_1$, provided $f_j \in L^q(B_1)$ for some $q > n$. Consider the problem of obtaining precise estimates on $C(p)$ in this case.

12B. Further results on quasi-linear systems

Regularity questions can become more complex when lower-order terms are added to systems of the form (12.1). In fact, there are extra complications even for solutions to a semilinear system of the form

$$Lu + B(x, u, \nabla u) = f, \tag{12b.1}$$

where L is a second-order, linear elliptic differential operator and $B(x, u, p)$ is smooth in its arguments. One limitation on what one could possibly prove is given by the following example of J. Frehse [Freh], namely that

$$u_1(x) = \sin \log \log |x|^{-1}, \quad u_2(x) = \cos \log \log |x|^{-1} \tag{12b.2}$$

provides a bounded, weak solution to the 2×2 system

$$\begin{aligned} \Delta u_1 + \frac{2(u_1 + u_2)}{1 + |u|^2} |\nabla u|^2 = 0, \\ \Delta u_2 + \frac{2(u_2 - u_1)}{1 + |u|^2} |\nabla u|^2 = 0, \end{aligned} \tag{12b.3}$$

belonging to $H^1(B)$, for any ball $B \subset \mathbb{R}^2$, centered at the origin, of radius $r < 1$. Evidently, u is not continuous at the origin; one can also see that ∇u does not belong to $L^p(B)$ for any $p > 2$. (After all, that would force u to be Hölder continuous.) Thus Theorem 12.4 and Proposition 12.5 do not extend to this case.

The following result shows that if a weak solution to such a semilinear system as (12b.1) has any Hölder continuity, then higher-order regularity results hold.

Proposition 12B.1. *Assume $u \in H^1$ solves (12b.1) and $B(x, u, p)$ is a smooth function of its arguments, satisfying*

$$|B(x, u, p)| \le C \langle p \rangle^2. \tag{12b.4}$$

Then, given $r > 0,\ s > -1$,

$$u \in C^r,\ f \in C^s_* \Longrightarrow u \in C^{s+2}_*. \tag{12b.5}$$

Proof. Write

$$u = Ef - EB(x, u, \nabla u), \quad \text{mod } C^\infty, \tag{12b.6}$$

where $E \in OPS^{-2}_{1,0}$ is a parametrix for the elliptic operator L. We have $Ef \in C^{s+2}_*$, and, since $u \in H^1 \Rightarrow B(x, u, \nabla u) \in L^1$, we have

$$EB(x, u, \nabla u) \in H^{2-\sigma, 1+\varepsilon}, \quad \forall\ \varepsilon > 0,\ \ \sigma > \frac{n\varepsilon}{1+\varepsilon}.$$

If $s \ge 0$, this implies

$$u \in H^{2-\sigma,1+\varepsilon} \cap H^{r-\sigma,p}, \tag{12b.7}$$

for all $p < \infty$, hence

$$u \in [H^{2-\sigma,1+\varepsilon}, H^{r-\sigma,p}]_\theta, \quad \forall\ \theta \in (0, 1). \tag{12b.8}$$

Results on such interpolation spaces follow from (6.30) of Chap. 13. If we set $\theta = 1/2$ and take p large enough, we have

$$u \in H^{1+r/2-\sigma,2+2\varepsilon}, \quad \forall\, \varepsilon \in (0,1), \ \ \sigma > \frac{n\varepsilon}{1+\varepsilon}. \tag{12b.9}$$

On the other hand, if we set $\theta = (1-\sigma)/(2-r)$, (assuming $r < 1$), we have

$$u \in H^{1,2q}, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}, \tag{12b.10}$$

hence

$$B(x,u,\nabla u) \in L^q, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}, \quad \text{e.g., } q = 1 + \frac{r}{2}. \tag{12b.11}$$

Another look at (12b.6) now yields

$$u \in H^{2,1+r/2} \cap H^{r-\sigma,p}, \quad \forall\, p < \infty, \tag{12b.12}$$

provided $s \geq 0$, which is an improvement of (12b.7). We can iterate this argument until we get (12b.5), provided $s \geq 0$.

If instead we merely assume $s > -1$, then, instead of (12b.7), we deduce from (12b.6) and $EB(x,u,\nabla u) \in H^{2-\sigma,1+\varepsilon}$ that

$$EB(x,u,\nabla u) \in H^{2-\sigma,1+\varepsilon} \cap H^{r-\sigma,p} \tag{12b.13}$$

and hence (parallel to (12b.8)–(12b.11)) that

$$\begin{aligned} EB(x,u,\nabla u) &\in \bigcap_{\theta \in (0,1)} [H^{2-\sigma,1+\varepsilon}, H^{r-\sigma,p}]_\theta \\ &\subset H^{1+r/2-\sigma,2} \cap H^{1,2+r}, \end{aligned} \tag{12b.14}$$

so another look at (12b.6) gives

$$u \in H^{1,2+r},$$

hence

$$B(x,u,\nabla u) \in L^{1+r/2}, \tag{12b.15}$$

so

$$EB(x,u,\nabla u) \in H^{2,1+r/2} \cap H^{r-\sigma,p}, \tag{12b.16}$$

and we can iterate this argument until (12b.5) is proved.

Note that Proposition 12B.1 applies to the semilinear system (11.25) for a harmonic map $u : \Omega \to X$, where X is a submanifold of $\mathbb{R}^N$:

$$\Delta u - \Gamma(u)(\nabla u, \nabla u) = 0. \tag{12b.17}$$

On the other hand, there are quasi-linear equations with a somewhat similar structure that also arise naturally in geometry, such as the system (4.94) satisfied by the metric tensor, in harmonic coordinates, when the Ricci tensor is given. This system has the following form, more general than (12b.1):

$$\sum \partial_j a^{jk}(x,u)\partial_k u + B(x,u,\nabla u) = f. \tag{12b.18}$$

We assume that $a^{jk}(x,u)$ and $B(x,u,p)$ are smooth in their arguments and that (12b.4) holds. Recall that we have established one regularity result for such a system in § 4, namely, if $n = \dim \Omega$ and $n < q < p < \infty$, then

$$u \in H^{1,q}, \ f \in H^{s,p} \Longrightarrow u \in H^{s+2,p} \tag{12b.19}$$

if $s \geq -1$. Here, we want to weaken the hypothesis that $u \in H^{1,q}$ for some $q > n$, which of course implies $u \in C^r, \ r = 1 - n/q$. We will establish the following:

Proposition 12B.2. *Assume that $u \in H^1$ solves (12b.18) and that $B(x,u,p)$ satisfies (12b.4). Also assume $u \in C^r$ for some $r > 0$. Then*

$$f \in L^1 \Longrightarrow u \in H^{2-\sigma,1+\varepsilon}, \quad \forall\, \varepsilon \in (0,1), \ \ \sigma > \frac{n\varepsilon}{1+\varepsilon}, \tag{12b.20}$$

and, if $1 < p < \infty$,

$$f \in L^p \Longrightarrow u \in H^{2,p}. \tag{12b.21}$$

More generally, for $s \geq 0$,

$$f \in H^{s,p} \Longrightarrow u \in H^{s+2,p}. \tag{12b.22}$$

To begin the proof, as in the demonstration of Proposition 4.9, we write

$$\sum a^{jk}(x,u)\,\partial_k u = A_j(u;x,D)u, \tag{12b.23}$$

mod C^∞, with

$$u \in C^r \Longrightarrow A_j(u;x,\xi) \in C^r S^1_{1,0} \cap S^1_{1,1} + S^{1-r}_{1,1}. \tag{12b.24}$$

Hence, given $\delta \in (0, 1)$,

$$\begin{gathered} A_j(u; x, \xi) = A_j^{\#}(x, \xi) + A_j^b(x, \xi), \\ A_j^{\#}(x, \xi) \in S^1_{1,\delta}, \qquad A_j^b(x, \xi) \in S^{1-r\delta}_{1,1}. \end{gathered} \tag{12b.25}$$

Thus we can write

$$\sum \partial_j a^{jk}(x, \xi)\, \partial_k u = P^{\#} u + P^b u, \tag{12b.26}$$

with

$$P^{\#} = \sum \partial_j A_j^{\#}(x, D) \in OPS^2_{1,\delta}, \quad \text{elliptic} \tag{12b.27}$$

and

$$P^b = \sum \partial_j A_j^b(x, D). \tag{12b.28}$$

Then we let

$$E^{\#} \in OPS^{-2}_{1,\delta} \tag{12b.29}$$

be a parametrix for $P^{\#}$, and we have

$$u = -E^{\#} P^b u + E^{\#} B(x, u, \nabla u) + E^{\#} f, \tag{12b.30}$$

mod C^∞, and if $u \in C^r$,

$$P^b : H^{\sigma, p} \longrightarrow H^{\sigma - 2 + r\delta, p}, \quad P^b : C^\sigma_* \longrightarrow C^{\sigma - 2 + r\delta}_*, \tag{12b.31}$$

provided $1 < p < \infty$ and $\sigma - 2 + r\delta > -1$, so

$$\sigma > 1 - r\delta. \tag{12b.32}$$

Therefore, our hypotheses on u imply

$$E^{\#} P^b u \in H^{1 + r\delta, 2}. \tag{12b.33}$$

Now, if $u \in H^1(\Omega)$, then (12b.4) implies

$$B(x, u, \nabla u) \in L^1, \tag{12b.34}$$

so, for small $\varepsilon > 0,\ \sigma > n\varepsilon/(1 + \varepsilon)$,

$$E^{\#} B(x, u, \nabla u) \in H^{2 - \sigma, 1 + \varepsilon}. \tag{12b.35}$$

Hence we have (12b.30), mod C^∞, with

(12b.36)
$$E^\# P^b u \in H^{1+r\delta,2}, \qquad E^\# B(x,u,\nabla u) \in H^{2-\sigma,1+\varepsilon}, \\ E^\# f \in H^{2-\sigma,1+\varepsilon}.$$

This implies

$$u \in H^{1+r\delta,1+\varepsilon},$$

hence, by (12b.31),

(12b.37)
$$E^\# P^b u \in H^{1+2r\delta,1+\varepsilon}.$$

Another look at (12b.30) gives

(12b.38)
$$\begin{aligned} u \in H^{1+2r\delta,1+\varepsilon} \quad & \text{if } 1+2r\delta \le 2-\sigma, \\ H^{2-\sigma,1+\varepsilon} \quad & \text{if } 1+2r\delta \ge 2-\sigma. \end{aligned}$$

If the first of these alternatives holds, then

$$E^\# P^b u \in H^{1+3r\delta,1+\varepsilon}.$$

We continue until the conclusion of (12b.20) is achieved.

Given that $u \in C^r$ and that (12b.20) holds, by interpolation we have

(12b.39)
$$u \in \big[H^{2-\sigma,1+\varepsilon}, H^{r-\sigma,p}\big]_\theta, \quad \forall\, \theta \in (0,1),$$

using $C^r_* \subset H^{r-\sigma,p}$, $\forall\, \sigma > 0$, $p < \infty$. If we take $\theta = 1/2$ we get

$$u \in H^{1+r/2-\sigma,q}, \quad \frac{1}{q} = \frac{1}{2+2\varepsilon} + \frac{1}{2p},$$

hence, taking p arbitrarily large, we have

(12b.40)
$$u \in H^{1+r/2-\sigma,2+2\varepsilon}, \quad \forall\, \varepsilon \in (0,1), \;\; \sigma > \frac{n\varepsilon}{1+\varepsilon}.$$

Note that this is an improvement of the original hypothesis that $u \in H^{1,2}$. On the other hand, if we take $\theta = (1-\sigma)/(2-r)$, we get

(12b.41)
$$u \in H^{1,2q}, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r},$$

so

(12b.42)
$$B(x,u,\nabla u) \in L^q, \quad \forall\, q < \frac{1-\frac{1}{2}r}{1-r}.$$

Hence

$$E^{\#}B(x,u,\nabla u)\in H^{2,q}. \tag{12b.43}$$

Meanwhile, by (12b.40),

$$E^{\#}P^{b}u\in H^{1+r/2+r\delta-\sigma,2}. \tag{12b.44}$$

On the other hand, if we set

$$q=1+\frac{r}{2}, \tag{12b.45}$$

which satisfies the condition in (12b.41), we can take $\theta\approx r/(2+r)$ in (12b.39) and get

$$u\in H^{\mu,q},\quad \forall\,\mu<\frac{4+r^2}{2+r}, \tag{12b.46}$$

hence

$$E^{\#}P^{b}u\in H^{\rho,q},\quad \forall\,\rho<\frac{4+r^2}{2+r}+r\delta. \tag{12b.47}$$

Note that

$$\frac{4+r^2}{2+r}+r\delta=2-r+r\delta+r^2-\frac{1}{4}r^3+\cdots, \tag{12b.48}$$

which is >2, for any given $r\in(0,1)$, if δ is taken close enough to 1. Now, another look at (12b.30) establishes the following special case of (12b.21):

$$1<p\le 1+\frac{r}{2},\quad f\in L^p(\Omega)\Longrightarrow u\in H^{2,p}. \tag{12b.49}$$

Under the hypotheses that $u\in C^r$ and that (12b.49) holds, we have, parallel to (12b.39),

$$u\in\left[H^{2,p},H^{r-\sigma,Q}\right]_{\theta},\quad \forall\,\theta\in(0,1), \tag{12b.50}$$

for all $\sigma>0,\ Q<\infty$. As before, we can take $\theta\approx 1/(2-r)$ and get

$$u\in H^{1,2q},\quad \forall\,q<\frac{1-\frac{1}{2}r}{1-r}p. \tag{12b.51}$$

Hence, parallel to (12b.43), and as before using $1+r/2<(1-r/2)/(1-r)$, we have

$$E^\# B(x, u, \nabla u) \in H^{2,(1+r/2)p}. \tag{12b.52}$$

Similarly, if we take $\theta \approx r/(2+r)$ in (12b.50), we get

$$u \in H^{\mu,(1+r/2)p}, \quad \forall\, \mu < \frac{4+r^2}{2+r}, \tag{12b.53}$$

and hence

$$E^\# P^b u \in H^{\rho,(1+r/2)p}, \quad \forall\, \rho < \frac{4+r^2}{2+r} + r\delta.$$

As before, given $r \in (0, 1)$, we can choose δ close enough to 1 that $\rho > 2$. Another look at (12b.30) establishes that

$$1 < p \le \Big(1 + \frac{r}{2}\Big)^2, \quad f \in L^p(\Omega) \Longrightarrow u \in H^{2,p}. \tag{12b.54}$$

Now we can iterate this argument repeatedly, and since, for all $r > 0$, we have $(1 + r/2)^k \to \infty$ as $k \to \infty$, we obtain (12b.21).

We next want to weaken the requirement of Hölder continuity on u.

Proposition 12B.3. *Let $u \in H^1(\Omega)$ solve (12b.18). Assume the very strong ellipticity condition*

$$a^{jk}_{\alpha\beta}(x, u)\zeta_{j\alpha}\zeta_{k\beta} \ge \lambda_0 |\zeta|^2, \quad \lambda_0 > 0. \tag{12b.55}$$

Also assume $B(x, u, \nabla u)$ is a quadratic form in ∇u. Assume furthermore that u is continuous on Ω. Then, locally, if $p > n/2$,

$$f \in M^p_2 \Longrightarrow \nabla u \in M^q_2, \quad \textit{for some } q > n. \tag{12b.56}$$

Hence $u \in C^r$, for some $r > 0$.

To begin, given $x_0 \in \Omega$, shrink Ω down to a smaller neighborhood, on which

$$|u(x) - u_0| \le E, \tag{12b.57}$$

for some $u_0 \in \mathbb{R}^M$ (if (12b.18) is an $M \times M$ system). We will specify E below. With the same notation as in (12.22), write

$$\big(\partial_j a^{jk}(x, u)\, \partial_k u, w\big)_{L^2} = -\int \langle \nabla u, \nabla w\rangle\, dx, \tag{12b.58}$$

so $a^{jk}_{\alpha\beta}(x, u)$ determines an inner product on $T^*_x \otimes \mathbb{R}^M$ for each $x \in \Omega$, in a fashion that depends on u, perhaps, but one has bounds on the set of inner products so arising. Now, if we let $\psi \in C^\infty_0(\Omega)$ and $w = \psi(x)^2(u - u_0)$, and take the inner product of (12b.18) with w, we have

$$\begin{aligned}\int \psi^2|\nabla u|^2\,dx + 2\int \psi(\nabla u)(\nabla\psi)(u-u_0)\,dx\\ -\int \psi^2(u-u_0)B(x,u,\nabla u)\,dx\\ = -\int \psi^2 f(u-u_0)\,dx.\end{aligned} \tag{12b.59}$$

Hence we obtain the inequality

$$\begin{aligned}&\int \psi^2\big[|\nabla u|^2 - |u-u_0|\cdot|B(x,u,\nabla u)| - \delta^2|\nabla u|^2\big]\,dx\\ &\le \frac{1}{\delta^2}\int |\nabla\psi|^2|u-u_0|^2\,dx + \int \psi^2|f|\cdot|u-u_0|\,dx,\end{aligned} \tag{12b.60}$$

for any $\delta \in (0,1)$. Now, for some $A < \infty$, we have

$$|B(x,u,\nabla u)| \le A|\nabla u|^2. \tag{12b.61}$$

Then we choose E in (12b.57) so that

$$EA \le 1-a < 1. \tag{12b.62}$$

Then take $\delta^2 = a/2$, and we have

$$\frac{a}{2}\int \psi^2|\nabla u|^2\,dx \le \frac{2}{a}\int |\nabla\psi|^2\cdot|u-u_0|^2\,dx + \int \psi^2|f|\cdot|u-u_0|\,dx. \tag{12b.63}$$

Now, given $x \in \Omega$, for $R <$ dist$(x,\partial\Omega)$, define $U(x,R)$ as in (12.67) by

$$U(x,R) = R^{-n}\int_{B_R(x)} |u(y)-u_{x,R}|^2\,dy, \tag{12b.64}$$

where, as before, $u_{x,R}$ is the mean value of $u\big|_{B_R(x)}$. The following result is analogous to Lemma 12.11. Let A_0 be the constant produced by Lemma 12.12, applied to the present case, and pick ρ such that $A_0\rho^2 \le 1/2$.

Lemma 12B.4. *Let $\overline{\mathcal{O}} \subset\subset \Omega$. There exist $R_0 > 0$, $\vartheta < 1$, and $C_0 < \infty$ such that if $x \in \overline{\mathcal{O}}$ and $r \le R_0$, then either*

$$U(x,r) \le C_0 r^{2(2-n/p)}, \tag{12b.65}$$

or

$$U(x,\rho r) \le \vartheta U(x,r). \tag{12b.66}$$

Proof. If not, there exist $x_\nu \in \overline{\mathcal{O}}$, $R_\nu \to 0$, $\vartheta_\nu \to 1$, and $u_\nu \in H^1(\Omega, \mathbb{R}^M)$ solving (12b.18) such that

$$U_\nu(x_\nu, R_\nu) = \varepsilon_\nu^2 > C_0 R_\nu^{2(2-n/p)} \tag{12b.67}$$

and

$$U_\nu(x_\nu, \rho R_\nu) > \vartheta_\nu U_\nu(x_\nu, R_\nu). \tag{12b.68}$$

The hypothesis that u is continuous implies $\varepsilon_\nu \to 0$. We want to obtain a contradiction.

As in (12.81), set

$$v_\nu(x) = \varepsilon_\nu^{-1}\big[u_\nu(x_\nu + R_\nu x) - u_{\nu x_\nu, R_\nu}\big]. \tag{12b.69}$$

Then v_ν solves

$$\begin{aligned} &\partial_j a^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big)\, \partial_k v_\nu^\beta \\ &+ \varepsilon_\nu B\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}, \nabla v_\nu(x)\big) = \frac{R_\nu^2}{\varepsilon_\nu} f. \end{aligned} \tag{12b.70}$$

Note that, by the hypothesis (12b.67),

$$\frac{R_\nu^2}{\varepsilon_\nu} < \frac{1}{C_0} R_\nu^{n/p}. \tag{12b.71}$$

Now set

$$V_\nu(0, r) = r^{-n} \int\limits_{B_r(0)} |v_\nu(y) - v_{\nu 0, r}|^2 \, dy. \tag{12b.72}$$

Then, as in (12.84), we have

$$V_\nu(0, 1) = \|v_\nu\|^2_{L^2(B_1(0))} = 1, \quad V_\nu(0, \rho) > \vartheta_\nu. \tag{12b.73}$$

Passing to a subsequence, we can assume that

$$v_\nu \to v \ \text{ weakly in } \ L^2\big(B_1(0), \mathbb{R}^M\big), \quad \varepsilon_\nu v_\nu \to 0 \ \text{ a.e. in } B_1(0). \tag{12b.74}$$

Also, as in (12.87), there is an array of constants $b^{jk}_{\alpha\beta}$ such that

$$a^{jk}_{\alpha\beta}\big(x_\nu + R_\nu x, \varepsilon_\nu v_\nu(x) + u_{\nu x_\nu, R_\nu}\big) \longrightarrow b^{jk}_{\alpha\beta} \quad \text{a.e. in } B_1(0), \tag{12b.75}$$

and this is bounded convergence.

We next need to estimate the L^2-norm of ∇v_ν, which will take just slightly more work than it did in (12.88).

Substituting $\varepsilon_\nu v_\nu\big((x-x_\nu)/R_\nu\big)+u_{\nu x_\nu,R_\nu}$ for $u_\nu(x)$ in (12b.63), and replacing u_0 by $u_{\nu x_\nu,R_\nu}$, we have

$$\begin{aligned}
&\frac{a}{2}\int \psi^2\Big|\nabla v_\nu\Big(\frac{x-x_\nu}{R_\nu}\Big)\Big|^2\,dx\\
&\quad\le \frac{2}{a}\int R_\nu^2|\nabla\psi|^2\,\Big|v_\nu\Big(\frac{x-x_\nu}{R_\nu}\Big)\Big|^2\,dx\\
&\qquad+\frac{R_\nu^2}{\varepsilon_\nu}\int \psi^2|f|\cdot\Big|v_\nu\Big(\frac{x-x_\nu}{R_\nu}\Big)\Big|\,dx,
\end{aligned}\tag{12b.76}$$

for $\psi\in C_0^\infty\big(B_{R_\nu}(x_\nu)\big)$. Actually, for this new value of u_0, the estimate (12b.57) might change to $|u(x)-u_0|\le 2E$, so at this point we strengthen the hypothesis (12b.62) to

$$2EA\le 1-a<1,\tag{12b.77}$$

in order to get (12b.76). Since $R_\nu^2/\varepsilon_\nu\le R_\nu^{n/p}/C_0$, we have, for $\Psi(x)=\psi(x_\nu+R_\nu x)\in C_0^\infty\big(B_1(0)\big)$,

$$\frac{a}{2}\int \Psi^2|\nabla v_\nu|^2\,dx\le\frac{2}{a}\int|\nabla\Psi|^2|v_\nu|^2\,dx+\frac{R_\nu^{n/p}}{C_0}\int\Psi^2|F|\cdot|v_\nu|\,dx,\tag{12b.78}$$

where $F(x)=f(x_\nu+R_\nu x)$.

Since $\|v_\nu\|_{L^2(B_1(0))}=1$, if $\Psi\le 1$, we have

$$\int\Psi^2|F|\cdot|v_\nu|\,dx\le\Big(\int_{B_1(0)}|F|^2\,dx\Big)^{1/2}\le C_1R_\nu^{-n/p}\tag{12b.79}$$

if $f\in M_2^p$, so we have

$$\frac{a}{2}\int\Psi^2|\nabla v_\nu|^2\,dx\le\frac{2}{a}\int|\nabla\Psi|^2|v_\nu|^2\,dx+\frac{C_1}{C_0}\|f\|_{M_2^p}.\tag{12b.80}$$

This implies that v_ν is bounded in $H^1\big(B_\rho(0)\big)$ for each $\rho<1$. Now, as in (12.89), we can pass to a further subsequence and obtain

$$\begin{aligned}
v_\nu&\longrightarrow v\ \text{ strongly in }\ L^2_{\text{loc}}\big(B_1(0)\big),\\
\nabla v_\nu&\longrightarrow\nabla v\ \text{ weakly in }\ L^2_{\text{loc}}\big(B_1(0)\big).
\end{aligned}\tag{12b.81}$$

Thus, as in (12.90), we can pass to the limit in (12b.70), to obtain

$$\partial_j b^{jk}_{\alpha\beta} \partial_k v^\beta = 0 \quad \text{on } B_1(0). \tag{12b.82}$$

Also, by (12b.73),

$$V(0,1) = \|v\|_{L^2(B_1(0))} \le 1, \quad V(0,\rho) \ge 1. \tag{12b.83}$$

This contradicts Lemma 12.12, which requires $V(0,\rho) \le (1/2)V(0,1)$.

Now that we have Lemma 12B.4, the proof of Proposition 12B.3 is easily completed, by estimates similar to those in (12.69)–(12.73).

We can combine Propositions 12B.2 and 12B.3 to obtain the following:

Corollary 12B.5. *Let $u \in H^1(\Omega) \cap C(\Omega)$ solve (12b.18). If the very strong ellipticity condition (12b.53) holds and $B(x,u,\nabla u)$ is a quadratic form in ∇u, then, given $p \ge n/2,\ q \in (1,\infty),\ s \ge 0$,*

$$f \in M^p_2 \cap H^{s,q} \Longrightarrow u \in H^{s+2,q}. \tag{12b.84}$$

We mention that there are improvements of Proposition 12B.3, in which the hypothesis that u is continuous is relaxed to the hypothesis that the local oscillation of u is sufficiently small (see [HW]). For a number of results in the case when the hypothesis (12b.4) is strengthened to

$$|B(x,u,p)| \le C\langle p\rangle^a,$$

for some $a < 2$, see [Gia]. Extensions of Corollary 12B.5, involving Morrey space estimates, can be found in [T2].

Corollary 12B.5 implies that any harmonic map (satisfying (12b.17)) is smooth wherever it is continuous. An example of a discontinuous harmonic map from $\mathbb{R}^3$ to the unit sphere $S^2 \subset \mathbb{R}^3$ is

$$u(x) = \frac{x}{|x|}. \tag{12b.85}$$

It has been shown by F. Helein [Hel2] that any harmonic map $u : \Omega \to M$ from a two-dimensional manifold Ω into a compact Riemannian manifold M is smooth. Here we will give the proof of Helein's first result of this nature:

Proposition 12B.6. *Let Ω be a two-dimensional Riemannian manifold and let*

$$u : \Omega \longrightarrow S^m \tag{12b.86}$$

be a harmonic map into the standard unit sphere $S^m \subset \mathbb{R}^{m+1}$. Then $u \in C^\infty(\Omega)$.

Proof. We are assuming that $u \in H^1_{\mathrm{loc}}(\Omega)$, that u satisfies (12b.86), and that the components u_j of $u = (u_1, \dots, u_{m+1})$ satisfy

$$\Delta u_j + u_j |\nabla u|^2 = 0. \tag{12b.87}$$

Here, Δu_j and $|\nabla u|^2 = \sum |\nabla u_\ell|^2$ are determined by the Riemannian metric on Ω, but the property of being a harmonic map is invariant under conformal changes in this metric (see Chap. 15, § 2, for more on this), so we may as well take Ω to be an open set in $\mathbb{R}^2$, and $\Delta = \partial_1^2 + \partial_2^2$ the standard Laplace operator. Now $|u(x)|^2 = 1$ a.e. on Ω implies

$$\sum_{j=1}^{m+1} u_j (\partial_i u_j) = 0, \quad i = 1, 2, \tag{12b.88}$$

and putting this together with (12b.87) gives

$$\Delta u_j = - \sum_{k=1}^{m+1} (u_j \nabla u_k - u_k \nabla u_j) \cdot \nabla u_k, \quad \forall\ j. \tag{12b.89}$$

On the other hand, a calculation gives

$$\operatorname{div}(u_j \nabla u_k - u_k \nabla u_j) = \sum_\ell \partial_\ell (u_j \partial_\ell u_k - u_k \partial_\ell u_j) = 0, \tag{12b.90}$$

for all j and k. Furthermore, since $u \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$,

$$u_j \nabla u_k - u_k \nabla u_j \in L^2_{\mathrm{loc}}(\Omega), \quad \nabla u_k \in L^2_{\mathrm{loc}}(\Omega). \tag{12b.91}$$

Now Proposition 12.14 of Chap. 13 implies

$$\sum_k (u_j \nabla u_k - u_k \nabla u_j) \cdot \nabla u_k = f_j \in \mathfrak{H}^1_{\mathrm{loc}}(\Omega), \tag{12b.92}$$

where $\mathfrak{H}^1_{\mathrm{loc}}(\Omega)$ is the local Hardy space, discussed in § 12 of Chap. 13. Also, by Corollary 12.12 of Chap. 13, when $\dim \Omega = 2$,

$$\Delta u_j = -f_j \in \mathfrak{H}^1_{\mathrm{loc}}(\Omega) \Longrightarrow u_j \in C(\Omega). \tag{12b.93}$$

Now that we have $u \in C(\Omega)$, Proposition 12B.6 follows from Corollary 12B.5.

If $\dim \Omega > 2$, there are results on partial regularity for harmonic maps $u : \Omega \to M$, for energy-minimizing harmonic maps [SU] and for "stationary" harmonic maps; see [Ev4] and [Bet]. See also [Si2], for an exposition. On the other hand, there is an example due to T. Riviere [Riv] of a harmonic map for which there is no partial regularity.

We mention another system of the type (12b.1), the 3×3 system

$$\Delta u = 2H u_x \times u_y \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{12b.94}$$

Here H is a real constant, Ω is a bounded open set in $\mathbb{R}^2$, and $g \in C^\infty(\overline{\Omega}, \mathbb{R}^3)$. We seek $u : \overline{\Omega} \to \mathbb{R}^3$. This equation arises in the study of surfaces in $\mathbb{R}^3$ of constant mean curvature H. In fact, if $\Sigma \subset \mathbb{R}^3$ is a surface and $u : \Omega \to \Sigma$ a conformal map (using, e.g., isothermal coordinates) then, by (6.10) and (6.15), Σ has constant mean curvature H if and only if (12b.94) holds. In one approach to the analogue of the Plateau problem for surfaces of mean curvature H, the problem (12b.94) plays a role parallel to that played by $\Delta u = 0$ in the study of the Plateau problem for minimal surfaces (the $H = 0$ case) in § 6. For this reason, in some articles (12b.94) is called the "equation of prescribed mean curvature," though that term is a bit of a misnomer.

The equation (12b.94) is satisfied by a critical point of the functional

$$J(u) = \int_\Omega \Big\{ \frac{1}{2} |\nabla u|^2 + \frac{2}{3} H (u \cdot u_x \times u_y) \Big\}\, dx\, dy, \tag{12b.95}$$

acting on the space

$$V = \{ u \in H^1(\Omega, \mathbb{R}^3) : u = g \text{ on } \partial\Omega \}. \tag{12b.96}$$

That J is well defined and smooth on V follows from the following estimate of Rado:

$$|V(u) - V(g)|^2 \le \frac{1}{32\pi} \big(\|\nabla u\|_{L^2}^2 + \|\nabla g\|_{L^2}^2 \big)^3, \tag{12b.97}$$

provided $u = g$ on $\partial\Omega$, where

$$V(u) = \int_\Omega (u \cdot u_x \times u_y)\, dx\, dy. \tag{12b.98}$$

The boundary problem (12b.94) is not solvable for all g, though it is known to be solvable provided

$$|H| \cdot \|g\|_{L^\infty} \le 1. \tag{12b.99}$$

We refer to [Str1] for a discussion of this and also a treatment of the Plateau problem for surfaces of mean curvature H, using (12b.94). Here we merely mention that given $u \in H^1(\Omega, \mathbb{R}^3)$, solving (12b.94), the fact that

$$u \in C(\overline{\Omega}, \mathbb{R}^3) \tag{12b.100}$$

then follows from Corollary 12.12 and Proposition 12.14 of Chap. 13, just as in (12b.93). Hence Corollary 12B.5 is applicable. This result, established by [Wen], was an important precursor to Proposition 12.13 of Chap. 13.

13. Elliptic regularity IV (Krylov–Safonov estimates)

In this section we obtain estimates for solutions to second-order elliptic equations of the form

$$Lu = f, \quad Lu = a^{jk}(x)\,\partial_j\partial_k u + b^j(x)\,\partial_j u + c(x)u, \tag{13.1}$$

on a domain $\Omega \subset \mathbb{R}^n$. We assume that a^{jk}, b^j, and c are real-valued and that $a^{jk} \in L^\infty(\Omega)$, with

$$\lambda|\xi|^2 \le a^{jk}(x)\xi_j\xi_k \le \Lambda|\xi|^2, \tag{13.2}$$

for certain $\lambda, \Lambda \in (0,\infty)$. We define

$$\mathcal{D} = \det\,(a^{jk}), \quad \mathcal{D}_* = \mathcal{D}^{1/n}. \tag{13.3}$$

A. Alexandrov [Al] proved that if $|b|/\mathcal{D}_* \in L^n(\Omega)$ and $c \le 0$ on Ω, then

$$u \in C(\overline{\Omega}) \cap H^{2,n}_{\mathrm{loc}}(\Omega), \quad Lu \ge f \text{ on } \Omega, \tag{13.4}$$

implies

$$\sup_{x\in\Omega} u(x) \le \sup_{y\in\partial\Omega} u^+(y) + C\|\mathcal{D}_*^{-1} f\|_{L^n(\Omega)}, \tag{13.5}$$

where $C = C\big(n, \operatorname{diam}\Omega, \|b/\mathcal{D}_*\|_{L^n}\big)$. We will not make use of this and will not include a proof, but we will establish the following result of I. Bakelman [B], essentially a more precise version of (13.5) for the special case $b^j = c = 0$ (under stronger regularity hypotheses on u). It is used in some proofs of (13.5) (see [GT]).

To formulate this result, set

$$\begin{aligned}\Gamma^+ = \{y \in \Omega : u(x) \le u(y) + p\cdot(x-y), \forall\, x \in \Omega,\\ \text{for some } p = p(y) \in \mathbb{R}^n\}.\end{aligned} \tag{13.6}$$

If $u \in C^1(\Omega)$, then y belongs to Γ^+ if and only if the graph of u lies everywhere *below* its tangent plane at $\big(y, u(y)\big)$. If $u \in C^2(\Omega)$, then u is concave on Γ^+, that is, $(\partial_j\partial_k u) \le 0$ on Γ^+.

Proposition 13.1. *If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, we have*

$$\sup_{x\in\Omega} u(x) \le \sup_{y\in\partial\Omega} u(y) + \frac{d}{nV_n^{1/n}} \left\| \mathcal{D}_*^{-1}(a^{jk}\partial_j\partial_k u)\right\|_{L^n(\Gamma+)}, \tag{13.7}$$

where $d =$ diam Ω, and V_n is the volume of the unit ball in $\mathbb{R}^n$.

To establish this, we use the matrix inequality

$$(\det A)(\det B) \le \Big(\frac{1}{n}\operatorname{Tr} AB\Big)^n, \tag{13.8}$$

for positive, symmetric, $n \times n$ matrices A and B. (See the exercise at the end of this section for a proof.) Setting

$$A = -H(u) = -\big(\partial_j\partial_k u(x)\big), \quad B = \big(a^{jk}(x)\big), \quad x \in \Gamma^+, \tag{13.9}$$

where $H(u)$ is the Hessian matrix, as in (3.7a), we have

$$|\det H(u)| \le \mathcal{D}^{-1}\Big(-\frac{1}{n}a^{jk}\,\partial_j\partial_k u\Big)^n \quad \text{on } \Gamma^+. \tag{13.10}$$

Thus Proposition 13.1 follows from

Lemma 13.2. *For $u \in C^2(\Omega) \cap C(\overline{\Omega})$, we have*

$$\sup_{x\in\Omega} u(x) \le \sup_{y\in\partial\Omega} u(y) + \frac{d}{V_n^{1/n}}\Big(\int_{\Gamma+} |\det H(u)|\,dx\Big)^{1/n}. \tag{13.11}$$

Proof. Replacing u by $u - \sup_{\partial\Omega} u$, it suffices to assume $u \le 0$ on $\partial\Omega$. Define $\chi(\Omega)$ to be $\bigcup_{y\in\Omega}\chi(y)$, where

$$\chi(y) = \{p \in \mathbb{R}^n : u(x) \le u(y) + p\cdot(x-y), \forall\, x \in \Omega\}, \tag{13.12}$$

so $\chi(y) \ne \emptyset \Leftrightarrow y \in \Gamma^+$. Also, if $u \in C^1(\Omega)$ (as we assume here),

$$\chi(y) = \{Du(y)\}, \ \text{ for } y \in \Gamma^+. \tag{13.13}$$

Thus the Lebesgue measure of $\chi(\Omega)$ is given by

$$\mathcal{L}^n\big(\chi(\Omega)\big) = \mathcal{L}^n\big(\chi(\Gamma^+)\big) = \mathcal{L}^n\big(Du(\Gamma^+)\big) \le \int_{\Gamma+} |\det H(u)|\,dx. \tag{13.14}$$

Thus it suffices to show that if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ and $u \le 0$ on $\partial\Omega$, then

$$\sup_{x\in\Omega} u(x) \le \frac{d}{V_n^{1/n}} \mathcal{L}^n\big(\chi(\Omega)\big). \tag{13.15}$$

This is basically a comparison result. Assume $\sup u > 0$ is attained at x_0. Let W_1 be the function on $\overline{\Omega}$ whose graph is the cone with apex at $\big(x_0, u(x_0)\big)$ and base $\partial\Omega \times \{0\}$. Then, if $\chi_{W_1}(y)$ denotes the function (13.12) with u replaced by W_1, we have

$$\chi_u(\Omega) \supset \chi_{W_1}(\Omega). \tag{13.16}$$

Similarly, if W_2 is the function on $B_d(x_0)$ whose graph is the cone with apex at $\big(x_0, u(x_0)\big)$ and base $\{x : |x - x_0| = d\} \times \{0\}$, then

$$\chi_{W_1}(\Omega) \supset \chi_{W_2}\big(B_d(x_0)\big). \tag{13.17}$$

Finally, the inequality

$$\sup W_2 \le \frac{d}{V_n^{1/n}} \mathcal{L}^n\big(\chi_{W_2}(B_d(x_0))\big) \tag{13.18}$$

is elementary, so we have (13.15), and hence Lemma 13.2 is proved.

We now make the assumption that

$$\frac{\Lambda}{\lambda} \le \gamma, \quad \left(\frac{|b|}{\lambda}\right)^2 \le \nu, \quad \frac{|c|}{\lambda} \le \nu, \tag{13.19}$$

and establish the following local maximum principle, following [GT].

Proposition 13.3. *Let $u \in H^{2,n}(\Omega)$, $Lu \ge f$, $f \in L^n(\Omega)$. Then, for any ball $B = B_{2R}(y) \subset \Omega$ and any $p \in (0, n]$, we have*

$$\sup_{x\in B_R(y)} u(x) \le C\left\{\left(\frac{1}{Vol(B)}\int_B (u^+)^p\, dx\right)^{1/p} + \frac{R}{\lambda}\|f\|_{L^n(B)}\right\}, \tag{13.20}$$

where $C = C(n, \gamma, \nu R^2, p)$.

Proof. Translating and dilating, we can assume without loss of generality that $0 \in \Omega$ and $B = B_1(0)$. We will also assume that $u \in C^2(\Omega) \cap H^{2,n}(\Omega)$, since if (13.20) is established in this case, the more general case follows by a simple approximation argument.

Given $\beta \geq 1$, define

$$\eta(x) = \big(1 - |x|^2\big)^\beta, \quad \text{for } |x| \leq 1. \tag{13.21}$$

Setting $v = \eta u$ on B, we have

$$\begin{aligned} a^{jk}\, \partial_j \partial_k v &= \eta a^{jk}\, \partial_j \partial_k u + 2a^{jk}(\partial_j \eta)(\partial_k u) + ua^{jk}\, \partial_j \partial_k \eta \\ &\geq \eta(f - b^j\, \partial_j u - cu) + 2a^{jk}(\partial_j \eta)(\partial_k u) + ua^{jk}\, \partial_j \partial_k \eta. \end{aligned} \tag{13.22}$$

Let Γ_v^+ be as in (13.6), but with u replaced by v, and Ω replaced by B. Clearly, $u \geq 0$ on Γ_v^+. We have

$$|Dv| \leq \frac{v}{1 - |x|} \quad \text{on } \Gamma_v^+, \tag{13.23}$$

so

$$\begin{aligned} |Du| = \eta^{-1}|Dv - uD\eta| &\leq \frac{1}{\eta}\Big(\frac{v}{1 - |x|} + u|D\eta|\Big) \\ &\leq 2(1 + \beta)\eta^{-1/\beta} u \quad \text{on } \Gamma_v^+. \end{aligned} \tag{13.24}$$

Hence

$$\begin{aligned} -a^{jk}\, \partial_j \partial_k v &\leq \Big\{\big(16\beta^2 + 2\beta\eta\big)\Lambda\eta^{-2/\beta} + 2\beta|b|\eta^{-1/\beta} + c\Big\}v + \eta f \\ &\leq C\lambda\eta^{-2/\beta} v + f, \end{aligned} \tag{13.25}$$

on Γ_v^+, where $C = C(n, \beta, \gamma, \nu)$. Of course, $a^{jk}\partial_j \partial_k v \leq 0$ on Γ_v^+. If $\beta \geq 2$, we have, upon applying Proposition 13.1 to v,

$$\begin{aligned} \sup_B\, v &\leq C\Big(\big\|\eta^{-2/\beta} v^+\big\|_{L^n(B)} + \frac{1}{\lambda}\|f\|_{L^n(B)}\Big) \\ &\leq C_1\left\{\big(\sup_B\, v^+\big)^{1-2/\beta}\big\|(u^+)^{2/\beta}\big\|_{L^n(B)} + \frac{1}{\lambda}\|f\|_{L^n(B)}\right\}. \end{aligned} \tag{13.26}$$

Choose $\beta = 2n/p \geq 2$, so we have

$$\sup_B\, v \leq C_1\left\{\big(\sup_B\, v^+\big)^{1-p/n}\|u^+\|_{L^p(B)}^{p/n} + \frac{1}{\lambda}\|f\|_{L^n(B)}\right\}. \tag{13.27}$$

(Here we allow $p < 1$, in which case $\|\cdot\|_{L^p}$ is not a norm, but (13.27) is still valid.) Using the elementary inequality

$$a^{1-p/n} b^{p/n} \leq \varepsilon a + \varepsilon^{-(n/p-1)} b, \quad \forall\, \varepsilon \in (0, \infty), \tag{13.28}$$

and taking $a = \sup_B v^+$, $b = \|u^+\|_{L^p(B)}$, and $\varepsilon = 1/2C_1$, we have (the $R = 1$ case of) (13.20), so Proposition 13.3 is proved.

Replacing u by $-u$, we have an estimate on $\sup_{B_R(y)} (-u)$ when $Lu \le f$. Thus, when $Lu = f$ and the hypotheses of Proposition 13.3 hold, we have

$$\sup_{B_R(y)} |u| \le C \left\{ \left(\frac{1}{\mathrm{Vol}(B)} \int_B |u|^p \, dx \right)^{1/p} + \frac{R}{\lambda} \|f\|_{L^n(B)} \right\}. \tag{13.29}$$

Next we establish a "weak Harnack inequality" of [KrS], which will lead to results on Hölder continuity of solutions of $Lu = f$. This result will also be applied directly in the next section, to results on solutions to certain completely nonlinear equations.

Proposition 13.4. *Assume $u \in H^{2,n}(\Omega)$, $Lu \le f$ in Ω, $f \in L^n(\Omega)$, and $u \ge 0$ on a ball $B = B_{2R}(y) \subset \Omega$. Then*

$$\left(\frac{1}{Vol(B_R)} \int_{B_R} u^p \, dx \right)^{1/p} \le C \left(\inf_{B_R} u + \frac{R}{\lambda} \|f\|_{L^n(B)} \right), \tag{13.30}$$

for some positive $p = p(n, \gamma, \nu R^2)$ and $C = C(n, \gamma, \nu R^2)$.

As before, there is no loss of generality in assuming $B = B_1(0)$. Also, replacing L and f by $\lambda^{-1} L$ and $\lambda^{-1} f$, we can assume $\lambda = 1$.

To begin the proof, take $\varepsilon > 0$ and set

$$\begin{aligned} \overline{u} = u + \varepsilon + \|f\|_{L^n(B)}, \quad & w = \log \frac{1}{\overline{u}}, \\ v = \eta w, \qquad & g = \frac{f}{\overline{u}}, \end{aligned} \tag{13.31}$$

where η is given by (13.21). Note that w is large (positive) where $\overline{u}$ is small. We have

$$\begin{aligned} -a^{jk} \partial_j \partial_k v &= -\eta a^{jk} \partial_j \partial_k w - 2a^{jk} (\partial_j \eta)(\partial_k w) - w a^{jk} \partial_j \partial_k \eta \\ &\le \eta \big[-a^{jk} (\partial_j w)(\partial_k w) + b^j \partial_j w + |c| + g \big] \\ &\qquad - 2a^{jk} (\partial_j \eta)(\partial_k w) - w a^{jk} \partial_j \partial_k \eta \\ &\le \frac{2}{\eta} a^{jk} (\partial_j \eta)(\partial_k \eta) - w a^{jk} \partial_j \partial_k \eta + (|b|^2 + |c| + g)\eta, \end{aligned} \tag{13.32}$$

where the last inequality is obtained via Cauchy's inequality, applied to the inner product $\langle V, W \rangle = V_j a^{jk} W_k$.

Now the form of η implies that $a^{jk}\,\partial_j\partial_k\eta \ge 0$ provided $2(\beta-1)a^{jk}x_jx_k + a^{jj}|x|^2 \ge a^{jj}$, and hence

$$2\beta|x|^2 \ge n\Lambda \Longrightarrow a^{jk}\,\partial_j\partial_k\eta \ge 0. \tag{13.33}$$

Thus, if $\alpha \in (0,1)$, then

$$\beta \ge \frac{n\gamma}{2\alpha},\quad |x| \ge \alpha \Longrightarrow a^{jk}\,\partial_j\partial_k\eta \ge 0. \tag{13.34}$$

Hence, on the set $B^+ = \{x \in B : w(x) > 0\}$, we have

$$\begin{aligned} -a^{jk}\,\partial_j\partial_k v &\le 4\beta^2\big(1-|x|^2\big)^{\beta-2}|x|^2 + v\chi_{B_\alpha}\sup_{B_\alpha}\left(-\frac{a^{jk}\,\partial_j\partial_k\eta}{\eta}\right) \\ &\qquad + \big(|b|^2+|c|+g\big)\eta \\ &\le 4\beta^2\Lambda + |b|^2 + |c| + g + \frac{2n\beta\Lambda}{1-\alpha^2}v\chi_{B_\alpha}. \end{aligned} \tag{13.35}$$

Note that $\|g\|_{L^n(B)} \le 1$. Thus Proposition 13.1 yields

$$\sup_B\, v \le C\big(1+\|v^+\|_{L^n(B_\alpha)}\big), \tag{13.36}$$

with $C = C(n,\alpha,\gamma,\nu)$.

Note that if u satisfies the hypotheses of Proposition 13.4 and $t \in (0,\infty)$, then u/t satisfies $L(u/t) \le f/t$, and the analogue of w in (13.31) is $w-k$, where $k = \log(1/t)$. The function g in (13.31) is unchanged, and, working through (13.32)–(13.36), we obtain the following extension of (13.36):

$$\sup_B\, \eta(w-k) \le C\big(1+\|\eta(w-k)^+\|_{L^n(B_\alpha)}\big), \quad \forall\, k \in \mathbb{R}, \tag{13.37}$$

with constants independent of k.

The next stage in the proof of Proposition 13.4 will involve a decomposition into cubes of the sort used for Calderon–Zygmund estimates in § 5 of Chap. 13. To set up some notation, given $y \in \mathbb{R}^n$, $R > 0$, let $Q_R(y)$ denote the open cube centered at y, of edge $2R$:

$$Q_R(y) = \{x \in \mathbb{R}^n : |x_j - y_j| < R,\ 1 \le j \le n\}. \tag{13.38}$$

If $\alpha < 1/\sqrt{n}$, then $Q_\alpha = Q_\alpha(0) \subset\subset B$.

The cube decomposition we will use in the proof of Lemma 13.5 below can be described in general as follows. Let Q_0 be a cube in $\mathbb{R}^n$, let $\varphi \ge 0$ be an element of $L^1(Q_0)$, and suppose $\int_{Q_0}\varphi\,dx \le t\mathcal{L}^n(Q_0)$, $t \in (0,\infty)$. Bisecting the edges of Q_0, we subdivide it into 2^n subcubes. Those subcubes that satisfy

$\int_Q \varphi\, dx \leq t\mathcal{L}^n(Q)$ are similarly subdivided, and this process is repeated indefinitely. Let $\mathcal{F}$ denote the set of subcubes so obtained that satisfy

$$\int_Q \varphi\, dx > t\mathcal{L}^n(Q);$$

we do not further subdivide these cubes. For each $Q \in \mathcal{F}$, denote by $\widetilde{Q}$ the subcube whose subdivision gives Q. Since $\mathcal{L}^n(\widetilde{Q})/\mathcal{L}^n(Q) = 2^n$, we see that

$$t < \frac{1}{\mathcal{L}^n(Q)} \int_Q \varphi\, dx \leq 2^n t, \quad \forall\, Q \in \mathcal{F}. \tag{13.39}$$

Also, setting $F = \bigcup_{Q\in\mathcal{F}} Q$ and $G = Q_0 \setminus F$, we have

$$\varphi \leq t, \quad \text{a.e. in } G. \tag{13.40}$$

This subdivision was also done in the proof of Lemma 5.5 in Chap. 13. Let us also set $\widetilde{F} = \bigcup_{Q\in\mathcal{F}} \widetilde{Q}$; since $Q \in \mathcal{F} \Rightarrow \widetilde{Q} \notin \mathcal{F}$, we have

$$\int_{\widetilde{F}} \varphi\, dx \leq t\mathcal{L}^n(\widetilde{F}). \tag{13.41}$$

In particular, when φ is the characteristic function χ_Γ of a measurable subset Γ of Q_0, of measure $\leq t \cdot \mathcal{L}^n(Q_0)$, we deduce from (13.40)–(13.41) that

$$\mathcal{L}^n(\Gamma) = \mathcal{L}^n(\Gamma \cap \widetilde{F}) \leq t\mathcal{L}^n(\widetilde{F}). \tag{13.42}$$

We have the following measure-theoretic result:

Lemma 13.5. *Let Q_0 be a cube in $\mathbb{R}^n$, $w \in L^1(Q_0)$, and, for $k \in \mathbb{R}$, set*

$$\Gamma_k = \{x \in Q_0 : w(x) \leq k\}. \tag{13.43}$$

Suppose there are positive constants $\delta < 1$ and C such that

$$\sup_{Q_0 \cap Q_{3r}(z)} (w - k) \leq C \tag{13.44}$$

whenever k and $Q = Q_r(z) \subset Q_0$ satisfy

$$\mathcal{L}^n(\Gamma_k \cap Q) \geq \delta\mathcal{L}^n(Q). \tag{13.45}$$

Then, for all $k \in \mathbb{R}$,

$$\sup_{Q_0} (w - k) \le C \left(1 + \frac{\log\bigl(\mathcal{L}^n(\Gamma_k)/\mathcal{L}^n(Q_0)\bigr)}{\log\ \delta}\right). \tag{13.46}$$

Proof. We show by induction that

$$\sup_{Q_0} (w - k) \le mC, \tag{13.47}$$

for any $m \in \mathbb{Z}^+$ and $k \in \mathbb{R}$ such that $\mathcal{L}^n(\Gamma_k) \ge \delta^m \mathcal{L}^n(Q_0)$. This is true by hypothesis if $m = 1$. Suppose that it holds for $m = M \in \mathbb{Z}^+$ and that $\mathcal{L}^n(\Gamma_k) \ge \delta^{M+1}\mathcal{L}^n(Q_0)$. Define $\widetilde{\Gamma}_k$ by

$$\widetilde{\Gamma}_k = \bigcup \bigl\{Q_{3r}(z) \cap Q_0 : \mathcal{L}^n\bigl(Q_r(z) \cap \Gamma_k\bigr) \ge \delta\, \mathcal{L}^n\bigl(Q_r(z)\bigr)\bigr\}. \tag{13.48}$$

Applying the estimate (13.42), with $t = \delta$, we see that either $\widetilde{\Gamma}_k = Q_0$ or

$$\mathcal{L}^n(\widetilde{\Gamma}_k) \ge \delta^{-1}\mathcal{L}^n(\Gamma_k) \ge \delta^M \ \mathrm{vol}(Q_0), \tag{13.49}$$

and hence, replacing k by $k + C$, we obtain

$$\sup_{Q_0}(w - k) \le (M + 1)C, \tag{13.50}$$

which verifies (13.47) for $m = M + 1$.

Now, the estimate (13.46) follows by choosing m appropriately, and the lemma is proved.

Returning to the estimation of the functions defined in (13.31), we see that (13.36) implies

$$\sup_B\ v \le C\bigl(1 + \|v^+\|_{L^n(Q_\alpha)}\bigr) \le C\Bigl(1 + \bigl[\mathrm{vol}(Q_\alpha^+)\bigr]^{1/n}\ \sup_B\ v^+\Bigr), \tag{13.51}$$

where $Q_\alpha = Q_\alpha(0)$, as stated below (13.38), and

$$Q_\alpha^+ = \{x \in Q_\alpha : v(x) > 0\} = \{x \in Q_\alpha : \overline{u}(x) < 1\}.$$

Hence, if C is the constant in (13.36),

$$\frac{\mathrm{vol}(Q_\alpha^+)}{\mathrm{vol}(Q_\alpha)} \le \Bigl(\frac{1}{4\alpha C}\Bigr)^n = \theta \Longrightarrow \sup_B\ v \le 2C. \tag{13.52}$$

Now choose $\alpha = 1/3n$, and take $\theta = (4\alpha C)^{-n}$, as in (13.52). Using the coordinate change $x \mapsto \alpha(x - z)/r$, we obtain for any cube $Q = Q_r(z)$ such that $B_{3nr}(z) \subset B$, the implication

$$\frac{\mathrm{vol}(Q^+)}{\mathrm{vol}(Q)} \le \theta \Longrightarrow \sup_{Q_{3r}(z)} w \le C(n,\gamma,\nu). \tag{13.53}$$

With α and θ as specified above, take $\delta = 1-\theta,\ Q_0 = Q_\alpha(0)$, and note that the estimate (13.53) holds also when w is replaced by $w-k$, and Q^+ is replaced by the set $\{x \in Q : w(x)-k > 0\}$, as a consequence of (13.37). Let

$$\mu(t) = \mathcal{L}^n\big(\{x \in Q_0 : \overline{u}(x) > t\}\big). \tag{13.54}$$

Setting $k = \log\ 1/t$, we have from Lemma 13.5 the estimate

$$\mu(t) \le C\big(\inf_{Q_0} t^{-1}\overline{u}\big)^\kappa, \quad \forall\, t > 0, \tag{13.55}$$

where $C = C(n,\gamma,\nu),\ \kappa = \kappa(n,\gamma,\nu)$. Replacing the cube Q_0 by the inscribed ball $B_\alpha(0),\ \alpha = 1/3n$, and using the identity

$$\int_{Q_0} (\overline{u})^p\, dx = p\int_0^\infty t^{p-1}\mu(t)\, dt, \tag{13.56}$$

we have

$$\int_{B_\alpha} (\overline{u})^p\, dx \le C\big(\inf_{B_\alpha} \overline{u}\big)^p, \quad \text{for } p = \frac{\kappa}{2}. \tag{13.57}$$

The inequality (13.30) then follows by letting $\varepsilon \to 0$ if we use a covering argument to extend (13.57) to arbitrary $\alpha < 1$ (especially, $\alpha = 1/2$) and use the coordinate transformation $x \mapsto (x-y)/2R$. Thus Proposition 13.4 is established.

Putting together (13.29) and (13.30), we have the following.

Corollary 13.6. *Assume $u \in H^{2,n}(\Omega),\ Lu = f$ on $\Omega,\ f \in L^n(\Omega)$, and $u \ge 0$ on a ball $B = B_{4R}(y) \subset \Omega$. Then*

$$\sup_{B_R(y)} u(x) \le C_1\Big(\inf_{B_{2R}(y)} u + \frac{R}{\lambda}\|f\|_{L^n(B_{4R})}\Big), \tag{13.58}$$

for some $C_1 = C_1(n,\gamma,\nu R^2)$. In particular, if $u \ge 0$ on Ω,

$$Lu = 0 \Longrightarrow \sup_{B_R(y)} u(x) \le C_1 \inf_{B_{2R}(y)} u(x). \tag{13.59}$$

We can use this to establish Hölder estimates on solutions to $Lu = f$. We will actually apply Corollary 13.6 to $L_1 = a^{jk}\,\partial_j\partial_k + b^j\,\partial_j$, so $L_1 u = f_1 = f - cu$. Suppose that

$$a = \inf_{B_{4R}(y)} u \leq \sup_{B_{4R}(y)} u = b. \tag{13.60}$$

Then $v = (u-a)/(b-a)$ is ≥ 0 on $B_{4R}(y)$, and $L_1 v = f_1/(b-a)$, so Corollary 13.6 yields

$$\sup_{B_R(y)} \frac{u-a}{b-a} \leq C_1 \Big(\inf_{B_{2R}(y)} \frac{u-a}{b-a} + \frac{R}{\lambda} \frac{1}{b-a} \| f - cu \|_{L^n(B_{4R})} \Big). \tag{13.61}$$

Without loss of generality, we can assume $C_1 \geq 1$. Now given this, one of the following two cases must hold:

$$C_1 \inf_{B_{2R}(y)} \frac{u-a}{b-a} \geq \frac{1}{2} \sup_{B_R(y)} \frac{u-a}{b-a}, \tag{i}$$

$$C_1 \inf_{B_{2R}(y)} \frac{u-a}{b-a} < \frac{1}{2} \sup_{B_R(y)} \frac{u-a}{b-a}. \tag{ii}$$

If case (i) holds, then either

$$\sup_{B_R(y)} \frac{u-a}{b-a} \leq \frac{1}{2} \quad \text{or} \quad \inf_{B_{2R}(y)} \frac{u-a}{b-a} \geq \frac{1}{4C_1},$$

and hence (since we are assuming $C_1 \geq 1$)

$$\text{(i)} \implies \operatorname*{osc}_{B_R(y)} u \leq \left(1 - \frac{1}{4C_1}\right) \operatorname*{osc}_{B_{4R}(y)} u. \tag{13.62}$$

If case (ii) holds, then

$$\sup_{B_R(y)} \frac{u-a}{b-a} \leq \frac{2R}{\lambda} \frac{1}{b-a} \| f - cu \|_{L^n(B_{4R})},$$

so

$$\text{(ii)} \implies \operatorname*{osc}_{B_R(y)} u \leq \frac{2R}{\lambda} \| f - cu \|_{L^n(B_{4R})}, \tag{13.63}$$

which is bounded by $C_2 R$ in view of the sup-norm estimate (13.29). Consequently, under the hypotheses of Corollary 13.6, we have

$$\operatorname*{osc}_{B_R(y)} u \leq \max\Big(C_2 R, \Big(1 - \frac{1}{C_1}\Big) \operatorname*{osc}_{B_{4R}(y)} u \Big), \tag{13.64}$$

with $C_1 = C_1(n, \gamma, \nu R_0^2)$, $C_2 = C_2(n, \gamma, \nu R_0^2)\big[\|f\|_{L^n(\Omega)} + \|u\|_{L^n(\Omega)}\big]$, given $B_{4R_0}(y) \subset \Omega$, $R \le R_0$. Therefore, we have the following:

Theorem 13.7. *Assume $u \in H^{2,n}(\Omega)$, $Lu = f$, and $f \in L^n(\Omega)$. Given $\mathcal{O} \subset\subset \Omega$, there is a positive $\mu = \mu(\mathcal{O}, \Omega, n, \gamma, \nu)$ such that*

$$\|u\|_{C^\mu(\mathcal{O})} \le A\big(\|u\|_{L^n(\Omega)} + \|f\|_{L^n(\Omega)}\big), \tag{13.65}$$

with $A = A(\mathcal{O}, \Omega, n, \gamma, \nu)$.

Some boundary regularity results follow fairly easily from the methods developed above. For the present, assume Ω is a smoothly bounded region in $\mathbb{R}^n$, that

$$u \in H^{2,n}(\Omega) \cap C(\overline{\Omega}), \quad u\big|_{\partial\Omega} \le 0, \tag{13.66}$$

and that $Lu = f$ on Ω. Let $B = B_{2R}(y)$ be a ball centered at $y \in \partial\Omega$. Then, extending (13.20), we have, for any $p \in (0, n]$,

$$\sup_{\Omega \cap B_R(y)} u \le C\left\{\Big(\frac{1}{\mathrm{vol}(B)} \int_{B\cap\Omega} (u^+)^p\, dx\Big)^{1/p} + \frac{R}{\lambda}\|f\|_{L^n(B\cap\Omega)}\right\}, \tag{13.67}$$

with $C = C(n, \gamma, \nu R^2, p)$. To establish this, extend u to be 0 on $B \setminus \Omega$. This extended function might not belong to $H^{2,n}(B)$, but the proof of Proposition 13.3 can still be seen to apply, given the following observation:

Lemma 13.8. *Assume that u satisfies the hypotheses of Proposition 13.1 and that $\Omega \subset \widetilde{\Omega}$, and set $u = 0$ on $\widetilde{\Omega} \setminus \Omega$. Then*

$$\sup_{\widetilde{\Omega}} u \le \sup_{\partial\widetilde{\Omega}} u + \frac{\widetilde{d}}{nV_n^{1/n}} \big\|\mathcal{D}_*^{-1}(a^{jk}\,\partial_j\partial_k u)\big\|_{L^n(\widetilde{\Gamma}^+)}, \tag{13.68}$$

where $\widetilde{d} =$ diam $\widetilde{\Omega}$, and $\widetilde{\Gamma}^+$ is the upper contact set of u, defined as in (13.6), with Ω, replaced by $\widetilde{\Omega}$.

Note that if $u(x) > 0$ anywhere on Ω, then $\widetilde{\Gamma}^+ \subset \Gamma^+$.

The following result extends Proposition 13.4.

Proposition 13.9. *Assume $u \in H^{2,n}(\Omega)$, $Lu = f$ on Ω, $u \ge 0$ on $B \cap \Omega$. Set*

$$m = \inf_{B\cap\partial\Omega} u, \tag{13.69}$$

and

$$\widetilde{u}(x) = \begin{cases} \min\big(m, u(x)\big), & x \in B \cap \Omega, \\ m, & x \in B \setminus \Omega. \end{cases} \tag{13.70}$$

Then

$$\left(\frac{1}{vol(B_R)} \int_{B_R} (\widetilde{u})^p \, dx\right)^{1/p} \le C\Big(\inf_{\Omega \cap B_R} u + \frac{R}{\lambda} \|f\|_{L^n(B\cap\Omega)}\Big), \tag{13.71}$$

for some positive $p = p(n, \gamma, \nu R^2)$ *and* $C = C(n, \gamma, \nu R^2)$.

Proof. One adapts the proof of Proposition 13.4, with u replaced by $\widetilde{u}$. One gets an estimate of the form (13.53), with w replaced by $w - k$, $k \ge -\log m$. From there, one gets an estimate of the form (13.55), for $0 < t \le m$. But $\mu(t) = 0$ for $t > m$, so (13.71) follows as before.

This leads as before to a Hölder estimate:

Proposition 13.10. *Assume* $u \in H^{2,n}(\Omega)$, $Lu = f \in L^n(\Omega)$, $u\big|_{\partial\Omega} = \varphi \in C^\beta(\partial\Omega)$, *and* $\beta > 0$. *Then there is a positive* $\mu = \mu(\Omega, n, \gamma, \nu, \beta)$ *such that*

$$\|u\|_{C^\mu(\overline{\Omega})} \le A\Big(\|u\|_{L^n(\Omega)} + \|f\|_{L^n(\Omega)} + \|\varphi\|_{C^\beta(\partial\Omega)}\Big), \tag{13.72}$$

with $A = A(\Omega, n, \gamma, \nu, \beta)$.

We next establish another type of boundary estimate, which will also be very useful in applications in the following sections. The following result is due to [Kry2]; we follow the exposition in [Kaz] of a proof of L. Caffarelli.

Proposition 13.11. *Assume* $u \in C^2(\overline{\Omega})$ *satisfies*

$$Lu = f, \quad u\big|_{\partial\Omega} = 0. \tag{13.73}$$

Assume

$$\|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} \le K. \tag{13.74}$$

Then there is a Hölder estimate for the normal derivative of u *on* $\partial\Omega$*:*

$$\|\partial_\nu u\|_{C^\alpha(\partial\Omega)} \le CK, \tag{13.75}$$

for some positive $\alpha = \alpha(\Omega, n, \nu, \lambda, \Lambda, K)$ *and* $C = C(\Omega, n, \nu, \lambda, \Lambda)$.

To prove this, we can flatten out a portion of the boundary. After having done so, absorb the terms $b^j(x)\partial_j u + c(x)u$ into f. It suffices to assume that

$$Lu = f \quad \text{on} \quad B^+, \qquad Lu = a^{jk}(x)\,\partial_j\partial_k u, \tag{13.76}$$

where

$$B^+ = \{x \in \mathbb{R}^n : |x| < 4,\ x_n \ge 0\},$$

and that

$$u = 0 \quad \text{on} \quad \Sigma = \{x \in \mathbb{R}^n : |x| < 4,\ x_n = 0\}, \tag{13.77}$$

and to show that there is an estimate

$$\|\partial_n u\|_{C^\alpha(\Gamma)} \le CK, \quad C = C(n, \lambda, \Lambda), \tag{13.78}$$

where K is as in (13.74), with Ω replaced by B^+, $\alpha = \alpha(n, \lambda, \Lambda, K) > 0$, and

$$\Gamma = \{x \in \Sigma : |x| \le 1\}. \tag{13.79}$$

Note that, for $(x', 0) \in \Sigma$,

$$\partial_n u(x', 0) = v(x', 0), \tag{13.80}$$

where

$$v(x) = x_n^{-1} u(x). \tag{13.81}$$

Let us fix some notation. Given $R \le 1$ and $\delta = \lambda/9n\Lambda < 1/2$, let

$$\begin{aligned} Q(R) &= \big\{x \in B^+ : |x'| \le R,\ 0 \le x_n \le \delta R\big\}, \\ Q^+(R) &= \Big\{x \in Q(R) : \frac{1}{2}\delta R \le x_n \le \delta R\Big\} \end{aligned} \tag{13.82}$$

(see Fig. 13.1). Then set

$$m_R = \inf_{Q(R)} v, \quad M_R = \sup_{Q(R)} v, \tag{13.83}$$

so $\operatorname{osc}_{Q(R)} v = M_R - m_R$. Before proving Proposition 13.11, we establish two lemmas.

Lemma 13.12. *Under the hypotheses (13.76) and (13.77), if also $u \ge 0$ on $Q(R)$, then*

$$\inf_{Q^+(R)} v \le \frac{2}{\delta} \inf_{Q(R/2)} v + \frac{R}{\lambda} \sup |f|. \tag{13.84}$$

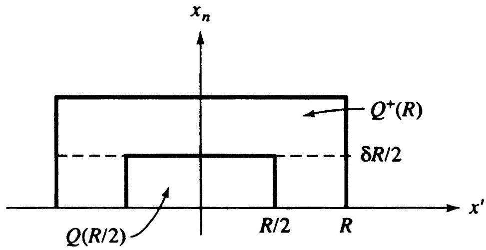

FIGURE 13.1 Setup for Boundary Estimate

Proof. Let $\gamma = \inf\{v(x) : |x'| \le R,\ x_n = \delta R\}$, and set

$$z(x) = \gamma x_n\Bigl(\delta - \frac{2\delta}{R^2}|x'|^2 + \frac{1}{R}x_n\Bigr) - \frac{1}{2\lambda}x_n\bigl(\delta R - x_n\bigr)\sup|f|. \tag{13.85}$$

Given $\delta \in (0, 1/2]$, we have the following behavior on $\partial Q(R)$:

$$\begin{aligned} &z(x) = 0, \quad \text{for } x = (x', 0), && \text{(bottom)},\\ &z(x) < 0 \quad \text{on } \{x \in Q(R) : |x'| = R\}, && \text{(side)},\\ &z(x) < 2\gamma\delta^2 R < \gamma\delta R \quad \text{on } \{x \in Q(R) : x_n = \delta R\} && \text{(top)}. \end{aligned} \tag{13.86}$$

Also,

$$Lz \le -\sup|f| \le f \quad \text{on } Q(R) \quad \text{if } \delta = \frac{\lambda}{9n\Lambda}. \tag{13.87}$$

Since $u \ge 0$ on $Q(R)$ and $u = x_n v \ge \gamma\delta R$ on the top of $Q(R)$, we have

$$L(u - z) \ge 0 \ \text{ on } \ Q(R), \quad u \ge z \ \text{ on } \ \partial Q(R). \tag{13.88}$$

Thus, by the maximum principle, $u \ge z$ on $Q(R)$, so $v(z) \ge z(x)/x_n$ on $Q(R)$. Hence

$$\inf_{Q(R/2)} v \ge \frac{\delta}{2}\Bigl(\gamma - \frac{R}{\lambda}\sup|f|\Bigr). \tag{13.89}$$

Since $\gamma \ge \inf_{Q^+(R)} v$, this yields (13.84).

Lemma 13.13. *If u satisfies (13.76) and (13.77) and $u \ge 0$ on $Q(2R)$, then*

$$\sup_{Q^+(R)} v \le C\Bigl(\inf_{Q^+(R)} v + R\sup|f|\Bigr), \tag{13.90}$$

with $C = C(n, \lambda, \Lambda, K)$.

Proof. By (13.58), if $x \in Q^+(R),\ r = \delta R/8$, we have

$$\sup_{B_r(x)} u \le C\Bigl(\inf_{B_r(x)} u + r^2 \sup |f|\Bigr). \tag{13.91}$$

Since $\delta R/2 \le x_n \le \delta R$ on $Q^+(R)$, (13.90) follows from this plus a simple covering argument.

We now prove Proposition 13.11. The various factors C_j will all have the form $C_j = C_j(n, \lambda, \Lambda, K)$. If we apply (13.90), with u replaced by $u - m_{2R}x_n \ge 0$, on $Q(2R)$, we obtain

$$\sup_{Q^+(R)} (v - m_{2R}) \le C_1\Bigl(\inf_{Q^+(R)} (v - m_{2R}) + R \sup |f|\Bigr). \tag{13.92}$$

By Lemma 13.12, this is

$$\begin{aligned} &\le C_2\Bigl(\inf_{Q(R/2)} (v - m_{2R}) + R \sup |f|\Bigr) \\ &= C_2\bigl(m_{R/2} - m_{2R} + R \sup |f|\bigr). \end{aligned} \tag{13.93}$$

Reasoning similarly, with u replaced by $M_{2R}x_n - u \ge 0$ on $Q(2R)$, we have

$$\sup_{Q^+(R)} (M_{2R} - v) \le C_2\bigl(M_{2R} - M_{R/2} + R \sup |f|\bigr). \tag{13.94}$$

Summing these two inequalities yields

$$M_{2R} - m_{2R} \le C_3\bigl[(M_{2R} - m_{2R}) - (M_{R/2} - m_{R/2}) + R \sup |f|\bigr], \tag{13.95}$$

which implies

$$\operatorname*{osc}_{Q(R/2)} v \le \vartheta \operatorname*{osc}_{Q(2R)} v + R \sup |f|, \tag{13.96}$$

with $\vartheta = 1 - 1/C_3 < 1$. This readily implies the Hölder estimate (13.78), proving Proposition 13.11.

Exercises

1. Prove the matrix inequality (13.8). (*Hint*: Set $C = A^{1/2} \ge 0$ and reduce (13.8) to

$$\frac{1}{n} \operatorname{Tr} X \ge (\det X)^{1/n}, \tag{13.97}$$

for $X = CBC \ge 0$. This is equivalent to the inequality

$$\frac{1}{n}(\lambda_1 + \cdots + \lambda_n) \ge (\lambda_1 \cdots \lambda_n)^{1/n}, \quad \lambda_j > 0, \tag{13.98}$$

which is called the *arithmetic-geometric mean inequality*. It can be deduced from the facts that $\log x$ is concave and that *any* concave function φ satisfies

$$\varphi\Big(\frac{1}{n}(\lambda_1 + \cdots + \lambda_n)\Big) \geq \frac{1}{n}\big[\varphi(\lambda_1) + \cdots + \varphi(\lambda_n)\big].) \tag{13.99}$$

14. Regularity for a class of completely nonlinear equations

In this section we derive Hölder estimates on the second derivatives of real-valued solutions to nonlinear PDE of the form

$$F(x, D^2u) = 0, \tag{14.1}$$

satisfying the following conditions. First we require uniform strong ellipticity:

$$\lambda|\xi|^2 \leq \partial_{\zeta_{jk}} F(x, u, \nabla u, \partial^2 u)\xi_j \xi_k \leq \Lambda|\xi|^2, \tag{14.2}$$

with $\lambda, \Lambda \in (0, \infty)$, constants. Next, we require that F be a *concave* function of ζ:

$$\partial_{\zeta_{jk}} \partial_{\zeta_{\ell m}} F(x, u, p, \zeta)\Xi_{jk}\Xi_{\ell m} \leq 0, \quad \Xi_{jk} = \Xi_{kj}, \tag{14.3}$$

provided $\zeta = \partial^2 u(x),\ p = \nabla u(x)$.

As an example, consider

$$F(x, u, p, \zeta) = \log\ \det \zeta - f(x, u, p). \tag{14.4}$$

Then $(D_\zeta F)\Xi = \mathrm{Tr}(\zeta^{-1}\Xi)$, so the quantity (14.3) is equal to

$$-\mathrm{Tr}\big(\zeta^{-1}\Xi\zeta^{-1}\Xi\big) = -\,\mathrm{Tr}\big(\zeta^{-1/2}\Xi\zeta^{-1}\Xi\zeta^{-1/2}\big), \quad \Xi^t = \Xi, \tag{14.5}$$

provided the real, symmetric, $n \times n$ matrix ζ is positive-definite, and $\zeta^{-1/2}$ is the positive-definite square root of ζ^{-1}. Then the function (14.4) satisfies (14.3), on the region where ζ is positive-definite. It also satisfies (14.2) for $\partial^2 u(x) = \zeta \in \mathcal{K}$, any compact set of positive-definite, real, $n \times n$ matrices. In particular, if $\mathcal{F}$ is a bounded set in $C^2(\overline{\Omega})$ such that $(\partial_j \partial_k u)$ is positive-definite for each $u \in \mathcal{F}$, and (14.1) holds, with $|f(x, u, \nabla u)| \leq C_0$, then (14.2) holds, uniformly for $u \in \mathcal{F}$.

We first establish interior estimates on solutions to (14.1). We will make use of results of § 13 to establish these estimates, following [Ev], with simplifications of [GT]. To begin, let $\mu \in \mathbb{R}^n$ be a unit vector and apply ∂_μ to (14.1), to get

$$F_{\zeta_{ij}}\,\partial_i \partial_j \partial_\mu u + F_{p_i}\,\partial_i \partial_\mu u + F_u\,\partial_\mu u + \mu^i\,\partial_{x_i} F = 0. \tag{14.6}$$

Then apply ∂_μ again, to obtain

$$\begin{aligned} F_{\zeta_{ij}}\,\partial_i\partial_j\partial_\mu^2 u + (\partial_{\zeta_{ij}}\partial_{\zeta_{k\ell}} F)(\partial_i\partial_j\partial_\mu u)(\partial_k\partial_\ell\partial_\mu u) \\ + A_\mu^{ij}(x, D^2u)\,\partial_i\partial_j\partial_\mu u + B_\mu(x, D^2u) = 0, \end{aligned} \tag{14.7}$$

where

$$A_\mu^{ij}(x, D^2u) = 2(\partial_{\zeta_{ij}}\partial_{p_k} F)(\partial_k\partial_\mu u) + 2(\partial_{\zeta_{ij}}\partial_u F)(\partial_\mu u) + 2\mu^k(\partial_{\zeta_{ij}}\partial_{x_k} F),$$

and $B_\mu(x, D^2u)$ also involves first- and second-order derivatives of F.

Given the concavity of F, we have the differential inequality

$$F_{\zeta_{ij}}\,\partial_i\partial_j\partial_\mu^2 u \geq -A_\mu^{ij}\,\partial_i\partial_j\partial_\mu u - B_\mu, \tag{14.8}$$

where $A_\mu^{ij} = A_\mu^{ij}(x, D^2u),\ B_\mu = B_\mu(x, D^2u)$. If we set

$$h_\mu = \frac{1}{2}\Big(1 + \frac{\partial_\mu^2 u}{1+M}\Big), \quad M = \sup_\Omega\, |\partial^2 u|, \tag{14.9}$$

then (14.8) implies

$$-F_{\zeta_{ij}}\,\partial_i\partial_j h_\mu \leq \frac{C}{1+M}\big(A_0|\partial^3 u| + B_0\big), \tag{14.10}$$

where

$$A_0 = A_0\big(\|u\|_{C^2(\overline{\Omega})}\big), \quad B_0 = B_0\big(\|u\|_{C^2(\overline{\Omega})}\big). \tag{14.11}$$

Now let $\{\mu_k : 1 \leq k \leq N\}$ be a collection of unit vectors, and set

$$h_k = h_{\mu_k}, \quad v = \sum_{k=1}^N h_k^2. \tag{14.12}$$

Use h_k in (14.10), multiply this by h_k, and sum over k, to obtain

$$\sum_{k=1}^N F_{\zeta_{ij}}(\partial_i h_k)(\partial_j h_k) - \frac{1}{2}F_{\zeta_{ij}}\,\partial_i\partial_j v \leq \frac{C}{1+M}\big(A_0|\partial^3 u| + B_0\big). \tag{14.13}$$

Make sure that $\{\mu_k : 1 \leq k \leq N\}$ contains the set

$$\mathfrak{U} = \{e_j : 1 \leq j \leq n\} \cup \{2^{-1/2}(e_i \pm e_j) : 1 \leq i < j \leq n\}, \tag{14.14}$$

where $\{e_j\}$ is the standard basis of $\mathbb{R}^n$. Consequently,

$$|\partial^3 u|^2 = \sum_{i,j,\ell} |\partial_i \partial_j \partial_\ell u|^2 \le 4(1+M)^2 \sum_{k=1}^N |\partial h_k|^2. \tag{14.15}$$

The ellipticity condition (14.2) implies

$$\sum_{k=1}^N F_{\zeta_{ij}} (\partial_i h_k)(\partial_j h_k) \ge \lambda \sum_{k=1}^N |\partial h_k|^2. \tag{14.16}$$

Now, take $\varepsilon \in (0,1)$, and set

$$w_k = h_k + \varepsilon v. \tag{14.17}$$

We have

$$\varepsilon\lambda \sum_{k=1}^N |\partial h_k|^2 - \frac{1}{2} F_{\zeta_{ij}}\, \partial_i \partial_j w_k \le C \left\{ A_0 \Big(\sum_{k=1}^N |\partial h_k|^2 \Big)^{1/2} + \frac{B_0}{1+M} \right\}. \tag{14.18}$$

Thus, by Cauchy's inequality,

$$F_{\zeta_{ij}} \partial_i \partial_j w_k \ge -\lambda \overline{\mu}, \quad \overline{\mu} = \frac{C_n}{\lambda} \Big(\frac{A_0^2}{\lambda\varepsilon} + \frac{B_0}{1+M} \Big). \tag{14.19}$$

We now prepare to apply Proposition 13.4. Let $B_R \subset B_{2R}$ be concentric balls in Ω, and set

$$\begin{aligned} W_{ks} &= \sup_{B_{sR}} w_k, \quad M_{ks} = \sup_{B_{sR}} h_k, \quad m_{ks} = \inf_{B_{sR}} h_k, \\ \omega(sR) &= \sum_{k=1}^N \operatorname*{osc}_{B_{sR}} h_k = \sum_{k=1}^N (M_{ks} - m_{ks}). \end{aligned} \tag{14.20}$$

Applying Proposition 13.4 to (14.19), we have

$$\left(\frac{1}{\operatorname{vol} B_R} \int_{B_R} \big(W_{k2} - w_k\big)^p \, dx \right)^{1/p} \le C(W_{k2} - W_{k1} + \overline{\mu} R^2), \tag{14.21}$$

where $p = p(n, \Lambda/\lambda) > 0,\ C = C(n, \Lambda/\lambda)$. Denote the left side of (14.21) by

$$\Phi_{p,R}(W_{k2} - w_k).$$

Note that

$$\begin{aligned} W_{k2} - w_k &\ge M_{k2} - h_k - 2\varepsilon\omega(2R), \\ W_{k2} - W_{k1} &\ge M_{k2} - M_{k1} + 2\varepsilon\omega(2R). \end{aligned} \tag{14.22}$$

Hence

$$\Phi_{p,R}(M_{k2} - h_k) \le C\big\{M_{k2} - M_{k1} + \varepsilon\omega(2R) + \overline{\mu}R^2\big\}. \tag{14.23}$$

Consequently,

$$\begin{aligned} \Phi_{p,R}\Big(\sum_k (M_{k2} - h_k)\Big) &\le N^{1/p} \sum_k \Phi_{p,R}(M_{k2} - h_k) \\ &\le \big\{(1+\varepsilon)\omega(2R) - \omega(R) + \overline{\mu}R^2\big\}. \end{aligned} \tag{14.24}$$

We want a complementary estimate on $\Phi_{p,R}(h_\ell - m_{\ell 2})$. We exploit the concavity of F in ζ again to obtain

$$\begin{aligned} F_{\zeta_{ij}}\big(y, D^2u(y)\big)&\big(\partial_i\partial_j u(y) - \partial_i\partial_j u(x)\big) \\ &\le F\big(y, Du(y), \partial^2 u(x)\big) - F\big(y, Du(y), \partial^2 u(y)\big) \\ &= F\big(y, Du(y), \partial^2 u(x)\big) - F\big(x, Du(x), \partial^2 u(x)\big) \\ &\le D_0|x - y|, \end{aligned} \tag{14.25}$$

where

$$D_0 = D_0\big(\|u\|_{C^2(\overline{\Omega})}\big). \tag{14.26}$$

The equality in (14.25) follows from $F(x, D^2u) = 0$. At this point, we impose a special condition on the unit vectors μ_k used to define h_k above. The following is a result of [MW]:

Lemma 14.1. *Given $0 < \lambda < \Lambda < \infty$, let $\mathcal{S}(\lambda, \Lambda)$ denote the set of positive-definite, real, $n \times n$ matrices with spectrum in $[\lambda, \Lambda]$. Then there exist $N \in \mathbb{Z}^+$ and $\lambda^* < \Lambda^*$ in $(0, \infty)$, depending only on n, λ, and Λ, and unit vectors $\mu_k \in \mathbb{R}^n$, $1 \le k \le N$, such that*

$$\{\mu_k : 1 \le k \le N\} \supset \mathfrak{U}, \tag{14.27}$$

where $\mathfrak{U}$ is defined by (14.14), and such that every $A \in \mathcal{S}(\lambda, \Lambda)$ can be written in the form

$$A = \sum_{k=1}^N \beta_k P_{\mu_k}, \quad \beta_k \in [\lambda^*, \Lambda^*], \tag{14.28}$$

where P_{μ_k} is the orthogonal projection of $\mathbb{R}^n$ onto the linear span of μ_k.

Proof. Let the set of real, symmetric, $n \times n$ matrices be denoted as Symm(n) $\approx \mathbb{R}^{n(n+1)/2}$. Note that $A \in$ Symm(n) belongs to $\mathcal{S}(\lambda, \Lambda)$ if and only if

$$\lambda |v|^2 \le v \cdot Av \le \Lambda |v|^2, \quad \forall\, v \in \mathbb{R}^n.$$

Thus $\mathcal{S}(\lambda, \Lambda)$ is seen to be a compact, convex subset of Symm(n). Also, $\mathcal{S}(\lambda, \Lambda)$ is contained in the *interior* of $\mathcal{S}(\lambda_1, \Lambda_1)$ if $0 < \lambda_1 < \lambda < \Lambda < \Lambda_1$.

It suffices to prove the lemma in the case $\Lambda = 1/2n$. Suppose $0 < \lambda < 1/2n$. By the spectral theorem for elements of Symm(n), $\mathcal{S}(\lambda/2, 1/2n)$ is contained in the interior of the convex hull $CH(\mathcal{P})$ of the set

$$\mathcal{P} = \{0\} \cup \{P_\mu : \mu \in S^{n-1} \subset \mathbb{R}^n\}.$$

Thus, there exists a *finite* subset $\mathfrak{A} \supset \mathfrak{U}$ of unit vectors such that $\mathcal{S}(\lambda/2, 1/2n)$ is contained in the interior of $CH(\mathcal{P}_0)$, with $\mathcal{P}_0 = \{0\} \cup \{P_\mu : \mu \in \mathfrak{A}\}$. Write $\mathfrak{A}$ as $\{\mu_k : 1 \le k \le N\}$. Then any element of $\mathcal{S}(\lambda/2, 1/2n)$ has a representation of the form $\sum_{k=1}^N \widetilde{\beta}_k P_{\mu_k}$, with $\widetilde{\beta}_k \in [0, 1]$.

Now, if we take $A \in \mathcal{S}(\lambda, 1/2n)$, it follows that

$$A - \sum_{k=1}^N \frac{\lambda}{2N} P_{\mu_k} \in \mathcal{S}\Big(\frac{\lambda}{2}, \frac{1}{2n}\Big),$$

so $A = \sum_{k=1}^N \big(\widetilde{\beta}_k + \lambda/2N\big) P_{\mu_k}$ has the form (14.28), with $\beta_k = \widetilde{\beta}_k + \lambda/2N \in \big[\lambda/2N, 2\big]$. This proves the lemma.

If we choose the set $\{\mu_k : 1 \le k \le N\}$ of unit vectors to satisfy the condition of Lemma 14.1, then

$$\begin{aligned} &F_{\zeta_{ij}}\big(y, D^2u(y)\big)\big(\partial_i \partial_j u(y) - \partial_i \partial_j u(x)\big) \\ &\quad = \sum_{k=1}^N \beta_k(y)\big(\partial^2_{\mu_k} u(y) - \partial^2_{\mu_k} u(x)\big) \\ &\quad = 2(1+M) \sum_{k=1}^N \beta_k(y)\big(h_k(y) - h_k(x)\big), \end{aligned} \tag{14.29}$$

with $\beta_k(y) \in [\lambda^*, \Lambda^*]$. Consequently, for $x \in B_{2R},\ y \in B_R$, we have from (14.25) that

$$\sum_{k=1}^N \beta_k(y)\big(h_k(y) - h_k(x)\big) \le C\lambda\widetilde{\mu}R, \quad \widetilde{\mu} = \frac{D_0}{\lambda(1+M)}. \tag{14.30}$$

Hence, for any $\ell \in \{1, \dots, N\}$,

$$
\begin{aligned}
h_\ell(y) - m_{\ell 2} &\le \frac{1}{\lambda^*}\Big\{C\lambda\widetilde{\mu}R + \Lambda^* \sum_{k\neq\ell}\big(M_{k2} - h_k(y)\big)\Big\} \\
&\le C\Big\{\widetilde{\mu}R + \sum_{k\neq\ell}\big(M_{k2} - h_k(y)\big)\Big\},
\end{aligned}
\tag{14.31}
$$

where $C = C(n, \Lambda/\lambda)$. We can use (14.24) to estimate the right side of (14.31), obtaining

$$
\Phi_{p,R}(h_\ell - m_{\ell 2}) \le C\big\{(1+\varepsilon)\omega(2R) - \omega(R) + \widetilde{\mu}R + \overline{\mu}R^2\big\}. \tag{14.32}
$$

Setting $\ell = k$, adding (14.32) to (14.23), and then summing over k, we obtain

$$
\omega(2R) \le C\big\{(1+\varepsilon)\omega(2R) - \omega(R) + \widetilde{\mu}R + \overline{\mu}R^2\big\}, \tag{14.33}
$$

and hence

$$
\omega(R) \le \Big(1 - \frac{1}{C} + \varepsilon\Big)\omega(2R) + \big(\widetilde{\mu}R + \overline{\mu}R^2\big). \tag{14.34}
$$

Now C is independent of ε, though $\overline{\mu}$ is not. Thus fix $\varepsilon = 1/2C$, to obtain

$$
\omega(R) \le \Big(1 - \frac{1}{2C}\Big)\omega(2R) + \big(\widetilde{\mu}R + \overline{\mu}R^2\big). \tag{14.35}
$$

From this it follows that if $B_{2R_0} \subset \Omega$ and $R \le R_0$, we have

$$
\operatorname*{osc}_{B_R} \partial^2 u \le C\Big(\frac{R}{R_0}\Big)^\alpha (1+M)\big(1 + \widetilde{\mu}R_0 + \overline{\mu}R_0^2\big), \tag{14.36}
$$

where C and α are positive constants depending only on n and Λ/λ. We have proved the following interior estimate:

Proposition 14.2. *Let $u \in C^4(\overline{\Omega})$ satisfy (14.1), and assume that (14.2) and (14.3) hold. Then, for any $\mathcal{O} \subset\subset \Omega$, there is an estimate*

$$
\|\partial^2 u\|_{C^\alpha(\mathcal{O})} \le C\big(\mathcal{O}, \Omega, n, \lambda, \Lambda, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}\big). \tag{14.37}
$$

In fact, examining the derivation of (14.36), we can specify the dependence on $\mathcal{O}, \Omega$. If $\mathcal{O}$ is a ball, and $|x - y| \ge \rho$ for all $x \in \mathcal{O},\ y \in \partial\Omega$, then

$$
\|\partial^2 u\|_{C^\alpha(\mathcal{O})} \le C\big(n, \lambda, \Lambda, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}\big)\rho^{-\alpha}. \tag{14.38}
$$

We now tackle global estimates on $\overline{\Omega}$ for solutions to the Dirichlet problem for (14.1). We first obtain estimates for $\partial^2 u\big|_{\partial\Omega}$.

Lemma 14.3. *Under the hypotheses of Proposition 14.2, if $u\big|_{\partial\Omega} = \varphi$, there is an estimate*

$$\|\partial^2 u\|_{C^\alpha(\partial\Omega)} \le C\big(\Omega, n, \lambda, \Lambda, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}, \|\varphi\|_{C^3(\partial\Omega)}\big). \tag{14.39}$$

Proof. Let $Y = b^\ell(x)\partial_\ell$ be a smooth vector field tangent to $\partial\Omega$, and consider $v = Yu$, which solves the boundary problem

$$F_{\zeta_{ij}}\,\partial_i\partial_j v = G(x), \quad v\big|_{\partial\Omega} = Y\varphi, \tag{14.40}$$

where

$$\begin{aligned} G(x) = 2F_{\zeta_{ij}}(\partial_i B^\ell)(\partial_j\partial_\ell u) &+ F_{\zeta_{ij}}(\partial_i\partial_j b^\ell)\partial_\ell u \\ &+ F_{p_i}(\partial_i b^\ell)(\partial_\ell u) - F_{p_i}\partial_i v - F_u v - b^\ell\partial_{x_\ell} F. \end{aligned} \tag{14.41}$$

The hypotheses give a bound on $\|G\|_{L^\infty(\Omega)}$ in terms of the right side of (14.39). If $\psi \in C^2(\overline{\Omega})$ denotes an extension of $Y\varphi$ from $\partial\Omega$ to $\overline{\Omega}$, then Proposition 13.11, applied to $v - \psi$, yields an estimate

$$\|\partial_\nu Y u\|_{C^\alpha(\partial\Omega)} \le C, \tag{14.42}$$

where C is of the form (14.39). On the other hand, the ellipticity of (14.1) allows one to solve for $\partial_\nu^2 u\big|_{\partial\Omega}$ in terms of quantities estimated in (14.42), plus $u\big|_{\partial\Omega}$ and $\nabla u\big|_{\partial\Omega}$, and second-order tangential derivatives of u, so (14.39) is proved.

We now want to estimate $|\partial_\gamma^2 u(x) - \partial_\gamma^2 u(x_0)|$, given $x_0 \in \partial\Omega,\ x \in \overline{\Omega},\ \gamma \in \mathbb{R}^n$ a unit vector. For simplicity, we will strengthen the concavity hypothesis (14.3) to *strong concavity*:

$$\partial_{\zeta_{jk}}\partial_{\zeta_{\ell m}} F(x, u, p, \zeta)\,\Xi_{jk}\Xi_{\ell m} \le -\lambda_0|\Xi|^2, \quad \Xi = \Xi^t, \tag{14.43}$$

for some $\lambda_0 > 0$, when $\zeta = \partial^2 u,\ p = \nabla u$. Then we can improve (14.8) to

$$F_{\zeta_{ij}}\,\partial_i\partial_j(\partial_\gamma^2 u) \le -A_\gamma^{ij}\,\partial_i\partial_j\partial_\gamma u - B_\gamma - \lambda_0|\partial^2\partial_\gamma u|^2 \le -C_1, \tag{14.44}$$

by Cauchy's inequality, where

$$C_1 = C_1\big(n, \lambda, \Lambda, \lambda_0, \|A_\gamma(x, D^2 u)\|_{L^\infty}, \|B_\gamma(x, D^2 u)\|_{L^\infty}\big).$$

Now the function

$$W(x) = C_2|x - x_0|^\alpha \quad (0 < \alpha < 1) \tag{14.45}$$

is concave on $\mathbb{R}^n \setminus \{x_0\}$, and if C_2 is sufficiently large, compared to $C_1 \cdot \operatorname{diam}(\Omega)^{2-\alpha}/\lambda$, we have

$$LW \le -C_1, \quad Lv = F_{\zeta_{ij}}\, \partial_i \partial_j v. \tag{14.46}$$

Hence, by the maximum principle,

$$\partial_\gamma^2 u \le \partial_\gamma^2(x_0) + W \ \text{ on } \partial\Omega \Longrightarrow \partial_\gamma^2 u \le \partial_\gamma^2 u(x_0) + W \ \text{ on } \Omega. \tag{14.47}$$

Now the estimate (14.39) implies that the hypothesis of (14.47) is satisfied, provided that also $C_2 \ge \|\partial^2 u\|_{C^\alpha(\partial\Omega)}$, so we have the one-sided estimate given by the conclusion of (14.47).

For the reverse estimate, use (14.25), with $y = x_0$, together with (14.29), to write

$$\sum_{k=1}^N \beta_k(x_0)\big(\partial_{\mu_k}^2 u(x_0) - \partial_{\mu_k}^2 u(x)\big) \le D_0|x - x_0|. \tag{14.48}$$

Recall that $\beta_k(x_0) \in [\lambda^*, \Lambda^*],\ \lambda^* > 0$. This together with (14.47) implies

$$|\partial_{\mu_k}^2 u(x) - \partial_{\mu_k}^2 u(x_0)| \le C_3|x - x_0|^\alpha, \tag{14.49}$$

with C_3 of the form (14.39), and we can express any $\partial_j \partial_\ell u$ as a linear combination of the $\partial_{\mu_k}^2 u$, to obtain the following:

Lemma 14.4. *If we have the hypotheses of Lemma 14.3, and we also assume (14.43), then there is an estimate*

$$|\partial^2 u(x) - \partial^2 u(x_0)| \le C|x - x_0|^\alpha, \quad x_0 \in \partial\Omega, \ \ x \in \overline{\Omega}, \tag{14.50}$$

with

$$C = C\big(\Omega, n, \lambda, \Lambda, \lambda_0, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}, \|\varphi\|_{C^3(\partial\Omega)}\big). \tag{14.51}$$

We now put (14.38) and (14.50) together to obtain a Hölder estimate for $\partial^2 u$ on $\overline{\Omega}$. To estimate $|\partial^2 u(x) - \partial^2 u(y)|$, given $x,\ y \in \overline{\Omega}$, suppose $\operatorname{dist}(x, \partial\Omega) + \operatorname{dist}(y, \partial\Omega) = 2\rho$, and consider two cases:

(i) $|x - y| < \rho^2$,
(ii) $|x - y| \ge \rho^2$.

In case (i), we can use (14.38) to deduce that

$$|\partial^2 u(x) - \partial^2 u(y)| \le C|x - y|^\alpha \rho^{-\alpha} \le C|x - y|^{\alpha/2}. \tag{14.52}$$

In case (ii), let $x' \in \partial\Omega$ minimize the distance from x to $\partial\Omega$, and let $y' \in \partial\Omega$ minimize the distance from y to $\partial\Omega$. Thus

$$\begin{aligned} &|x-x'| \le 2\rho \le 2|x-y|^{1/2}, \qquad |y-y'| \le 2\rho \le 2|x-y|^{1/2}, \\ &|x'-y'| \le |x-y| + |x'-x| + |y'-y| \le |x-y| + 4|x-y|^{1/2}. \end{aligned} \tag{14.53}$$

Thus

$$\begin{aligned} |\partial^2 u(x) - \partial^2 u(y)| &\le |\partial^2 u(x) - \partial^2 u(x')| + |\partial^2 u(x') - \partial^2 u(y')| \\ &\quad + |\partial^2 u(y') - \partial^2 u(y)| \\ &\le \widetilde{C}|x-x'|^\alpha + \widetilde{C}|x'-y'|^\alpha + \widetilde{C}|y'-y|^\alpha \\ &\le C|x-y|^{\alpha/2}. \end{aligned} \tag{14.54}$$

In (14.52) and (14.54), C has the form (14.51). Taking $r = \alpha/2$, we have the following global estimate:

Proposition 14.5. *Let $u \in C^4(\overline{\Omega})$ satisfy (14.1), with $u\big|_{\partial\Omega} = \varphi$. Assume the ellipticity hypothesis (14.2) and the strong concavity hypothesis (14.43). Then there is an estimate*

$$\|u\|_{C^{2+r}(\overline{\Omega})} \le C\big(\Omega, n, \lambda, \Lambda, \lambda_0, \|F\|_{C^2}, \|u\|_{C^2(\overline{\Omega})}, \|\varphi\|_{C^3(\partial\Omega)}\big), \tag{14.55}$$

for some $r > 0$, depending on the same quantities as C.

Now that we have this estimate, the continuity method yields the following existence result. For $\tau \in [0,1]$, consider a family of boundary problems

$$F_\tau(x, D^2u) = 0 \text{ on } \Omega, \quad u\big|_{\partial\Omega} = \varphi_\tau. \tag{14.56}$$

Assume F_τ and φ_τ are smooth in all variables, including τ. Also, assume that the ellipticity condition (14.2) and the strong concavity condition (14.43) hold, uniformly in τ, for any smooth solution u_τ.

Theorem 14.6. *Assume there is a uniform bound in $C^2(\overline{\Omega})$ for any solution $u_\tau \in C^\infty(\overline{\Omega})$ of (14.56). Also assume that $\partial_u F_\tau \le 0$. Then, if (14.56) has a solution in $C^\infty(\overline{\Omega})$ for $\tau = 0$, it has a smooth solution for $\tau = 1$.*

With some more work, one can replace the strong concavity hypothesis (14.43) by (14.3); see [CKNS].

There is an interesting class of elliptic PDE, known as *Bellman equations*, for which $F(x, u, p, \zeta)$ is concave but not strongly concave in ζ, and also it is Lipschitz but not C^∞ in its arguments; see [Ev2] for an analysis.

Verifying the hypothesis in Theorem 14.6 that u_τ is bounded in $C^2(\overline{\Omega})$ can be a nontrivial task. We will tackle this, for Monge–Ampere equations, in the next section.

Exercises

1. Discuss the Dirichlet problem for
$$\Delta u + \partial_1^2 u + \frac{1}{2}\big(1 + (\Delta u)^2\big)^{1/2} = \sigma e^u,$$
for $\sigma \ge 0$.

15. Monge–Ampere equations

Here we look at equations of Monge–Ampere type:

$$\det H(u) - F(x, u, \nabla u) = 0 \text{ on } \Omega, \quad u = \varphi \text{ on } \partial\Omega, \tag{15.1}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^n$, which we will assume to be strongly convex. As in (3.7a), $H(u) = (\partial_j \partial_k u)$ is the Hessian matrix. We assume $F(x, u, \nabla u) > 0$, say $F(x, u, \nabla u) = \exp f(x, u, \nabla u)$, and look for a convex solution to (15.1). It is convenient to set

$$G(u) = \log \det H(u) - f(x, u, \nabla u), \tag{15.2}$$

so (15.1) is equivalent to $G(u) = 0$ on Ω, $u = \varphi$ on $\partial\Omega$. Note that

$$DG(u)v = g^{jk}\, \partial_j \partial_k v - (\partial_{p_j} f)(x, u, \nabla u)\, \partial_j v - (\partial_u f)(x, u, \nabla u) v, \tag{15.3}$$

where (g^{jk}) is the inverse matrix of $(\partial_j \partial_k u)$, which we will also denote as (g_{jk}). We will assume

$$(\partial_u f)(x, u, p) \ge 0, \tag{15.4}$$

this hypothesis being equivalent to $(\partial_u F)(x, u, p) \ge 0$.

The hypotheses made above do not suffice to guarantee that (15.1) has a solution. Consider the following example:

$$\det H(u) - K\big(1 + |\nabla u|^2\big)^2 = 0 \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{15.5}$$

where Ω is a domain in $\mathbb{R}^2$. Compare with (3.41). Let K be a positive constant. If there is a convex solution u, the surface $\Sigma = \{(x, u(x)) : x \in \Omega\}$ is a surface in $\mathbb{R}^3$ with Gauss curvature K. If Ω is convex, then the Gauss map $N : \Sigma \to S^2$ is one-to-one and the image $N(\Sigma)$ has area equal to $K \cdot \text{Area}(\Omega)$. But $N(\Sigma)$ must be contained in a hemisphere of S^2, so we must have $K \cdot \text{Area}(\Omega) \le 2\pi$. We deduce that if $K \cdot \text{Area}(\Omega) > 2\pi$, then (15.5) has no solution.

To avoid this obstruction to existence, we hypothesize that there exists $u^b \in C^\infty(\overline{\Omega})$, which is convex and satisfies

$$\log \det H(u^b) - f(x, u^b, \nabla u^b) \ge 0 \text{ on } \Omega, \quad u^b = \varphi \text{ on } \partial\Omega. \tag{15.6}$$

We call u^b a *lower solution* to (15.1). Note that the first part of (15.6) is equivalent to $\det H(u^b) \geq F(x, u^b, \nabla u^b)$. In such a case, we will use the method of continuity and seek a convex $u_\sigma \in C^\infty(\overline{\Omega})$ solving

$$\begin{aligned} &\log \det H(u_\sigma) - f(x, u_\sigma, \nabla u_\sigma) \\ &= (1-\sigma)\big[\log \det H(u^b) - f(x, u^b, \nabla u^b)\big] \\ &= (1-\sigma)h(x), \end{aligned} \tag{15.7}$$

for $\sigma \in [0,1]$ and $u_\sigma = \varphi$ on $\partial\Omega$. Note that $u_0 = u^b$ solves (15.7) for $\sigma = 0$. If such u_σ exists for all $\sigma \in [0,1]$, then $u = u_1$ is the desired solution to (15.1).

Let J be the largest interval in $[0,1]$, containing 0, such that (15.7) has a convex solution $u_\sigma \in C^\infty(\overline{\Omega})$ for all $\sigma \in J$. Since the linear operator in (15.3) is elliptic and invertible (by the maximum principle) under the hypothesis (15.4), the same sort of argument used in the proof of Lemma 10.1 shows that J is open, and the real work is to show that J is closed. In this case, we need to obtain bounds on u_σ in $C^{2+\mu}(\overline{\Omega})$, for some $\mu > 0$, in order to apply the regularity theory of § 8 and conclude that J is closed.

Lemma 15.1. *Given $\sigma \leq \tau \in J$, we have*

$$u^b \leq u_\sigma \leq u_\tau \quad \text{on } \Omega. \tag{15.8}$$

Proof. The operator $G(u)$ satisfies the hypotheses of Proposition 10.8; since $u^b = u_\sigma = u_\tau$ on $\partial\Omega$, (15.8) follows.

In particular, taking $\sigma = \tau$, we have *uniqueness* of the solution $u_\sigma \in C^\infty(\overline{\Omega})$ to (15.7).

Next we record some estimates that are simple consequences of convexity alone:

Lemma 15.2. *Assume Ω is convex. For any $\sigma \in J$,*

$$u_\sigma \leq \sup_{\partial\Omega} \varphi \quad \text{on } \Omega \tag{15.9}$$

and

$$\sup_{x \in \Omega} |\nabla u_\sigma(x)| \leq \sup_{y \in \partial\Omega} |\nabla u_\sigma(y)|. \tag{15.10}$$

Thus we will have a bound on u_σ in $C^1(\overline{\Omega})$ if we bound ∇u_σ on $\partial\Omega$. Since $u_\sigma\big|_{\partial\Omega} = \varphi \in C^\infty(\partial\Omega)$, it remains to bound the normal derivative $\partial_\nu u_\sigma$ on $\partial\Omega$. Assume ∂_ν points out of Ω. Then (15.8) implies

$$\partial_\nu u_\sigma(y) \leq \partial_\nu u^b(y), \quad \forall\, y \in \partial\Omega. \tag{15.11}$$

On the other hand, a lower bound on $\partial_\nu u_\sigma(y)$ follows from convexity alone. In fact, if $\nu(y)$ is the outward normal to $\partial\Omega$ at y, say $\widetilde{y} = y - \ell(y)\nu(y)$ is the other point in $\partial\Omega$ through which the normal line passes. Then convexity of u_σ implies

$$u_\sigma\big(sy + (1-s)\widetilde{y}\big) \le s\varphi(y) + (1-s)\varphi(\widetilde{y}), \tag{15.12}$$

for $0 \le s \le 1$. Noting that $\ell(y) = |y - \widetilde{y}|$, we have

$$\partial_\nu u_\sigma(y) \ge \frac{\varphi(\widetilde{y}) - \varphi(y)}{|\widetilde{y} - y|}.$$

Thus we have the next result:

Lemma 15.3. *If Ω is convex, then, for any $\sigma \in J$,*

$$\sup_{\overline{\Omega}} |\nabla u_\sigma| \le Lip^1(\varphi) + \sup_{\overline{\Omega}} |\nabla u^b|. \tag{15.13}$$

Here, $\mathrm{Lip}^1(\varphi)$ denotes the Lipschitz constant of φ:

$$\mathrm{Lip}^1(\varphi) = \sup_{y,y'\in\partial\Omega} \frac{|\varphi(y) - \varphi(y')|}{|y - y'|}. \tag{15.14}$$

We now look for C^2-bounds on solutions to (15.7). For notational simplicity, we write (15.7) as

$$\log\, \det H(u) - f(x, u, \nabla u) = 0, \quad u\big|_{\partial\Omega} = \varphi, \tag{15.15}$$

where the second term on the left is

$$f_\sigma(x, u, \nabla u) = f(x, u, \nabla u) + (1-\sigma)h(x),$$

and we drop the σ. By (15.4) and (15.6), we have $f(x, u, p) > 0$ and $(\partial_u f)(x, u, p) \le 0$.

Since u is convex, it suffices to estimate pure second derivatives $\partial_\gamma^2 u$ from above. Following [CNS], who followed [LiP2], we make use of the function

$$w = e^{\beta|\nabla u|^2/2}\, \partial_\gamma^2 u,$$

where β is a constant that will be chosen later. Suppose this is maximized, among all unit $\gamma \in \mathbb{R}^n$, $x \in \overline{\Omega}$, at $\gamma = \gamma_0$, $x = x_0$. Rotating coordinates, we can assume $\big(g_{jk}(x_0)\big) = \big(\partial_j \partial_k u(x_0)\big)$ is in diagonal form and $\gamma_0 = (1, 0, \dots, 0)$. Set $u_{11} = \partial_1^2 u$, so we take

$$w = e^{\beta|\nabla u|^2/2} u_{11} = \psi(\nabla u) u_{11}. \tag{15.16}$$

We now derive some identities and inequalities valid on all of Ω.

Differentiating (15.15), we obtain

$$
\begin{aligned}
&g^{ij}\,\partial_i\partial_j\partial_\ell u = \partial_\ell f(x,u,\nabla u),\\
&g^{ij}\,\partial_i\partial_j u_{11} = g^{i\ell}g^{jm}(\partial_i\partial_j\partial_1 u)(\partial_k\partial_m\partial_1 u) + \partial_1^2 f,
\end{aligned}
\tag{15.17}
$$

where (g^{ij}) is the inverse matrix to $(g_{ij}) = (\partial_i\partial_j u)$, as above. Also, a calculation gives

$$
\begin{aligned}
w^{-1}\,\partial_i w &= (\log\psi)_{p_k}\,\partial_i\partial_k u + u_{11}^{-1}(\partial_i\partial_1^2 u),\\
w^{-1}\,\partial_i\partial_j w &= w^{-2}(\partial_i w)(\partial_j w) + (\log\psi)_{p_k p_\ell}(\partial_i\partial_k u)(\partial_j\partial_\ell u)\\
&\quad + (\log\psi)_{p_k}(\partial_i\partial_j\partial_k u) + u_{11}^{-1}\,\partial_i\partial_j u_{11} - u_{11}^{-2}(\partial_i\partial_1^2 u)(\partial_j\partial_1^2 u).
\end{aligned}
\tag{15.18}
$$

Forming $w^{-1}\,g^{ij}\partial_i\partial_j w$ and using (15.17) to rewrite the term $u_{11}^{-1}g^{ij}\,\partial_i\partial_j u_{11}$, we obtain

$$
\begin{aligned}
&\psi^{-1}g^{ij}\,\partial_i\partial_j w\\
&\quad\ge u_{11}\Big[(\log\psi)_{p_k p_\ell}g^{ij}(\partial_i\partial_k u)(\partial_j\partial_\ell u) + (\log\psi)_{p_k}g^{ij}\,\partial_i\partial_j\partial_k u\Big]\\
&\qquad + g^{ik}\,g^{i\ell}(\partial_i\partial_j\partial_1 u)(\partial_k\partial_\ell\partial_1 u) - u_{11}^{-1}g^{ij}(\partial_i\partial_1^2 u)(\partial_j\partial_1^2 u) + \partial_1^2 f.
\end{aligned}
\tag{15.19}
$$

Now we have $(\log\psi)_{p_k} = \beta p_k$ and $(\log\psi)_{p_k p_\ell} = \beta\delta^{k\ell}$, and hence

$$
(\log\psi)_{p_k p_\ell}g^{ij}(\partial_i\partial_k u)(\partial_j\partial_\ell u) = \beta\delta^{k\ell}\,\delta^j{}_k(\partial_j\partial_\ell u) = \beta\Delta u.
\tag{15.20}
$$

Let us assume the following bounds hold on $f(x,u,p)$:

$$
|(\nabla f)(x,u,p)| \le \mu, \quad |(\partial^2 f)(x,u,p)| \le \mu.
\tag{15.21}
$$

Using the first identity in (15.17), we have

$$
\begin{aligned}
&u_{11}(\log\psi)_{p_k}g^{ij}\,\partial_i\partial_j\partial_k u + \partial_1^2 f\\
&\ge f_{p_i}(w^{-1}\partial_i w)u_{11} - C\big[1 + |\partial^2 u|^2 + \beta(1+|\partial^2 u|)\big],
\end{aligned}
\tag{15.22}
$$

with $C = C\big(\mu, \|\nabla u\|_{L^\infty(\Omega)}\big)$.

Now, let us look at x_0, where, recall, $e^{\beta|\nabla u|^2/2}\partial_1^2 u$ is maximal, among all values of $e^{\beta|\nabla u(x)|^2/2}\partial_\gamma^2 u(x)$. If $x_0 \in \Omega$ (i.e., $x_0 \notin \partial\Omega$), then $\partial_i w(x_0) = 0$ and the left side of (15.19) is ≤ 0 at x_0. Furthermore, due to the diagonal nature of (g^{ij}) at x_0, we easily verify that $g^{11}g^{ij}\zeta_{i1}\zeta_{j1} \le g^{ij}g^{k\ell}\zeta_{ik}\zeta_{j\ell}$, and hence

$$
u_{11}^{-1}g^{ij}(\partial_i\partial_1^2 u)(\partial_j\partial_1^2 u) \le g^{ik}g^{j\ell}(\partial_i\partial_j\partial_1 u)(\partial_k\partial_\ell\partial_1 u),
\tag{15.23}
$$

at x_0. Thus the evaluation of (15.19) at x_0 implies the estimate

$$0 \geq \beta(\partial_1^2 u)(\Delta u) - \mu - C\big[1 + |\partial^2 u|^2 + \beta(1 + |\partial^2 u|)\big] \tag{15.24}$$

if $x_0 \notin \partial\Omega$. Hence, with $X = \partial_1^2 u(x_0)$,

$$(\beta - C_1)X^2 \leq \beta C_2(1 + X) + \mu, \tag{15.25}$$

where C_1 and C_2 depend on μ and $\|\nabla u\|_{L^\infty}$, but not on β. Taking β large, we obtain a bound on X:

$$\partial_1^2 u(x_0) \leq C\big(\mu, \|\nabla u\|_{L^\infty(\Omega)}\big) \quad \text{if } x_0 \notin \partial\Omega. \tag{15.26}$$

On the other hand, if $\sup w$ is achieved on $\partial\Omega$, we have

$$\sup_{x,\gamma} |\partial_\gamma^2 u(x)| \leq \sup_{\partial\Omega} |\partial^2 u| \cdot \exp\big(\beta\|\nabla u\|_{L^\infty}\big).$$

This establishes the following.

Lemma 15.4. *If $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ solves (15.15) and the hypotheses above hold, then*

$$\sup_{\overline{\Omega}} |\partial^2 u| \leq C\big(\mu, \|\nabla u\|_{L^\infty(\Omega)}\big)\Big[1 + \sup_{\partial\Omega} |\partial^2 u|\Big]. \tag{15.27}$$

To estimate $\partial^2 u$ at a boundary point $y \in \partial\Omega$, suppose coordinates are rotated so that $\nu(y)$ is parallel to the x_n-axis. Pick vector fields Y_j, tangent to $\partial\Omega$, so that $Y_j(y) = \partial_j$, $1 \leq j \leq n-1$. Then we easily get

$$|\partial_j \partial_k u(y)| \leq |Y_j Y_k \varphi(y)| + C|\nabla u(y)|, \quad 1 \leq j, \ k \leq n-1. \tag{15.28}$$

In fact, for later reference, we note the following. Suppose Y_j is the vector field tangent to $\partial\Omega$, equal to ∂_j at y, and obtained by parallel transport along geodesics emanating from y. If $Y_k = b_k^\ell \partial_\ell$, then

$$\begin{aligned} Y_j Y_k u(y) &= \partial_j \partial_k u(y) + \big(\partial_j b_k^\ell(y)\big)\,\partial_\ell u(y) \\ &= \partial_j \partial_k u(y) + \big(\nabla^0_{\partial_j} Y_k\big)u(y), \end{aligned} \tag{15.29}$$

where ∇^0 is the standard flat connection on $\mathbb{R}^n$. If ∇ is the Levi–Civita connection on $\partial\Omega$, we have $\nabla_{\partial_j} Y_k = 0$ at y, hence $\nabla^0_{\partial_j} Y_k = -\widetilde{II}(\partial_j, \partial_k)\,\partial_\nu$ at y, where $\partial_\nu = -N$ is the outward-pointing normal and $\widetilde{II}$ is the second fundamental form of $\partial\Omega$; see § 4 of Appendix C. Hence

$$\partial_j \partial_k u(y) = Y_j Y_k u(y) + \widetilde{II}(\partial_j, \partial_k)\,\partial_\nu u(y), \quad 1 \leq j, \ k \leq n-1. \tag{15.30}$$

Later it will be important to note that strong convexity of $\partial\Omega$ implies positive definiteness of $\widetilde{II}$.

We next need to estimate $\partial_n Y_k u(y),\ 1 \le k \le n-1$. If $Y_k = b_k^\ell(x)\,\partial_\ell$, then $v_k = Y_k u$ satisfies the equation

$$g^{ij}\,\partial_i\partial_j v_k - f_{p_i}\,\partial_i v_k = A(x) + g^{ij}\,B_{ij}(x), \tag{15.31}$$

where

$$\begin{aligned} A(x) &= 2\partial_i b_k^i + f_{x_\ell} b_k^\ell + f_u v_k + f_{p_i}(\partial_i b_k^\ell)\,\partial_\ell u, \\ B_{ij}(x) &= (\partial_i\partial_j b_k^\ell)\,\partial_\ell u, \end{aligned} \tag{15.32}$$

and $v_k\big|_{\partial\Omega} = Y_k\varphi$. This follows by multiplying the first identity in (15.17) by b_k^ℓ and summing over ℓ; one also makes use of the identity $g^{ij}\,\partial_j\partial_\ell u = \delta^i{}_\ell$.

We first derive a boundary gradient estimate for $v_k = Y_k u$ when (15.15) takes the simpler form

$$\log\,\det H(u) - f(x,u) = 0, \quad u\big|_{\partial\Omega} = \varphi; \tag{15.33}$$

that is, ∇u is not an argument of f. Here, we follow [Au]. We assume $\varphi \in C^\infty(\overline{\Omega})$, set

$$w_k = Y_k(u-\varphi) = v_k - Y_k\varphi, \tag{15.34}$$

then let α and β be real numbers, to be fixed below, and set

$$\widetilde{w}_k = w_k + \alpha h + \beta(u-\varphi). \tag{15.35}$$

Here, $h \in C^\infty(\overline{\Omega})$ is picked to vanish on $\partial\Omega$ and satisfy a strong convexity condition:

$$(\partial_i\partial_j h) \ge I, \quad h\big|_{\partial\Omega} = 0. \tag{15.36}$$

The hypothesis that $\overline{\Omega}$ is strongly convex is equivalent to the existence of such a function.

Now, a calculation using (15.31) (and noting that in this case $f_{p_i} = 0$) gives

$$g^{ij}\,\partial_i\partial_j\widetilde{w}_k = A(x) + n\beta + g^{ij}\widetilde{B}_{ij}(x), \quad \widetilde{w}_k\big|_{\partial\Omega} = 0, \tag{15.37}$$

where $A(x)$ is as in (15.32) (with the last term equal to zero), and

$$\widetilde{B}_{ij}(x) = B_{ij}(x) - \partial_i\partial_j Y_k\varphi + \alpha\,\partial_i\partial_j h - \beta\,\partial_i\partial_j\varphi. \tag{15.38}$$

We now choose α and β. Pick $\beta = \beta_0$, so large that $A(x) + n\beta_0 \ge 0$. This done, pick $\alpha = \alpha_0$, so large that $(\widetilde{B}_{ij}) \ge 0$. Then $\widetilde{w}_{k0}$, defined by (15.34) with $\alpha = \alpha_0,\ \beta = \beta_0$, satisfies

$$g^{ij}\,\partial_i\partial_j\widetilde{w}_{k0} \ge 0, \quad \widetilde{w}_{k0}\big|_{\partial\Omega} = 0. \tag{15.39}$$

Similarly, pick $\beta = \beta_1$ sufficiently negative that $A(x) + n\beta_1 \le 0$, and then pick $\alpha = \alpha_1$ sufficiently negative that $(\widetilde{B}_{ij}) \le 0$. Then, $\widetilde{w}_{k1}$, defined by (15.35) with $\alpha = \alpha_1$ and $\beta = \beta_1$, satisfies

$$g^{ij}\partial_i\partial_j\widetilde{w}_{k1} \le 0, \quad \widetilde{w}_{k1}\big|_{\partial\Omega} = 0. \tag{15.40}$$

The maximum principle implies $\widetilde{w}_{k0} \le 0$ and $\widetilde{w}_{k1} \ge 0$; hence

$$Y_k\varphi - \alpha_1 h - \beta_1(u-\varphi) \le Y_k u \le Y_k\varphi - \alpha_0 h - \beta_0(u-\varphi). \tag{15.41}$$

Thus, if ∂_ν denotes the normal derivative at $\partial\Omega$,

$$|\partial_\nu Y_k u| \le (\alpha_0-\alpha_1)|\partial_\nu h| + (\beta_0-\beta_1)|\partial_\nu u - \partial_\nu\varphi| + |\partial_\nu Y_k\varphi|, \tag{15.42}$$

when u solves (15.33).

In view of the example (15.5), for a surface with Gauss curvature K, we have ample motivation to estimate the normal derivative of $Y_k u$ when u solves the more general equation (15.15). We now tackle this, following [CNS].

Generally, if $w_k = Y_k(u-\varphi)$, (15.31) yields

$$\begin{aligned} &g^{ij}\,\partial_i\partial_j w_k - f_{p_i}\,\partial_i w_k \\ &\quad = \big[A(x) + f_{p_i}\partial_i Y_k\varphi\big] + g^{ij}\big[B_{ij}(x) - \partial_i\partial_j Y_k\varphi\big] = \Phi(x). \end{aligned} \tag{15.43}$$

Note that, given a bound for u in $C^1(\overline{\Omega})$, we have

$$|\Phi(x)| \le C + C g^{jj}, \tag{15.44}$$

where g^{jj} is the trace of (g^{ij}).

Translate coordinates so that $y = 0$. Recall that we assume $\nu(y)$ is parallel to the x_n-axis. Assume $x_n \ge 0$ on $\overline{\Omega}$. As above, assume $h \in C^\infty(\overline{\Omega})$ satisfies (15.36). Take $\mu \in (0, 1/4)$ and $M \in (0,\infty)$, and set $h_\mu(x) = h(x) - \mu|x|^2$. We have

$$\begin{aligned} &(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \\ &\quad = g^{ij}\,\partial_i\partial_j h_\mu - f_{p_i}\,\partial_i h_\mu + 2Mg^{nn} - 2Mf_{p_n}x_n \\ &\quad \ge \Big(\frac{1}{2}g^{jj} + 2Mg^{nn}\Big) - \big(Mf_{p_n}x_n + f_{p_i}\,\partial_i h_\mu\big). \end{aligned} \tag{15.45}$$

The arithmetic-geometric mean inequality implies

$$\big(M\sigma_1 \cdots \sigma_n\big)^{1/n} \leq \frac{1}{n}\Big(\sum_{j<n} \sigma_j + M\sigma_n\Big),$$

and if the eigenvalues of (g^{ij}) are $\sigma_n \leq \cdots \leq \sigma_1$, we have $g^{nn} \geq \sigma_n$, and hence

$$\big[M \det(g^{ij})\big]^{1/n} \leq \frac{1}{n}\big(g^{jj} + Mg^{nn}\big). \tag{15.46}$$

Given a positive lower bound on $\det(g^{ij}) = 1/F(x,u,\nabla u)$, we have

$$\frac{1}{2}g^{jj} + 2Mg^{nn} \geq cg^{jj} + c_1 M^{1/n}. \tag{15.47}$$

Hence (15.45) implies

$$(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \geq cg^{jj} + c_1M^{1/n} - c_2 - c_3Mx_n. \tag{15.48}$$

At this point, fix M sufficiently large that $c_1M^{1/n} \geq 1 + c_2$, so that

$$(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \geq 1 + cg^{jj} - c_3Mx_n \quad \text{on } \Omega. \tag{15.49}$$

Now, let

$$\mathcal{O}_\varepsilon = \{x \in \Omega : 0 < x_n < \varepsilon\},$$

as illustrated in Fig. 15.1. We can then pick ε sufficiently small that (e.g., with $\mu = 1/8$)

$$(g^{ij}\,\partial_i\partial_j - f_{p_i}\,\partial_i)(h_\mu + Mx_n^2) \geq cg^{jj} + \frac{1}{2} \quad \text{on } \mathcal{O}_\varepsilon. \tag{15.50}$$

Note that the function h has the property $\nabla h \neq 0$ on $\partial\Omega$. Thus, after possibly further shrinking ε, we have

$$h_\mu + Mx_n^2 \leq \begin{matrix} 0 & \text{on } \partial\mathcal{O}_\varepsilon \cap \partial\Omega, \\ -c_4 < 0 & \text{on } \Omega \cap \{x_n = \varepsilon\}. \end{matrix} \tag{15.51}$$

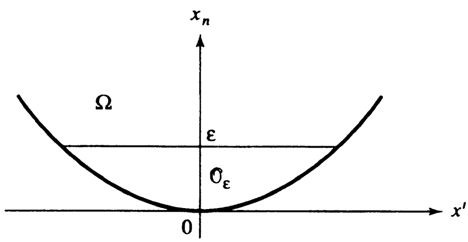

FIGURE 15.1 Setup for Normal Derivative Estimate

With $\varepsilon > 0$ so fixed, we can then pick A sufficiently large (depending on $\|u\|_{C^1(\overline{\Omega})}$) that $c_4 A \geq \|Y_k u\|_{L^\infty(\Omega)}$; hence

$$\begin{aligned} w_k + A(h_\mu + M x_n^2) &\leq 0, \\ w_k - A(h_\mu + M x_n^2) &\geq 0 \end{aligned} \tag{15.52}$$

on $\partial\mathcal{O}_\varepsilon$. We can also pick A so large that (by (15.50) and (15.43)–(15.44))

$$\begin{aligned} (g^{ij}\partial_i\partial_j - f_{p_i}\partial_i)\big(w_k + A(h_\mu + M x_n^2)\big) &\geq 0, \\ (g^{ij}\partial_i\partial_j - f_{p_i}\partial_i)\big(w_k - A(h_\mu + M x_n^2)\big) &\leq 0 \end{aligned} \tag{15.53}$$

on $\mathcal{O}_\varepsilon$. The maximum principle then implies that (15.52) holds on $\mathcal{O}_\varepsilon$. Thus

$$|\partial_n Y_k u(y)| \leq A|\partial_n h_\mu(y)|. \tag{15.54}$$

This completes our estimation of $\partial_n Y_k u(y)$, begun at (15.31).

We prepare to tackle the estimation of $\partial_n^2 u(y)$. A key ingredient will be a positive *lower* bound on $\partial_j^2 u(y)$, for $1 \leq j \leq n-1$. In order to get this, we make a further (temporary) hypothesis, namely that there is a strictly convex function $u^\# \in C^\infty(\overline{\Omega})$ satisfying

$$\log\det H(u^\#) - f(x, u^\#, \nabla u^\#) \leq 0 \text{ on } \Omega, \quad u^\#\big|_{\partial\Omega} = \varphi. \tag{15.55}$$

The function $u^\#$ is called an *upper solution* to (15.1). The proof of (15.8) yields

$$u^b \leq u_\sigma \leq u_\tau \leq u^\# \quad \text{on } \Omega, \tag{15.56}$$

for $\sigma \leq \tau \in J$. In the present context, where we have dropped the σ and where $u \in C^\infty(\overline{\Omega})$ is a solution to (15.15), this means $u^b \leq u \leq u^\#$ on Ω. Consequently, complementing (15.11), we have

$$\partial_\nu u \geq \partial_\nu u^\# \quad \text{on } \partial\Omega. \tag{15.57}$$

Now let Y_j be the vector field tangent to $\partial\Omega$, equal to ∂_j at y, used in (15.30). We have

$$\partial_j^2 u(y) = Y_j^2 u(y) + \kappa_j \partial_\nu u(y), \quad \kappa_j = \widetilde{II}(\partial_j, \partial_j) > 0, \tag{15.58}$$

for $1 \leq j \leq n-1$, by (15.30), assuming $\partial\Omega$ is strongly convex. There is a similar identity for $\partial_j^2 u^\#(y)$. Since $u = u^\# = \varphi$ on $\partial\Omega$, subtraction yields

$$\partial_j^2 u(y) = \partial_j^2 u^\#(y) + \kappa_j\big[\partial_\nu u(y) - \partial_\nu u^\#(y)\big] \geq \partial_j^2 u^\#(y), \tag{15.59}$$

for $1 \le j \le n-1$, the inequality following from (15.57). Since $u^{\#}$ is assumed to be a given strongly convex function, this yields a positive lower bound:

$$\partial_j^2 u(y) \ge K_0 > 0, \quad 1 \le j \le n-1. \tag{15.60}$$

Now we can get an upper bound on $\partial_n^2 u(y)$. Rotating the $x_1 \dots x_{n-1}$ coordinate axes, we can assume $\big(\partial_j \partial_k u(y)\big)_{1 \le j,k \le n-1}$ is diagonal. Then, at y,

$$\det H(u) = (\partial_n^2 u) \prod_{j=1}^{n-1} (\partial_j^2 u) + \varkappa(\partial^2 u), \tag{15.61}$$

where $\varkappa$ is an n-linear form in $\partial^2 u(y)$ that does *not* contain $\partial_n^2 u(y)$. Since $\det H(u) = f(x, u, \nabla u)$ and we have estimates on ∇u, as well as $\partial_j \partial_k u(y)$ for $\partial_j \partial_k \ne \partial_n^2$, we deduce that

$$K_0^{n-1} \partial_n^2 u(y) \le K_1. \tag{15.62}$$

This completes the estimation of $\|u\|_{C^2(\overline{\Omega})}$.

Once we have a bound in $C^2(\overline{\Omega})$ for solutions to (15.15), we can apply Theorem 14.6 to deduce the existence of a solution $u \in C^\infty(\overline{\Omega})$ to (15.1). We thus have the following:

Proposition 15.5. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded, open set with strongly convex boundary. Consider the Dirichlet problem (15.1), with $\varphi \in C^\infty(\partial\Omega)$. Assume $F(x, u, p)$ is a smooth function of its arguments satisfying*

$$F(x, u, p) > 0, \quad \partial_u F(x, u, p) \ge 0.$$

Furthermore, assume (15.1) has a lower solution u^b, and an upper solution $u^{\#} \in C^\infty(\overline{\Omega})$. Then (15.1) has a unique convex solution $u \in C^\infty(\overline{\Omega})$.

After a little more work, we will show that we need not assume the existence of an upper solution $u^{\#}$. Note that $u^{\#}$ was not needed for the estimates of

$$s_0 = \sup\ |u|, \quad s_1 = \sup\ |\nabla u|$$

in Lemmas 15.1–15.3. Thus, if we take a constant a satisfying

$$0 < a < \inf\{F(x, u, p) : x \in \overline{\Omega},\ |u| \le s_0,\ |p| \le s_1\},$$

then any smooth, strongly convex $u^{\#}$ satisfying

$$\det H(u^{\#}) \le a \ \text{ on } \ \Omega, \quad u^{\#}\big|_{\partial\Omega} = \varphi, \tag{15.63}$$

will serve as an upper solution to (15.1). Thus, for arbitrary $a > 0$, we want to produce $u^\# \in C^\infty(\overline{\Omega})$, which is strongly convex and satisfies (15.63). For this purpose, it is more than sufficient to have the following result, which is of interest in its own right.

Proposition 15.6. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded, open set with strongly convex boundary. Let $\varphi \in C^\infty(\partial\Omega)$ be given and assume $F \in C^\infty(\overline{\Omega})$ is positive. Then there is a unique convex solution $u \in C^\infty(\overline{\Omega})$ to*

$$\text{(15.64)} \qquad \det H(u) = F(x), \quad u\big|_{\partial\Omega} = \varphi.$$

Proof. First, note that (15.64) always has a lower solution. In fact, if you extend φ to an element of $C^\infty(\overline{\Omega})$ and let $h \in C^\infty(\overline{\Omega})$ be as in (15.36), then $u^b = \varphi + \tau h$ will work, for sufficiently large τ.

Following the proof of Proposition 15.5, we see that to establish Proposition 15.6, it suffices to obtain an a priori estimate in $C^2(\overline{\Omega})$ for a solution to (15.64). All the arguments used above to establish Proposition 15.5 apply in this case, up to the use of $u^\#$, in (15.55)–(15.59), to establish the estimate (15.60), namely,

$$\text{(15.65)} \qquad \partial_j^2 u(y) \geq K_0 > 0, \quad 1 \leq j \leq n-1.$$

Recall that y is an arbitrarily selected point in $\partial\Omega$, and we have rotated coordinates so that the normal $\nu(y)$ to $\partial\Omega$ is parallel to the x_n-axis. If we establish (15.65) in this case, without using the hypothesis that an upper solution exists, then the rest of the previous argument giving an estimate in $C^2(\overline{\Omega})$ will work, and Proposition 15.6 will be proved.

We establish (15.65), following [CNS], via a certain barrier function. It suffices to treat the case $j = 1$. We can also assume that y is the origin in $\mathbb{R}^n$ and that, near y, $\partial\Omega$ is given by

$$\text{(15.66)} \qquad x_n = \rho(x') = \sum_{j=1}^{n-1} B_j x_j^2 + O(|x'|^3), \quad B_j > 0,$$

where $x' = (x_1, \dots, x_{n-1})$.

Note that adding a linear term to u leaves the left side of (15.64) unchanged and also has no effect on $\partial_j^2 u$. Thus, without loss of generality, we can assume that

$$\text{(15.67)} \qquad u(0) = 0, \quad \partial_j u(0) = 0, \ 1 \leq j \leq n-1.$$

We have, on $\partial\Omega$,

$$\text{(15.68)} \qquad u = \varphi = \frac{1}{2} \sum_{j,k<n} \gamma_{jk} x_j x_k + \varkappa_3(x') + O(|x|^4),$$

where $\varkappa_3(x')$ is a polynomial, homogeneous of degree 3 in x'.

Now consider

$$\tilde{u}(x) = u(x) - \lambda x_n, \quad \lambda = B_1^{-1}\gamma_{11}. \tag{15.69}$$

This function satisfies $\det H(\tilde{u}) = F(x)$. Looking at $\tilde{u}\big|_{\partial\Omega} = \varphi - \lambda\rho(x')$, we see that the coefficients of x_1^2 cancel out here. We claim there is an estimate of the form

$$\tilde{u}\big|_{\partial\Omega} \le \sum_{1<j\le n} a_{1j}x_1x_j + C\Big(\sum_{1<k<n} x_k^2 + |x|^4\Big). \tag{15.70}$$

Indeed, in light of our remark about the disappearance of x_1^2, we need only worry about a multiple of x_1^3, which can be dominated on $\partial\Omega$ by a term of the form $a_{1n}x_1x_n$ plus a multiple of the quantity in parentheses in (15.70).

The barrier function will take the form

$$W(x) = \frac{1}{2B}\sum_{1<j\le n}(a_{1j}x_1 + Bx_j)^2 + \delta|x|^2 - \varepsilon x_n. \tag{15.71}$$

Take $B >> C$, then fix $\delta > 0$ small, and take $\varepsilon << \delta$. We can do this in such a fashion as to arrange

$$W \ge \tilde{u} \quad \text{on } \partial\Omega. \tag{15.72}$$

Note that 2δ is the smallest eigenvalue of $H(W)$, and all the other eigenvalues are bounded above independently of $\delta \in (0,1)$, so choosing δ small enough gives

$$\det H(W) < F(x) \quad \text{on } \Omega. \tag{15.73}$$

Then W is an upper barrier for $\tilde{u}$; the maximum principle yields

$$\tilde{u} \le W \quad \text{on } \Omega. \tag{15.74}$$

Consequently,

$$\partial_n\tilde{u}(0) \le \partial_n W(0) = -\varepsilon. \tag{15.75}$$

As noted above, our construction (15.69) yields

$$\partial_1^2\tilde{u}\big(x', \rho(x')\big) = 0, \ \text{ at } x' = 0, \tag{15.76}$$

that is, $\partial_1^2\tilde{u} + (\partial_n\tilde{u})\partial_1^2\rho = 0$, at $x' = 0$. Hence

$$\partial_1^2 u(0) = \partial_1^2\tilde{u}(0) = -\partial_n\tilde{u}(0)\cdot\partial_1^2\rho(0) \ge \varepsilon\partial_1^2\rho(0). \tag{15.77}$$

This proves the $j = 1$ case of (15.65), as needed, so Proposition 15.6 is proved.

In light of the comments made after the statement of Proposition 15.5, we have

Corollary 15.7. *In Proposition 15.5, the hypothesis that there exists an upper solution $u^\#$ can be omitted.*

There are some results for Monge–Ampere equations on nonconvex domains; see [GS] and [HRS].

In addition to the Monge–Ampere equations studied here, there are *complex* Monge–Ampere equations, whose study has been very important in complex function theory and differential geometry; see [Au, BT, CKNS, Fef, Yau1].

Exercises

1. Let $\Omega \subset \mathbb{R}^2$ be a strongly convex, smoothly bounded region. Let us assume that $F \in C^\infty(\overline{\Omega}),\ \varphi \in C^\infty(\partial\Omega)$, and $F > 0$. Show that

$$\det H(u) = F(x) \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = \varphi,$$

has exactly two solutions in $C^\infty(\overline{\Omega})$, one convex and one concave.

2. Suppose the hypothesis $\partial_u F(x, u, p) \geq 0$ in Proposition 15.5 is dropped. Establish the existence of solutions, using the Leray–Schauder theory.
3. Given Ω as in Proposition 15.5, $\varphi \in C^\infty(\partial\Omega)$, show that there exists $K_0 > 0$ such that, for all $K \in (0, K_0)$, there is a unique convex solution $u_K \in C^\infty(\overline{\Omega})$ to

$$\det H(u_K) = K\big(1 + |\nabla u_K|^2\big)^{(n+2)/2} \ \text{ on } \ \Omega, \quad u_K\big|_{\partial\Omega} = \varphi. \tag{15.78}$$

(*Hint*: Show that the convex solution to (15.64), with $F = 1$, yields a lower solution for (15.78), provided $K > 0$ is sufficiently small.)

Note that the graph of u_K is a surface with Gauss curvature K.

4. With u_K as in Exercise 3, show that there is $u_0 \in \mathrm{Lip}^1(\overline{\Omega})$ such that

$$u_K \nearrow u_0 \quad \text{as} \quad K \searrow 0. \tag{15.79}$$

In what sense can you say that u_0 solves

$$\det H(u_0) = 0 \ \text{ on } \ \Omega, \quad u_0\big|_{\partial\Omega} = \varphi? \tag{15.80}$$

See [RT] and [TU] for more on (15.80).

16. Elliptic equations in two variables

We have seen in § 12 that results on quasi-linear, uniformly elliptic equations for real-valued functions on a domain Ω are obtained more easily when $\dim \Omega = 2$ than when $\dim \Omega \geq 3$ and have extensions to systems that do not work in higher dimensions. Here we will obtain results on completely nonlinear equations for functions of two variables which are more general than those established in § 14 for functions of n variables. The key is the following result of Morrey on linear equations with bounded measurable coefficients, whose conclusion is stronger than that of Theorem 13.7:

Theorem 16.1. *Assume $u \in C^2(\Omega)$ and $Lu = f$ on $\Omega \subset \mathbb{R}^2$, where*

$$Lu = \sum_{j,k=1}^{2} a^{jk}(x)\, \partial_j \partial_k u. \tag{16.1}$$

Assume $a^{jk} = a^{kj}$ are measurable on Ω and

$$\lambda|\xi|^2 \le a^{jk}(x)\xi_j\xi_k \le \Lambda|\xi|^2, \tag{16.2}$$

for some $\lambda, \Lambda \in (0,\infty)$. Pick $p > 2$. Then, for $\mathcal{O} \subset\subset \Omega$, there is a $\mu > 0$ such that

$$\|u\|_{C^{1+\mu}(\mathcal{O})} \le C\big[\|u\|_{H^1(\Omega)} + \|f\|_{L^p(\Omega)}\big], \tag{16.3}$$

where $C = C(\mathcal{O}, \Omega, p, \lambda, \Lambda)$.

Proof. Let $V_j = \partial_j u$. Then these functions satisfy the divergence-form equations

$$\begin{aligned} &\partial_1\Big(\frac{a^{11}}{a^{22}}\partial_1 V_1 + 2\frac{a^{12}}{a^{22}}\partial_2 V_1\Big) + \partial_2\big(\partial_2 V_1\big) = \partial_1\Big(\frac{f}{a^{22}}\Big), \\ &\partial_1\big(\partial_1 V_2\big) + \partial_2\Big(\frac{a^{22}}{a^{11}}\partial_2 V_2 + 2\frac{a^{12}}{a^{11}}\partial_1 V_2\Big) = \partial_2\Big(\frac{f}{a^{11}}\Big). \end{aligned} \tag{16.4}$$

Proposition 9.8 applies to each of these equations, yielding

$$\|V_j\|_{C^\mu(\mathcal{O})} \le C\big[\|V_j\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)}\big]. \tag{16.5}$$

This yields the desired estimate (16.3).

Morrey's original proof of Theorem 16.1 came earlier than the DeGiorgi-Nash-Moser estimate used in the proof above. Instead, he used estimates on quasi-conformal mappings (see [Mor2]).

We apply Theorem 16.1 to estimates for real-valued solutions to equations of the form

$$F(x, u, \nabla u, \partial^2 u) = f \quad \text{on } \Omega \subset \mathbb{R}^2, \tag{16.6}$$

where $F = F(x, u, p, \zeta)$ is a smooth function of its arguments satisfying the ellipticity condition

$$\begin{aligned} \lambda|\xi|^2 \le \sum \frac{\partial F}{\partial \zeta_{jk}}(x, u, p, \zeta)\xi_j\xi_k \le \Lambda|\xi|^2, \\ 0 < \lambda = \lambda(u, p, \zeta), \quad \Lambda = \Lambda(u, p, \zeta). \end{aligned} \tag{16.7}$$

For $h > 0,\ \ell = 1, 2$, set

$$V_{\ell h}(x) = h^{-1}\big(u(x + he_\ell) - u(x)\big). \tag{16.8}$$

Then $V_{\ell h}$ satisfies the equation

$$\sum_{j,k} a^{jk}_{\ell h}(x)\partial_j \partial_k V_{\ell h} = g_{\ell h}(x) \tag{16.9}$$

on $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^2 \setminus \Omega) > h\}$, where the coefficients $a^{jk}_{\ell h}(x)$ are given by

$$a^{jk}_{\ell h}(x) = \int_0^1 \frac{\partial F}{\partial \zeta_{jk}}\big(x + she_\ell, \dots, s\partial^2 \tau_{\ell h} u + (1 - s)\partial^2 u\big)\, ds, \tag{16.10}$$

with $\tau_{\ell h} u(x) = u(x + he_\ell)$, and the functions $g_{\ell h}(x)$ are given by

$$\begin{aligned} g_{\ell h}(x) = &- \sum_j \left[\int_0^1 \frac{\partial F}{\partial p_j}\big(x + she_\ell, \dots, s\partial^2 \tau_{\ell h} u + (1 - s)\partial^2 u\big)\ ds\right] \partial_j V_{\ell k} \\ &- \int_0^1 \frac{\partial F}{\partial u}\big(x + she_\ell, \dots, s\partial^2 \tau_{\ell h} + (1 - s)\partial^2 u\big)\, ds\ V_{\ell h} \\ &- \int_0^1 \frac{\partial F}{\partial x_\ell}\big(x + she_\ell, \dots, s\partial^2 \tau_{\ell h} u + (1 - s)\partial^2 u\big)\, ds \\ &+ h^{-1}\big(f(x + he_\ell) - f(x)\big). \end{aligned} \tag{16.11}$$

Theorem 16.1 then yields an estimate

$$\|V_{\ell h}\|_{C^{1+\mu}(\mathcal{O})} \le C\big[\|V_{\ell h}\|_{L^2(\Omega)} + \|g_{\ell h}\|_{L^p(\Omega)}\big], \tag{16.12}$$

with $C = C(\mathcal{O}, \Omega, p, \lambda, \Lambda, \|u\|_{C^2(\overline{\Omega})})$. Note that

$$\|g_{\ell h}\|_{L^p(\Omega)} \le C\big(\|u\|_{C^2(\overline{\Omega})}\big) + \big\|h^{-1}(\tau_{\ell h} f - f)\big\|_{L^p(\Omega)}. \tag{16.13}$$

Letting $h \to 0$, we have the following:

Theorem 16.2. *Assume that $\Omega \subset \mathbb{R}^2$, that $u \in C^2(\overline{\Omega})$ solves (16.6), that the ellipticity condition (16.7) holds, and that $f \in H^{1,p}(\Omega)$, for some $p > 2$. Then, given $\mathcal{O} \subset\subset \Omega$, there is a $\mu > 0$ such that $u \in C^{2+\mu}(\mathcal{O})$ and*

$$\|u\|_{C^{2+\mu}(\mathcal{O})} \le C\big[1 + \|f\|_{H^{1,p}(\Omega)}\big], \tag{16.14}$$

where

$$C = C\big(\mathcal{O}, \Omega, p, \lambda, \Lambda, \|u\|_{C^2(\overline{\Omega})}\big). \tag{16.15}$$

For estimates up to the boundary, we use the following complement to Theorem 16.1:

Proposition 16.3. *If $u \in C^2(\overline{\Omega})$ and the hypotheses of Theorem 16.1 hold, then there is an estimate*

$$\|u\|_{C^{1+\mu}(\overline{\Omega})} \le C\big[\|u\|_{H^{1,p}(\Omega)} + \|\varphi\|_{C^2(\partial\Omega)} + \|f\|_{L^p(\Omega)}\big], \tag{16.16}$$

where $\varphi = u\big|_{\partial\Omega}$ and $C = C(\Omega, p, \lambda, \Lambda)$.

Proof. Given $y \in \partial\Omega$, locally flatten $\partial\Omega$ near y, using a coordinate change, transforming it to the x_1-axis. In the new coordinates, u satisfies an elliptic equation of the form

$$\widetilde{a}^{jk}\partial_j\partial_k u = f - \widetilde{b}^j\partial_j u = \widetilde{f}. \tag{16.17}$$

Then $\widetilde{V}_1 = \partial_1 u$ satisfies an analogue of the first equation in (16.4), while $\widetilde{V}_1 = \partial_1\varphi$ on the flattened part of $\partial\Omega$. Thus Proposition 9.9 (or rather the local version mentioned at the end of § 9) yields an estimate on $\widetilde{V}_1$ in $C^\mu(U \cap \overline{\Omega})$, for some neighborhood U of y in $\mathbb{R}^2$.

Thus, for any smooth vector field X on $\mathbb{R}^2$, tangent to $\partial\Omega$, we have an estimate on $\|Xu\|_{C^\mu(\overline{\Omega})}$ by the right side of (16.16). Furthermore, by Proposition 9.9, there is a Morrey space estimate

$$\|\nabla Xu\|_{M_2^q(\Omega)} \le \text{ RHS}, \tag{16.18}$$

for some $q > 2$, where "RHS" stands for the right side of (16.16). We may as well assume $q \le p$, so $\widetilde{f} \in L^p(\Omega) \subset M_2^q(\Omega)$. Then (16.17) and (16.18) together imply

$$\|\partial_j\partial_k u\|_{M_2^q(\Omega)} \le \text{ RHS}, \tag{16.19}$$

for all $j, k \le 2$, which in turn implies (16.16).

We now establish the following:

Theorem 16.4. *Assume that $\Omega \subset \mathbb{R}^2$ and that $u \in C^3(\overline{\Omega})$ solves (16.6), with the ellipticity condition (16.7), with $f \in H^{1,p}(\Omega)$ for some $p > 2$, and $u\big|_{\partial\Omega} = \varphi$. Then, for some $\mu > 0$, there is an estimate*

$$\|u\|_{C^{2+\mu}(\overline{\Omega})} \le C\big[1 + \|\varphi\|_{C^3(\partial\Omega)} + \|f\|_{H^{1,p}(\Omega)}\big], \tag{16.20}$$

where

$$C = C\big(\Omega, p, \lambda, \Lambda, \|u\|_{C^2(\overline{\Omega})}\big). \tag{16.21}$$

Proof. If $X = b^\ell \partial_\ell$ is a smooth vector field in $\mathbb{R}^2$, tangent to $\partial\Omega$, then Xu satisfies

$$\begin{aligned} F_{\zeta_{jk}}\,\partial_j\partial_k(Xu) = &- F_{p_j}\,\partial_j(Xu) - F_u\,Xu + F_{\zeta_{jk}}(\partial_j\partial_k b^\ell)(\partial_\ell u) \\ &+ 2F_{\zeta_{jk}}(\partial_j b^\ell)(\partial_k\partial_\ell u) + F_{p_j}(\partial_j b^\ell)(\partial_\ell u) + Xf, \end{aligned} \tag{16.22}$$

and $Xu = X\varphi$ on $\partial\Omega$. Thus Proposition 16.3 applies. We have a $C^{1+\mu}(\overline{\Omega})$-estimate on Xu, and even better, a Morrey space estimate:

$$\|\partial_j\partial_k Xu\|_{M_2^q(\Omega)} \le \text{ RHS}, \tag{16.23}$$

for some $q > 2$, and for all $j, k \le 2$, where "RHS" now stands for the right side of (16.20).

The proof is almost done. Parallel to (16.22), we have, for any ℓ,

$$F_{\zeta_{jk}}\,\partial_j\partial_k\partial_\ell u = -F_{p_j}\,\partial_j\partial_\ell u - F_u\,\partial_\ell u + \partial_\ell f. \tag{16.24}$$

Thus we can solve for $\partial_j\partial_k\partial_\ell u$ in terms of functions of the form $\partial_j\partial_k Xu$ and other terms estimable in the $M_2^q(\Omega)$-norm by the right side of (16.20). Hence we have (16.20), and even the stronger estimate

$$\|\partial^3 u\|_{M_2^q(\Omega)} \le \text{ RHS}. \tag{16.25}$$

From this result the continuity method readily gives the following:

Theorem 16.5. *Let Ω be a smoothly bounded domain in $\mathbb{R}^2$. Let the function $F_\sigma(x, u, p, \zeta)$ depend smoothly on all its arguments, for $\sigma \in [0,1]$, and let $\varphi_\sigma \in C^\infty(\overline{\Omega})$ have smooth dependence on σ. Assume that, for each $\sigma \in [0,1]$,*

$$\partial_u F_\sigma(x, u, p, \zeta) \le 0$$

and that the ellipticity condition (16.7) holds. Also assume that, for any solution $u_\sigma \in C^\infty(\overline{\Omega})$ to the equation

$$F_\sigma(x, u_\sigma, \nabla u_\sigma, \partial^2 u_\sigma) = 0 \text{ on } \Omega, \quad u_\sigma\big|_{\partial\Omega} = \varphi_\sigma, \tag{16.26}$$

there is a $C^2(\overline{\Omega})$*-bound:*

$$\|u_\sigma\|_{C^2(\overline{\Omega})} \le K. \tag{16.27}$$

If (16.26) has a solution in $C^\infty(\overline{\Omega})$ *for* $\sigma = 0$*, then it has a solution in* $C^\infty(\overline{\Omega})$ *for* $\sigma = 1$.

Exercises

1. In the proof of Theorem 16.1, can you replace the use of Proposition 9.8 by a result analogous to Proposition 12.5?
2. Suppose that, in (16.7), λ and Λ are independent of ζ. Obtain a variant of Theorem 16.5 in which (16.27) is weakened to a bound in $C^1(\overline{\Omega})$.

A. Morrey spaces

Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $p \in [1,\infty)$, one says $f \in M^p(\mathbb{R}^n)$ provided that

$$R^{-n} \int_{B_R} |f(x)|\, dx \le C\, R^{-n/p}, \tag{A.1}$$

for all balls B_R of radius $R \le 1$ in $\mathbb{R}^n$. More generally, if $1 \le q \le p$ and $f \in L^q_{\text{loc}}(\mathbb{R}^n)$, we will say $f \in M^p_q(\mathbb{R}^n)$ provided that, for all such B_R,

$$R^{-n} \int_{B_R} |f(x)|^q\, dx \le C\, R^{-nq/p}. \tag{A.2}$$

The spaces $M^p_q(\mathbb{R}^n)$ are called *Morrey spaces*. If we set $\delta_R f(x) = f(Rx)$, the left side of (A.2) is equal to $\int_{B_1} |\delta_R f(x)|^q\, dx$, so an equivalent condition is

$$\|\delta_R f\|_{L^q(B_1)} \le C' R^{-n/p}, \tag{A.3}$$

for all balls B_1 of radius 1, and for all $R \in (0,1]$. It follows from Hölder's inequality that

$$L^p_{\text{unif}}(\mathbb{R}^n) = M^p_p(\mathbb{R}^n) \subset M^p_q(\mathbb{R}^n) \subset M^p(\mathbb{R}^n).$$

We can give an equivalent characterization of M^p in terms of the heat kernel. Let $p_r(\xi) = e^{-|r\xi|^2}$. Then, given $f \in L^1_{\text{unif}}(\mathbb{R}^n)$,

$$f \in M^p(\mathbb{R}^n) \Longleftrightarrow p_r(D)|f| \le C\, r^{-n/p}, \tag{A.4}$$

for $0 < r \le 1$. To see the implication $\Rightarrow$, given $x \in \mathbb{R}^n$ write $f = f_1 + f_2$, where f_1 is the restriction of f to the unit ball $B_1(x)$ centered at x, and f_2 is the restriction of f to the complement. That $p_r(D)|f_1|(x) \le C r^{-n/p}$, for $r \in (0,1]$, follows easily from the characterization (A.1) and the formula

$$p_r(D)\delta_x(y) = (4\pi r^2)^{-n/2}\, e^{-|x-y|^2/4r^2},$$

while this formula also implies that $p_r(D)|f_2|(x)$ is rapidly decreasing as $r \searrow 0$. The implication $\Leftarrow$ is similarly easy to verify. Note that

$$f \text{ satisfies (A.4)} \Longrightarrow |p_r(D)f| \le C r^{-n/p}. \tag{A.5}$$

Recall the Zygmund spaces $C^r_*(\mathbb{R}^n)$, $r \in \mathbb{R}$, introduced in § 8 of Chap. 13, with norms defined as follows. Let $\Psi_0(\xi) \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 for $|\xi| \le 1$, set $\Psi_k(\xi) = \Psi_0(2^{-k}\xi)$, and let $\psi_k(\xi) = \Psi_k(\xi) - \Psi_{k-1}(\xi)$. The set $\{\psi_k(\xi)\}$ is a Littlewood–Paley partition of unity. One sets

$$\|f\|_{C^r_*} = \sup_k\, 2^{kr}\|\psi_k(D)f\|_{L^\infty}. \tag{A.6}$$

For $r \in (0,\infty) \setminus \mathbb{Z}^+$, C^r_* coincides with the Hölder space C^r, and C^1_* is the classical Zygmund space. As shown in Chap. 13, one has, for all $m,\ r \in \mathbb{R}$,

$$P \in OPS^m_{1,0} \Longrightarrow P : C^r_* \longrightarrow C^{r-m}_*. \tag{A.7}$$

The following relation exists between Zygmund spaces and Morrey spaces. From (A.4)–(A.5) we readily obtain the inclusion

$$M^p(\mathbb{R}^n) \subset C_*^{-n/p}(\mathbb{R}^n). \tag{A.8}$$

From this we deduce a result known as *Morrey's lemma*:

Lemma A.1. *If $p > n$, then, for $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\nabla f \in M^p(\mathbb{R}^n) \Longrightarrow f \in C^r_{loc}(\mathbb{R}^n), \quad r = 1 - \frac{n}{p} \in (0,1). \tag{A.9}$$

Proof. We can write

$$f = \sum_{j=1}^n B_j(\partial_j f) + Rf, \quad B_j \in OPS^{-1}(\mathbb{R}^n), \ \ R \in OPS^{-\infty}(\mathbb{R}^n). \tag{A.10}$$

Then (A.7)–(A.8) imply that $B_j\, \partial_j f \in C^r_*(\mathbb{R}^n)$, if the hypothesis of (A.9) holds.

If $\Omega \subset \mathbb{R}^n$ is a bounded region, we say $f \in M_q^p(\Omega)$ if $\tilde{f} \in M_q^p(\mathbb{R}^n)$, where $\tilde{f}(x) = f(x)$ for $x \in \Omega,\ 0$ for $x \notin \Omega$. If $\partial\Omega$ is smooth, it is easy to extend (A.9) to the implication (for $p > n$):

$$\nabla f \in M^p(\Omega) \Longrightarrow f \in C^r(\overline{\Omega}), \quad r = 1 - \frac{n}{p} \in (0,1), \tag{A.11}$$

via a simple reflection argument (across $\partial\Omega$).

One also considers homogeneous versions of Morrey spaces. If $p \in (1,\infty)$ and $1 \le q \le p,\ f \in L^q_{\text{loc}}(\mathbb{R}^n)$, we say $f \in \mathcal{M}_q^p(\mathbb{R}^n)$ provided (A.2) holds for all $R \in (0,\infty)$, not just for $R \le 1$. Note that if we set

$$\|f\|_{\mathcal{M}_q^p} = \sup_R \ R^{n/p}\Bigl(R^{-n}\int\limits_{B_R} |f(x)|^q \, dx\Bigr)^{1/q}, \tag{A.12}$$

where R runs over $(0,\infty)$ and B_R over all balls of radius R, then

$$\|\delta_r f\|_{\mathcal{M}_q^p} = r^{-n/p}\ \|f\|_{\mathcal{M}_q^p}, \tag{A.13}$$

where $\delta_r f(x) = f(rx)$. This is the same type of scaling as the $L^p(\mathbb{R}^n)$-norm. It is clear that compactly supported elements of $M_q^p(\mathbb{R}^n)$ and of $\mathcal{M}_q^p(\mathbb{R}^n)$ coincide. In a number of references, including [P], $\mathcal{M}_q^p$ is denoted $\mathcal{L}_{q,\lambda}$, with $\lambda = n\bigl(1 - q/p\bigr)$.

The following refinement of Morrey's lemma is due to S. Campanato.

Proposition A.2. *Given $p \in [1,\infty),\ s \in (0,1)$, assume that $u \in L^p_{loc}(\mathbb{R}^n)$ and that, for each ball $B_R(x)$ with $R \le 1$, there exists $\alpha \in \mathbb{C}$ such that*

$$\int\limits_{B_R(x)} |u(y) - \alpha|^p \, dy \le CR^{n+ps}. \tag{A.14}$$

Then

$$u \in C^s_{loc}(\mathbb{R}^n). \tag{A.15}$$

Proof. Pick $\varphi \in C_0^\infty(\mathbb{R}^n)$ to be a radial function, supported on $|x| \le 1$, such that $\widehat{\varphi}(\xi) \ge 0$, and let $\psi = \Delta\varphi$, so $\int \psi \, dx = 0$. It suffices to show that

$$\bigl|(\psi_R * u)(x)\bigr| \le CR^s, \quad R \le 1, \tag{A.16}$$

where $\psi_R(x) = R^{-n}\psi(R^{-1}x)$. Note that, for fixed $x,\ R,\ \alpha = \alpha\bigl(B_R(x)\bigr)$, we have

$$(\psi_R * u)(x) = \psi_R * (u - \alpha)(x), \tag{A.17}$$

so

$$
\begin{aligned}
&\left|(\psi_R * u)(x)\right| \\
&\le \|\psi_R\|_{L^{p'}(B_R(0))}\|u-\alpha\|_{L^p(B_R(x))} \\
&\le \Bigl(\int\limits_{B_R(0)} R^{-np'}|\psi(R^{-1}y)|^{p'}\,dy\Bigr)^{1/p'}\Bigl(\int\limits_{B_R(x)} |u(y)-\alpha|^p\,dy\Bigr)^{1/p} \\
&\le C\ R^{-n}\cdot R^{n/p'}\cdot R^{n/p}\cdot R^s = R^s,
\end{aligned}
\tag{A.18}
$$

as desired.

B. Leray–Schauder fixed-point theorems

We will demonstrate several fixed-point theorems that are useful for nonlinear PDE. The first, known as *Schauder's fixed-point theorem*, is an infinite dimensional extension of Brouwer's fixed-point theorem, which we recall.

Proposition B.1. *If K is a compact, convex set in a finite-dimensional vector space V, and $F : K \to K$ is a continuous map, then F has a fixed point.*

This was proved in § 19 of Chap. 1, specifically when K was the closed unit ball in $\mathbb{R}^n$. Now, given any compact convex $K \subset V$, if we translate it, we can assume $0 \in K$. Let W denote the smallest vector space in V that contains K; say $\dim_{\mathbb{R}} W = n$. Thus there is a basis of W, of the form $E \subset K$. Clearly, the convex hull of E has nonempty interior in W. From here, it is easily established that K is homeomorphic to the closed unit ball in $\mathbb{R}^n$.

A quicker reduction to the case of a ball goes like this. Put an inner product on V, and say a ball $B \subset V$ contains K. Let $\psi : B \to K$ map a point x to the point in K closest to x. Then consider a fixed point of $F \circ \psi : B \to K \subset B$.

The following is Schauder's generalization:

Theorem B.2. *If K is a compact, convex set in a Banach space V, and $F : K \to K$ is a continuous map, then F has a fixed point.*

Proof. Whether or not V has a countable dense set, K certainly does; say $\{v_j : j \in \mathbb{Z}^+\}$ is dense in K. For each $n \ge 1$, let V_n be the linear span of $\{v_1, \dots, v_n\}$ and $K_n \subset K$ the closed, convex hull of $\{v_1, \dots, v_n\}$. Thus K_n is a compact, convex subset of V_n, a linear space of dimension $\le n$.

We define continuous maps $Q_n : K \to K_n$ as follows. Cover K by balls of radius δ_n centered at the points $v_j,\ 1 \le j \le n$. Let $\{\varphi_{nj} : 1 \le j \le n\}$ be a partition of unity subordinate to this cover, satisfying $0 \le \varphi_j \le 1$. Then set

$$
Q_n(v) = \sum_{j=1}^{n} \varphi_{nj}(v)v_j, \quad Q_n : K \to K_n. \tag{B.1}
$$

Since $\varphi_{nj}(v) = 0$ unless $\|v - v_j\| \le \delta_n$, it follows that

$$\|Q_n(v) - v\| \le \delta_n. \tag{B.2}$$

The denseness of $\{v_j : j \in \mathbb{Z}^+\}$ in K implies we can take $\delta_n \to 0$ as $n \to \infty$.

Now consider the maps $F_n : K_n \to K_n$, given by $F_n = Q_n \circ F\big|_{K_n}$. By Proposition B.1, each F_n has a fixed point $x_n \in K_n$. Now

$$Q_n F(x_n) = x_n \Longrightarrow \|F(x_n) - x_n\| \le \delta_n. \tag{B.3}$$

Since K is compact, (x_n) has a limit point $x \in K$ and (B.3) implies $F(x) = x$, as desired.

It is easy to extend Theorem B.2 to the case where V is a Fréchet space, using a translation-invariant distance function. In fact, a theorem of Tychonov extends it to general locally convex V.

The following slight extension of Theorem B.2 is technically useful:

Corollary B.3. *Let E be a closed, convex set in a Banach space V, and let $F : E \to E$ be a continuous map such that $F(E)$ is relatively compact. Then F has a fixed point.*

Proof. The closed, convex hull K of $F(E)$ is compact; simply consider $F\big|_K$, which maps K to itself.

Corollary B.4. *Let B be the open unit ball in a Banach space V. Let $F : \overline{B} \to V$ be a continuous map such that $F(\overline{B})$ is relatively compact and $F(\partial B) \subset B$. Then F has a fixed point.*

Proof. Define a map $G : \overline{B} \to \overline{B}$ by

$$G(x) = F(x) \quad \text{if } \|F(x)\| \le 1, \quad G(x) = \frac{F(x)}{\|F(x)\|} \quad \text{if } \|F(x)\| \ge 1.$$

Then $G : \overline{B} \to \overline{B}$ is continuous and $G(\overline{B})$ is relatively compact. Corollary B.3 implies that G has a fixed point; $G(x) = x$. The hypothesis $F(\partial B) \subset B$ implies $\|x\| < 1$, so $F(x) = G(x) = x$.

The following Leray-Schauder theorem is the one we directly apply to such results as Theorem 1.10. The argument here follows [GT].

Theorem B.5. *Let V be a Banach space, and let $F : [0,1] \times V \to V$ be a continuous, compact map, such that $F(0, v) = v_0$ is independent of $v \in V$. Suppose there exists $M < \infty$ such that, for all $(\sigma, x) \in [0,1] \times V$,*

$$F(\sigma, x) = x \Longrightarrow \|x\| < M. \tag{B.4}$$

Then the map $F_1 : V \to V$ given by $F_1(v) = F(1, v)$ has a fixed point.

Proof. Without loss of generality, we can assume $v_0 = 0$ and $M = 1$. Let B be the open unit ball in V. Given $\varepsilon \in (0, 1]$, define $G_\varepsilon : \overline{B} \to V$ by

$$G_\varepsilon(x) = F\left(\frac{1 - \|x\|}{\varepsilon}, \frac{x}{\|x\|}\right) \quad \text{if} \quad 1 - \varepsilon \le \|x\| \le 1,$$
$$F\Big(1, \frac{x}{1 - \varepsilon}\Big) \qquad \text{if} \quad \|x\| \le 1 - \varepsilon.$$

Note that $G_\varepsilon(\partial B) = 0$. For each $\varepsilon \in (0, 1]$, Corollary B.4 applies to G_ε. Hence each G_ε has a fixed point $x(\varepsilon)$. Let $x_k = x(1/k)$, and set

$$\sigma_k = k\big(1 - \|x_k\|\big) \quad \text{if} \quad 1 - \frac{1}{k} \le \|x_k\| \le 1,$$
$$1 \qquad \text{if} \quad \|x_k\| \le 1 - \frac{1}{k},$$

so $\sigma_k \in (0, 1]$ and $F(\sigma_k, x_k) = x_k$. Passing to a subsequence, we have $(\sigma_k, x_k) \to (\sigma, x)$ in $[0, 1] \times \overline{B}$, since the map F is compact.

We claim $\sigma = 1$. Indeed, if $\sigma < 1$, then $\|x_k\| \ge 1 - 1/k$ for large k, hence $\|x\| = 1$ and $F(\sigma, x) = x$, contradicting (B.4) (with $M = 1$). Thus $\sigma_k \to 1$ and we have $F(1, x) = x$, as desired.

There are more general results, involving Leray-Schauder "degree theory," which can be found in [Schw, Ni6, Deim].

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15

Nonlinear Parabolic Equations

Introduction

We begin this chapter with some general results on the existence and regularity of solutions to semilinear parabolic PDE, first treating the pure initial-value problem in § 1, for PDE of the form

$$\frac{\partial u}{\partial t} = Lu + F(t, x, u, \nabla u), \quad u(0) = f, \tag{0.1}$$

where u is defined on $[0, T) \times M$, and M has no boundary. Some of the results established in § 1 will be useful in the next chapter, on nonlinear, hyperbolic equations. We also give a precursor to results on the global existence of weak solutions, which will be examined further in Chap. 17, in the context of the Navier–Stokes equations for fluids.

In § 2 we present a useful geometrical application of the theory of semilinear PDE, to the study of harmonic maps between compact Riemannian manifolds when the target space has negative curvature.

In § 3 we extend some of the results of § 1 to the case $\partial M \neq \emptyset$, when boundary conditions are placed on u. Section 4 is devoted to the study of reaction-diffusion equations, of the form

$$\frac{\partial u}{\partial t} = Lu + X(u), \tag{0.2}$$

where u takes values in $\mathbb{R}^\ell$ and X is a vector field on $\mathbb{R}^\ell$. Such systems arise in models of chemical reactions and in mathematical biology. One way to analyze the interplay of diffusion and the reaction due to $X(u)$ in (0.2) is via a nonlinear Trotter product formula, discussed in § 5.

In § 6 we examine a model for the melting of ice. The source of the nonlinearity in this problem is different from those considered in §§ 1–5; it is due to the equations specifying the interface where water meets ice, a "moving boundary."

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In §§7–9 we study quasi-linear parabolic PDE, beginning with fairly elementary results in § 7. The estimates established there need to be strengthened in order to be useful for global existence results. One stage of such strengthening is done in § 8, using the paradifferential operator calculus developed in § 10 of Chap. 13. We also include here some results on completely nonlinear parabolic equations and on quasi-linear systems that are "Petrowski-parabolic."

The next stage of strengthening consists of Nash–Moser estimates, carried out in § 9 and then applied to some global existence results. This theory mainly applies to scalar equations, but we also point out some $\ell \times \ell$ systems to which the Nash–Moser estimates can be applied, including some systems of reaction-diffusion equations in which there is nonlinear diffusion as well as nonlinear interaction.

1. Semilinear parabolic equations

In this section we look at equations of the form

$$\frac{\partial u}{\partial t} = Lu + F(t, x, u, \nabla u), \quad u(0) = f, \tag{1.1}$$

for $u(t, x)$, a function on $[0, T] \times M$. We assume M has no boundary; the case $\partial M \neq \emptyset$ will be treated in § 3. Generally, L will be a second-order, negative-semidefinite, elliptic differential operator (e.g., $L = \nu\Delta$), where Δ is the Laplace operator on a complete Riemannian manifold M and ν is a positive constant. We suppose F is C^∞ in its arguments.

We will begin with very general considerations, which often apply to an even more general class of linear operators L. For short, we suppress (t, x)-variables and set

$$\Phi(u) = F(u, \nabla u).$$

We convert (1.1) to the integral equation

$$u(t) = e^{tL} f + \int_0^t e^{(t-s)L} \Phi(u(s))\, ds = \Psi u(t). \tag{1.2}$$

We want to set up a Banach space $C([0, T], X)$ preserved by the map Ψ and establish that (1.2) has a solution via the contraction mapping principle. We assume that $f \in X$, a Banach space of functions, and that there is another Banach space Y such that the following four conditions hold:

$$e^{tL} : X \to X \text{ is a strongly continuous semigroup, for } t \geq 0, \tag{1.3}$$

$$\Phi : X \to Y \text{ is Lipschitz, uniformly on bounded sets,} \tag{1.4}$$

$$e^{tL} : Y \to X, \quad \text{for } t > 0, \tag{1.5}$$

and, for some $\gamma < 1$,

$$\|e^{tL}\|_{\mathcal{L}(Y,X)} \le C\, t^{-\gamma}, \quad \text{for } t \in (0,1]. \tag{1.6}$$

We will give a variety of examples later. Given these conditions, it is easy to see that Ψ acts on $C([0,T],X)$, for each $T > 0$. Fix $\alpha > 0$, and set

$$Z = \{u \in C([0,T],X) : u(0) = f,\ \|u(t) - f\|_X \le \alpha\}. \tag{1.7}$$

We want to pick T small enough that $\Psi : Z \to Z$ is a contraction. By (1.3), we can choose T_1 so that $\|e^{tL}f - f\|_X \le \alpha/2$ for $t \in [0,T_1]$. Now, if $u \in Z$, then, by (1.4), we have a bound $\|\Phi(u(s))\|_Y \le K_1$, for $s \in [0,T_1]$, so, using (1.6), we have

$$\Big\| \int_0^t e^{(t-s)L}\Phi\big(u(s)\big)\, ds \Big\|_X \le C_\gamma t^{1-\gamma} K_1. \tag{1.8}$$

If we pick $T_2 \le T_1$ small enough, this will be $\le \alpha/2$ for $t \in [0,T_2]$; hence $\Psi : Z \to Z$, provided $T \le T_2$.

To arrange that Ψ be a contraction, we again use (1.4) to obtain

$$\|\Phi\big(u(s)\big) - \Phi\big(v(s)\big)\|_Y \le K\|u(s) - v(s)\|_X,$$

for $u, v \in Z$. Hence, for $t \in [0,T_2]$,

$$\begin{aligned} \|\Psi(u)(t) - \Psi(v)(t)\|_X &= \Big\| \int_0^t e^{tL}\big[\Phi(u(s)) - \Phi(v(s))\big]\, ds \Big\|_X \\ &\le C_\gamma t^{1-\gamma} K \ \sup \|u(s) - v(s)\|_X; \end{aligned} \tag{1.9}$$

and now if $T \le T_2$ is chosen small enough, we have $C_\gamma T^{1-\gamma} K < 1$, making Ψ a contraction mapping on Z. Thus Ψ has a unique fixed point u in Z, solving (1.2). We have proved the following:

Proposition 1.1. *If X and Y are Banach spaces for which (1.3)–(1.6) hold, then the parabolic equation (1.1), with initial data $f \in X$, has a unique solution $u \in C([0,T],X)$, where $T > 0$ is estimable from below in terms of $\|f\|_X$.*

As an example, let M be a compact Riemannian manifold, and consider

$$X = C^1(M), \quad Y = C(M). \tag{1.10}$$

In this case we have the conditions (1.3)–(1.6) if $L = \Delta$. In particular,

$$\|e^{t\Delta}\|_{\mathcal{L}(C,C^1)} \le C\, t^{-1/2}, \quad \text{for } t \in (0,1]. \tag{1.11}$$

Thus we have short-time solutions to (1.1) with $f \in C^1(M)$.

It will be useful to weaken the hypothesis (1.3) a bit. Consider a pair of Banach spaces X and Z of functions, or distributions, on M, such that there are continuous inclusions

$$C_0^\infty(M) \subset X \subset Z \subset \mathcal{D}'(M).$$

We will say that a function $u(t)$ taking values in X, for $t \in I$, some interval in $\mathbb{R}$, belongs to $\mathcal{C}(I,X)$ provided $u(t)$ is locally bounded in X, and $u \in C(I,Z)$. More generally, this defines $\mathcal{C}(I,X)$ for any locally compact Hausdorff space I. Then we say e^{tL} is an almost continuous semigroup on X provided e^{tL} is uniformly bounded on X for $t \in [0,T]$, given $T < \infty$, $e^{(s+t)L}u = e^{sL}e^{tL}u$, for each $u \in X,\ s,t \in [0,\infty)$, and

$$u \in X \Longrightarrow e^{tL}u \in \mathcal{C}([0,\infty),X).$$

Examples include $e^{t\Delta}$ on $L^\infty(M)$ and on Hölder spaces $C^r(M),\ r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, when M is compact. The space $\mathcal{C}(I,X)$ may depend on the choice of Z, but we omit reference to Z in the notation. For example, when we consider $e^{t\Delta}$ on $L^\infty(M)$, with M compact, we might fix $p < \infty$ and take $Z = L^p(M)$.

The proof of Proposition 1.1 readily extends to the following variant:

Proposition 1.1A. *Let X and Y be Banach spaces for which (1.4)–(1.6) hold. In place of (1.3), we assume e^{tL} is an almost continuous semigroup on X. Also, we augment (1.4) with the condition that $\Phi : \mathcal{C}(I,X) \to \mathcal{C}(I,Y)$. Then the initial-value problem (1.1), given $f \in X$, has a unique solution $u \in \mathcal{C}([0,T],X)$, where $T > 0$ is estimable from below in terms of $\|f\|_X$.*

As examples, we can consider

$$X = C^{r+1}(M), \quad Y = C^r(M), \tag{1.12}$$

$r \geq 0$. If r is not an integer, these are Hölder spaces. We have, for any $s > 0$,

$$\|e^{t\Delta}\|_{\mathcal{L}(C^r, C^{r+s})} \leq C_s\, t^{-s/2}, \quad 0 < t \leq 1. \tag{1.13}$$

It follows from (1.2) that if $f \in C^{r+1}$ and one has a solution u in the space $\mathcal{C}([0,T],C^{r+1})$, then actually, for each $t > 0$, $u(t) \in C^{r+s}$ for every $s < 2$. We can iterate this argument repeatedly, and also, via the PDE (1.1), obtain the regularity of t-derivatives of u, proving:

Proposition 1.2. *Given $f \in C^1(M),\ L = \Delta$, the equation (1.1) has, for some $T > 0$, a unique solution*

$$u \in C\big([0,T],C^1(M)\big) \cap C^\infty\big((0,T] \times M\big). \tag{1.14}$$

A number of different pairs X and Y can be constructed; it is particularly of interest to have results for cases other than $X = C^1(M),\ Y = C(M)$, as these are often useful for establishing the existence of global solutions. When

(1.4) holds depends on the nature of the nonlinearity in (1.1). We list here some estimates that bear on when (1.6) holds, in case $L = \Delta$. The bound in the right column is on the operator norm over $0 < t \le 1$. In the cases listed here, we assume that $p \ge q$, and $s \ge r$.

(1.15)

Y	X	bound on $\|e^{t\Delta}\|_{\mathcal{L}(Y,X)}$
$L^q(M)$	$L^p(M)$	$C t^{-(n/2)(1/q-1/p)}$;
$H^{r,p}(M)$	$H^{s,p}(M)$	$C t^{-(1/2)(s-r)}$;
$H^{r,q}(M)$	$H^{s,p}(M)$	$C t^{-(n/2)(1/q-1/p)-(1/2)(s-r)}$;

We now take a look at the case $F(u, \nabla u) = \sum_j \partial_j F_j(u)$ of (1.1), with $L = \nu\Delta$; that is,

$$\frac{\partial u}{\partial t} = \nu \Delta u + \sum_j \partial_j F_j(u), \quad u(0) = f. \tag{1.16}$$

For simplicity, we take $M = \mathbb{T}^n$. The limiting case $\nu = 0$ of this, which we will consider in §5 of the next chapter, includes important cases of quasi-linear, hyperbolic equations. We will assume each F_j is smooth in its arguments (u can take values in $\mathbb{R}^K$) and satisfies estimates

$$|F_j(u)| \le C\langle u\rangle^p, \quad |\nabla F_j(u)| \le C\langle u\rangle^{p-1}, \tag{1.17}$$

for some $p \in [1, \infty)$. We will show that the Banach spaces

$$X = L^q(M), \quad Y = H^{-1,q/p}(M) \tag{1.18}$$

satisfy the conditions (1.3)–(1.6) for a certain range of q. First, we need $q \ge p$, so $q/p \ge 1$ in (1.18). Only (1.4) and (1.6) need to be investigated. For (1.4) we need $F_j : L^q \to L^{q/p}$ to be locally Lipschitz. To get this, write

$$\begin{aligned} F_j(u) - F_j(v) &= G_j(u,v)(u-v), \\ G_j(u,v) &= \int_0^1 F_j'\bigl(su + (1-s)v\bigr)\, ds. \end{aligned} \tag{1.19}$$

By (1.17), we have an estimate on $\|G_j(u,v)\|_{L^{q/(p-1)}}$, and, by the generalized Hölder inequality,

$$\|F_j(u) - F_j(v)\|_{L^{q/p}} \le \|G_j\|_{L^{q/(p-1)}} \|u - v\|_{L^q}, \tag{1.20}$$

so we have (1.4). To check (1.6), we use the third estimate in (1.15), to get

$$\|e^{t\Delta}\|_{\mathcal{L}(H^{-1,q/p}, L^q)} \le C\, t^{-(n/2)(p/q-1/q)-1/2}, \tag{1.21}$$

for $0 < t \le 1$, so we require $n(p-1)/q < 1$. Therefore, we have part of the following result:

Proposition 1.3. *Under the hypothesis (1.17), if $f \in L^q(M)$, the PDE (1.16) has a unique solution $u \in C([0,T], L^q(M))$, provided*

$$q \ge p \ \ and \ \ q > n(p-1). \tag{1.22}$$

Furthermore, $u \in C^\infty((0,T] \times M)$.

It remains to establish the smoothness. First, replacing L^q by L^{q_1} in (1.21), we see that, for any $t \in (0,T]$, $u(t) \in L^{q_1}$ for all $q_1 < q/(p-q/n)$. As $p-q/n < 1$, this means q_1 exceeds q by a factor > 1. Iterating this gives $u(t) \in L^{q_j}$, where q_j exceeds q_{j-1} by increasing factors. Once you have $q_j > np$, the next iteration gives $u(t) \in C^r(M)$, for some $r > 0$. Now, consider the spaces

$$X = C^r(M), \quad Y = H^{r-1-\varepsilon,q}(M), \tag{1.23}$$

where q is chosen very large, and $\varepsilon > 0$ very small. The fact that $u \mapsto F_j(u)$ is locally Lipschitz from $C^r(M)$ to $C^r(M)$, hence to $H^{r-\varepsilon,p}(M)$, gives (1.4) in this case, and estimates from the third line of (1.15), together with Sobolev imbedding theorems, give (1.6), and furthermore establish that actually, for each $t > 0$, $u(t) \in C^{r_1}(M)$, for $r_1 - r > 0$, estimable from below. Repeating this argument a finite number of times, we obtain $u(t) \in C^{r_j}(M)$, with $r_j > 1$. At this point, the regularity result of Proposition 1.2 applies.

We can now establish a global existence theorem for solutions to (1.16).

Proposition 1.4. *Suppose F_j satisfy (1.17) with $p = 1$. Then, given $f \in L^2(M)$, the equation (1.16) has a unique solution*

$$u \in C\big([0,\infty), L^2(M)\big) \cap C^\infty\big((0,\infty) \times M\big), \tag{1.24}$$

provided, when u takes values in $\mathbb{R}^K$, $F_j(u) = \big(F^1{}_j(u), \dots, F^K{}_j(u)\big)$, that

$$\frac{\partial F^k{}_j}{\partial u_i} = \frac{\partial F^i{}_j}{\partial u_k}, \quad 1 \le i,\ k \le K. \tag{1.25}$$

Proof. We have $u \in C([0,T], L^2) \cap C^\infty((0,T) \times M)$, since (1.22) holds with $q = 2$. To get global existence, it therefore suffices to bound $\|u(t)\|_{L^2}$; we prove this is nonincreasing. Indeed, for $t > 0$,

$$\begin{aligned} \frac{d}{dt}\, \|u(t)\|_{L^2}^2 &= 2\big(u(t), \sum \partial_j F_j(u(t))\big) - 2\nu \|\nabla u(t)\|_{L^2}^2 \\ &\le 2\big(u(t), \sum \partial_j F_j(u(t))\big). \end{aligned} \tag{1.26}$$

Now by (1.25) there exist smooth G_j such that $F^k{}_j = \partial G_j/\partial u_k$, and hence the right side of (1.26) is equal to

$$-2\sum\int \partial_j G_j(u)\,du = 0. \tag{1.27}$$

The proof is complete.

The hypothesis (1.25) implies that the $\nu = 0$ analogue of (1.16) is a symmetric hyperbolic system, as will be seen in the next chapter.

The condition $p = 1$ for (1.17) is rather restrictive. In the case of a *scalar* equation, we can eliminate this restriction, at least for bounded initial data, obtaining the following important existence theorem.

Proposition 1.5. *If (1.16) is scalar and $f \in L^\infty(M)$, then there is a unique solution*

$$u \in L^\infty([0,\infty)\times M)\cap C^\infty((0,\infty)\times M),$$

such that, as $t \searrow 0$, $u(t) \to f$ in $L^p(M)$ for all $p < \infty$.

Proof. Suppose $\|f\|_{L^\infty} \le M$. Alter $F_j(u)$ on $|u| \ge M + 1/2$, obtaining $\tilde{F}_j(u)$, constant on $u \le -M-1$ and on $u \ge M+1$. Then Proposition 1.4 yields a global solution u to the modified PDE. This u solves

$$\frac{\partial u}{\partial t} = \nu\Delta u + \sum a_j(t,x)\partial_j u, \quad a_j(t,x) = \tilde{F}_j'(u(t,x)), \tag{1.28}$$

so the maximum principle for linear parabolic equations applies; $\|u(t)\|_{L^\infty}$ is nonincreasing. Thus $\|u(t)\|_{L^\infty} \le M$ for all t, and hence u solves the original PDE.

The solution operator produced from Proposition 1.5 has an important L^1-contractive property, which will be useful for passing to the $\nu = 0$ limit in § 6 of the next chapter. We present an elegant demonstration from [Ho].

Proposition 1.6. *Let u_j be solutions to the equation (1.16) in the scalar case, with initial data $u_j(0) = f_j \in L^\infty(M)$. Then, for each $t > 0$,*

$$\|u_1(t) - u_2(t)\|_{L^1(M)} \le \|f_1 - f_2\|_{L^1(M)}. \tag{1.29}$$

Proof. Set $v = u_1 - u_2$. Then v solves

$$\frac{\partial v}{\partial t} = \nu\Delta v + \sum \partial_j\big[\Phi_j(u_1,u_2)v\big], \tag{1.30}$$

with

$$\Phi_j(u_1,u_2) = \int_0^1 F_j'\big(su_1 + (1-s)u_2\big)\,ds,$$

so $F_j(u_1)-F_j(u_2) = \Phi_j(u_1,u_2)(u_1-u_2)$. Set $G_j(t,x) = \Phi_j(u_1,u_2)$. Now, for given $T > 0$, let w solve the backward evolution equation

$$\frac{\partial w}{\partial t} = -\nu\Delta w + \sum G_j(t,x)\partial_j w, \quad w(T) = w_0 \in C^\infty(M). \tag{1.31}$$

Then $w(t)$ is well defined for $t \le T$, and the maximum principle yields

$$\|w(t)\|_{L^\infty} \le \|w_0\|_{L^\infty}, \quad \text{for } t \le T. \tag{1.32}$$

Note that $\|v(T)\|_{L^1}$ is the sup of $(v(T), w_0)$ over $\|w_0\|_{L^\infty} \le 1$. Now, for $t \in (0,T)$, we have

$$\begin{aligned}\frac{d}{dt}(v,w) &= (\nu\Delta v, w) + \sum\bigl(\partial_j(G_j v), w\bigr) \\ &\quad - (v, \nu\Delta w) + \sum(v, G_j\partial_j w) = 0.\end{aligned} \tag{1.33}$$

Since $\bigl(v(0), w(0)\bigr) \le \|v(0)\|_{L^1}\|w(0)\|_{L^\infty}$, this proves (1.29).

We next produce global *weak* solutions to (1.16), for $K \times K$ systems, with the symmetry hypothesis (1.25), in case (1.17) holds with $p = 2$. As before, we take $M = \mathbb{T}^n$. We will use a version of what is sometimes called a Galerkin method to produce a sequence of approximations, converging to a solution to (1.16).

Give $\varepsilon > 0$, define the projection P_ε on $L^2(M)$ by

$$P_\varepsilon f(x) = \sum_{|k|\le 1/\varepsilon} \hat{f}(k)e^{ik\cdot x},$$

where, for $k \in \mathbb{Z}^n$, $\hat{f}(k)$ form the Fourier coefficients of f. Consider the initial-value problem

$$\frac{\partial u_\varepsilon}{\partial t} = \nu P_\varepsilon\Delta P_\varepsilon u_\varepsilon + P_\varepsilon\sum\partial_j F_j(P_\varepsilon u_\varepsilon), \quad u_\varepsilon(0) = P_\varepsilon f. \tag{1.34}$$

We take $f \in L^2(M)$. For each $\varepsilon \in (0,1]$, ODE theory gives a unique short-time solution, satisfying $u_\varepsilon(t) = P_\varepsilon u_\varepsilon(t)$. Furthermore,

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{L^2}^2 = 2\nu(P_\varepsilon\Delta P_\varepsilon u_\varepsilon, u_\varepsilon) + 2\sum\bigl(P_\varepsilon\partial_j F_j(P_\varepsilon u_\varepsilon), u_\varepsilon\bigr). \tag{1.35}$$

The first term on the right is $-2\nu\|\nabla P_\varepsilon u_\varepsilon(t)\|_{L^2}^2 \le 0$. The last term is equal to

$$\begin{aligned} 2\sum\big(\partial_j F_j(P_\varepsilon u_\varepsilon), P_\varepsilon u_\varepsilon\big) &= -2\sum\big(F_j(P_\varepsilon u_\varepsilon), \partial_j P_\varepsilon u_\varepsilon\big) \\ &= -2\sum\int \partial_j\big[G_j(P_\varepsilon u_\varepsilon)\big]\,dx \\ &= 0, \end{aligned} \tag{1.36}$$

where G_j is as in (1.27). We deduce that

$$\|u_\varepsilon(t)\|_{L^2} \le \|f\|_{L^2}. \tag{1.37}$$

Hence, for each $\varepsilon > 0$, (1.34) is solvable for all $t > 0$, and

$$\{u_\varepsilon : \varepsilon \in (0,1]\} \text{ is bounded in } L^\infty(\mathbb{R}^+, L^2(M)). \tag{1.38}$$

Note that further use of (1.35)–(1.37) gives

$$2\nu\int_0^T \|\nabla P_\varepsilon u_\varepsilon(t)\|_{L^2}^2\,dt = \|P_\varepsilon f\|^2 - \|u_\varepsilon(T)\|_{L^2}^2, \tag{1.39}$$

for any $T \in (0,\infty)$. Hence, for each bounded interval $I = [0,T]$, since $P_\varepsilon u_\varepsilon = u_\varepsilon$,

$$\{u_\varepsilon\} \text{ is bounded in } L^2(I, H^1(M)). \tag{1.40}$$

Given that $|F_j(u)| \le C\langle u\rangle^2$, it follows from (1.38) that

$$\begin{aligned} \{F_j(P_\varepsilon u_\varepsilon)\} \ \text{ is bounded in } \ & L^\infty(\mathbb{R}^+, L^1(M)) \\ & \subset L^\infty(\mathbb{R}^+, H^{-n/2-\delta}(M)), \end{aligned} \tag{1.41}$$

for each $\delta > 0$. Now using the evolution equation (1.34) for $\partial u_\varepsilon/\partial t$ and (1.40)–(1.41), we conclude that

$$\left\{\frac{\partial u_\varepsilon}{\partial t}\right\} \text{ is bounded in } L^2(I, H^{-n/2-1-\delta}(M)), \tag{1.42}$$

hence

$$\{u_\varepsilon\} \text{ bounded in } H^1(I, H^{-n/2-1-\delta}(M)). \tag{1.43}$$

Now we can interpolate between (1.40) and (1.43) to obtain

$$\{u_\varepsilon\} \text{ bounded in } H^s(I, H^{1-s(n/2+1+\delta)}(M)), \tag{1.44}$$

for each $s \in [0, 1]$. Now if we pick $s > 0$ very small and apply Rellich's theorem, we deduce that

$$\{u_\varepsilon : 0 < \varepsilon \leq 1\} \text{ is compact in } L^2(I, H^{1-\gamma}(M)), \tag{1.45}$$

for all $\gamma > 0$.

The rest of the argument is easy. Given $T < \infty$, we can pick a sequence $u_k = u_{\varepsilon_k},\ \varepsilon_k \to 0$, such that

$$u_k \to u \ \text{ in } L^2([0,T], H^{1-\gamma}(M)), \ \text{ in norm.} \tag{1.46}$$

We can arrange that this hold for all $T < \infty$, by a diagonal argument. We can also assume that u_k is weakly convergent in each space specified in (1.38) and (1.40), and that $\partial u_k/\partial t$ is weakly convergent in the space given in (1.42). From (1.46) we deduce

$$F_j(P_{\varepsilon_k} u_{\varepsilon_k}) \to F_j(u) \ \text{ in } L^1([0,T], L^1(M)), \ \text{ in norm,} \tag{1.47}$$

as $k \to \infty$, hence

$$\partial_j F_j(P_{\varepsilon_k} u_{\varepsilon_k}) \to \partial_j F_j(u) \ \text{ in } L^1([0,T], H^{-1,1}(M)). \tag{1.48}$$

Using $H^{-1,1}(M) \subset H^{-n/2-1-\delta}(M)$, we see that each term in (1.34) converges as $\varepsilon_k \to 0$. We have proved the following:

Proposition 1.7. *If $|F_j(u)| \leq C\langle u\rangle^2$ and $|\nabla F_j(u)| \leq C\langle u\rangle$, then, for each $f \in L^2(M)$, a $K \times K$ system of the form (1.16), satisfying the symmetry hypothesis (1.25), possesses a global weak solution*

$$\begin{aligned} u \in L^\infty(\mathbb{R}^+, L^2(M)) &\cap L^2_{loc}(\mathbb{R}^+, H^1(M)) \\ &\cap Lip_{loc}\big(\mathbb{R}^+, H^{-2}(M) + H^{-n/2-1-\delta}(M)\big). \end{aligned} \tag{1.49}$$

When reading the discussion of the Navier–Stokes equations in Chap. 17, one will note a similar argument establishing a classical result of Hopf on global weak solutions to that system.

Exercises

1. Verify the estimates on operator norms of $e^{t\Delta} : Y \to X$ listed in (1.15).
2. Show that, given $f \in C^1(M),\ M = \mathbb{T}^n$, the solution to

$$\frac{\partial u}{\partial t} = \nu\Delta u + F(t, u, \nabla u), \quad u(0) = f,$$

   continues to exist as long as $\|u(t)\|_{L^\infty}$ does not blow up, provided this equation is *scalar* and $F_u \leq 0$. (*Hint*: Derive a PDE for $u_j = \partial u/\partial x_j$ and apply the maximum principle.)

3. Generalize the treatment of PDE (1.16), done above in the case $M = \mathbb{T}^n$, to the following situation for a general compact Riemannian manifold M:

$$\frac{\partial u}{\partial t} = \nu \Delta u + \operatorname{div} F(u),$$

where
(a) u is scalar and $F(u) = F(x,u) \in T_x M$, for $x \in M,\ u \in \mathbb{R}$,
(b) u is a vector field and $F(u) = F(x,u) \in \otimes^2 T_x M$, for $x \in M,\ u \in T_x M$.
Consider other generalizations.

4. More generally, extend the treatment of (1.16) to

$$\frac{\partial u}{\partial t} = \nu \Delta u + \mathcal{D} F(u), \quad u(0) = f,$$

where $\mathcal{D}$ is a first-order differential operator on the compact Riemannian manifold M, and F satisfies the estimates (1.17). What additional properties must $\mathcal{D}$ have for Proposition 1.4 to extend?

5. For given $\nu,\ \varepsilon > 0,\ k \in \mathbb{Z}^+$, consider the $(4k)$-order operator

$$L = \nu \Delta - \varepsilon \Delta^{2k}.$$

Show that $\|e^{tL}\|_{\mathcal{L}(Y,X)}$ satisfies estimates of the form (1.15), where, in the right column, one replaces $t^{-\gamma}$ by $t^{-\gamma/2k}$, and C depends on ν and ε.

6. Consider the PDE

$$\frac{\partial u}{\partial t} = \nu \Delta u - \varepsilon \Delta^{2k} u + \sum \partial_j F_j(u), \quad u(0) = f. \tag{1.50}$$

Suppose F_j satisfies (1.17), that is,

$$|F_j(u)| \le C \langle u \rangle^p, \quad |\nabla F_j(u)| \le C \langle u \rangle^{p-1}. \tag{1.51}$$

Show that there is a unique local solution

$$u \in C([0,T], L^q(M)) \cap C^\infty((0,T] \times M)$$

given $f \in L^q(M)$, provided

$$q \ge p \ \text{ and } \ q > \frac{n(p-1)}{4k-1}.$$

7. Suppose that (1.51) holds with $p = 2$ and that $\dim M = n < 8k - 2$. Suppose also that the symmetry hypothesis (1.25) holds (for a $K \times K$ system), so $F^k{}_j(u) = \partial G_j / \partial u_k$. Given $f \in L^2(M)$, show that (1.50) has a unique global solution $u \in C([0,\infty), L^2(M)) \cap C^\infty((0,\infty) \times M)$, and $\|u(t)\|_{L^2} \le \|f\|_{L^2}^2$, for $t > 0$.

8. Let $u = u_\varepsilon$ be the solution to (1.50) under the hypotheses of Exercise 7. We take $\nu > 0$ fixed and let $\varepsilon \searrow 0$. Obtain bounds on $\{u_\varepsilon : \varepsilon \in (0,1]\}$ which imply that a subsequence converges to a weak solution u_0 of the $\varepsilon = 0$ case of (1.50), thus providing another proof of Proposition 1.7. (*Hint*: Start with the following analogue of (1.39):

$$2\nu \int_0^T \|\nabla u_\varepsilon(t)\|_{L^2}^2 \, dt + 2\varepsilon \int_0^T \|\Delta^k u_\varepsilon(t)\|_{L^2}^2 \, dt = \|f\|_{L^2}^2 - \|u_\varepsilon(T)\|_{L^2}^2.)$$

9. Let u be a smooth solution for $0 < t < T$ of the system (1.16), under the hypothesis (1.25) of Proposition 1.4, namely,

$$\frac{\partial u}{\partial t} = \nu\Delta u + \sum \partial_j F_j(u), \quad \partial_{u_i} F^k{}_j = \partial_{u_k} F^i{}_j, \quad u(0) = f. \tag{1.52}$$

Thus $F^k{}_j = \partial_{u_k} G_j$. Show that

$$\begin{aligned}\frac{d}{dt}\|\nabla_x u(t)\|_{L^2}^2 &= -2\nu\|\Delta u\|_{L^2}^2 - 2\sum \int \Bigl(\partial_{u_k}\partial_{u_i} G_j(u)\Bigr)\, \partial_j u_k\, \partial_\ell^2 u_i\, dx \\ &\le -2\nu\|\Delta u\|_{L^2}^2 + \frac{1}{\nu}\Phi\bigl(\|u(t)\|_{L^\infty}\bigr)^2 \|\nabla_x u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2,\end{aligned}$$

where

$$\Phi(M) = \sup_{|u|\le M} \|\partial_{u_i}\partial_{u_k} G_j(u)\| = \sup_{|u|\le M} \|\partial_{u_k} F_j(u)\|. \tag{1.53}$$

Integrating this and using the estimate on $\nu\int_0^t \|\nabla_x u(\tau)\|_{L^2}^2\, d\tau$ that follows from (1.26)–(1.27), deduce that, for $0 < t < T$,

$$\|\nabla_x u(t)\|_{L^2}^2 + \nu\int_0^t \|\Delta u(\tau)\|_{L^2}^2\, d\tau \le \|\nabla_x f\|_{L^2}^2 + \nu^{-2}\Psi(u,t)^2\|f\|_{L^2}^2, \tag{1.54}$$

where $\Psi(u,t) = \Phi(M)$, with $M = \sup\{\|u(\tau)\|_{L^\infty} : 0 \le \tau \le t\}$.

10. In the context of Exercise 9, suppose that the space dimension is $n = 1$. Note that in this case, $\|u(t)\|_{L^\infty}^2 \le C\|\nabla_x u(t)\|_{L^2}\|u(t)\|_{L^2} + C\|u(t)\|_{L^2}^2$. Show that under the hypothesis
$$\Phi(M)M^{-2} \longrightarrow 0, \ \text{ as } M \to +\infty,$$
we have a bound on $\|u(t)\|_{H^1}$ as $t \nearrow T$, and hence a global existence result for (1.52). Compare with [Smo], p. 427.

11. Solve the system

$$\begin{aligned} u_t = u_{xx} + u(u_x^2 + v_x^2), &\quad v_t = v_{xx} + v(u_x^2 + v_x^2), \\ u(0,x) = A\cos kx, &\quad v(0,x) = A\sin kx, \end{aligned} \tag{1.55}$$

with $A > 1$. Here, $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Show that the maximal t-interval of existence in $\mathbb{R}^+$ is $[0, C(A)/k^2)$. This example is given in [LSU] and attributed to E. Heinz.

12. Consider the multidimensional "Burger's equation"

$$u_t + \nabla_u u = \Delta u, \quad u(0,x) = f(x), \tag{1.56}$$

for $u(t,x) : \mathbb{R}^+ \times \mathbb{T}^n \to \mathbb{R}^n$. Show that, for each $t > 0$,

$$\sup_{x\in\mathbb{T}^n} |u_j(t,x)| \le \sup_{x\in\mathbb{T}^n} |f_j(x)|, \quad 1 \le j \le n.$$

Deduce that (1.56) has a global solution. (*Hint*: Show that

$$\frac{d}{dt}\|u(t)\|_{H^k}^2 \le C\sum_{j,\ell}\sum_{|\alpha|+|\beta|\le k} \|(D^\alpha u_j)(D^\beta u_\ell)\|_{L^2}\|u\|_{H^{k+1}} - 2\|\nabla u\|_{H^k}^2,$$

and use Proposition 3.6 of Chap. 13 to estimate $\|(D^\alpha u_j)(D^\beta u_\ell)\|_{L^2}$.) Note that the case $n = 1$ is also treated by Proposition 1.5.

2. Applications to harmonic maps

Let M and N be compact Riemannian manifolds. Using Nash's result, proved in §5 of Chap. 14, we take N to be isometrically imbedded in some Euclidean space; $N \subset \mathbb{R}^k$. A harmonic map $u : M \to N$ is a critical point for the energy functional

$$E(u) = \frac{1}{2} \int_M |\nabla u(x)|^2 \, dV(x), \tag{2.1}$$

among all such maps. In the integrand, we use the natural square norm on $T_x^* M \otimes T_{u(x)} N \subset T_x^* M \otimes \mathbb{R}^k$. The quantity (2.1) clearly depends only on the metrics on M and N, not on the choice of isometric imbedding of N into Euclidean space.

If u_s is a smooth family of maps from M to N, then

$$\frac{d}{ds} \, E(u_s)\big|_{s=0} = -\int v(x) \Delta u(x) \, dV, \tag{2.2}$$

where $u = u_0$, and $v(x) = (\partial/\partial s) u_s(x) \in T_{u(x)} N$. One can vary u_0 so that v is any map $M \to \mathbb{R}^k$ such that $v(x) \in T_{u(x)} N$, so the stationary condition is that

$$\Delta u(x) \perp T_{u(x)} N, \quad \text{for all } x \in M. \tag{2.3}$$

We can rewrite the stationary condition (2.3) by a process similar to that used in (11.12)–(11.14) in Chap. 1. Suppose that, near a point $z \in N \subset \mathbb{R}^k$, N is given by

$$f_\ell(y) = 0, \quad 1 \le \ell \le L, \tag{2.4}$$

where $L = k - \dim N$, with $\nabla f_\ell(y)$ linearly independent in $\mathbb{R}^k$, for each y near z. If $u : M \to N$ is smooth and $u(x)$ is close to z, then we have

$$\sum_\nu \frac{\partial f_\ell}{\partial u_\nu} \frac{\partial u_\nu}{\partial x_j} = 0, \quad 1 \le \ell \le L, \ 1 \le j \le m, \tag{2.5}$$

where $(x_1, \dots, x_m)$ is a local coordinate system on M. Hence

$$\sum_\nu \frac{\partial f_\ell}{\partial u_\nu} \Delta u_\nu = - \sum_{\mu,\nu,j,k} g^{jk} \frac{\partial^2 f_\ell}{\partial u_\mu \partial u_\nu} \frac{\partial u_\mu}{\partial x_k} \frac{\partial u_\nu}{\partial x_j}. \tag{2.6}$$

Since $\{\nabla_y f_\ell(y) : 1 \le \ell \le L\}$ is a basis of the orthogonal complement in $\mathbb{R}^k$ of $T_y N$, it follows that, for smooth $u : M \to N$, the normal component of Δu depends only on the first-order derivatives of u, and is quadratic in ∇u; that is, we have a formula

$$(\Delta u)^N = \Gamma(u)(\nabla u, \nabla u). \tag{2.7}$$

Thus the stationary condition (2.3) for u is equivalent to

$$\Delta u - \Gamma(u)(\nabla u, \nabla u) = 0. \tag{2.8}$$

Denote the left side of (2.8) by $\tau(u)$; it follows from (2.7) that, given $u \in C^2(M, N)$, $\tau(u)$ is tangent to N at $u(x)$.

J. Eells and J. Sampson [ES] proved the following result.

Theorem 2.1. *Suppose N has negative sectional curvature everywhere. Then, given $v \in C^\infty(M, N)$, there exists a harmonic map $w \in C^\infty(M, N)$ which is homotopic to v.*

As in [ES], the existence of w will be established via solving the PDE

$$\frac{\partial u}{\partial t} = \Delta u - \Gamma(u)(\nabla u, \nabla u), \quad u(0) = v. \tag{2.9}$$

It will be shown that under the hypothesis of negative sectional curvature on N, there is a smooth solution to (2.9) for all $t \geq 0$ and that, for a sequence $t_k \to \infty$, $u(t_k)$ tends to the desired w. In outline, our treatment follows that presented in [J2], with some simplifications arising from taking N to be imbedded in $\mathbb{R}^k$ (as in [Str]), and also some simplifications in the use of parabolic theory.

The local solvability of (2.9) follows directly from Proposition 1.2. Since $\tau(u)$ is tangent to N for $u \in C^\infty(M, N)$, it follows that $u(t) : M \to N$ for each t in the interval $[0, T)$ on which the solution to (2.9) exists. To get global existence for (2.9), it suffices to estimate $\|u(t)\|_{C^1}$.

In order to estimate $\nabla_x u$, we use a differential inequality for the energy density

$$e(t, x) = \frac{1}{2}|\nabla_x u(t, x)|^2. \tag{2.10}$$

In fact, there is the identity

$$\begin{aligned} \frac{\partial e}{\partial t} - \Delta e = &- |{}^N\nabla^2 u|^2 - \frac{1}{2}\langle du \cdot Ric^M(e_j), du \cdot e_j\rangle \\ &+ \frac{1}{2}\langle R^N(du \cdot e_j, du \cdot e_k)du \cdot e_k, du \cdot e_j\rangle, \end{aligned} \tag{2.11}$$

where $\{e_j\}$ is an orthonormal frame at $T_x M$ and we sum over repeated indices. The operator ${}^N\nabla^2$ is obtained from the second covariant derivative:

$${}^N\nabla^2 u(x) : \otimes^2 T_x M \longrightarrow T_{u(x)} N.$$

See the exercises for a derivation of (2.11).

Given that N has negative sectional curvature, (2.11) implies the inequality

$$\frac{\partial e}{\partial t} - \Delta e \leq ce. \tag{2.12}$$

If $f(t,x) = e^{-ct} e(t,x)$, we have $\partial f/\partial t - \Delta f \leq 0$, and the maximum principle yields $f(t,x) \leq \|f(0,\cdot)\|_{L^\infty}$, hence

$$e(t,x) \leq e^{ct} \|\nabla v\|^2_{L^\infty}. \tag{2.13}$$

This C^1-estimate implies the global existence of a solution to (2.9), by Proposition 1.2.

For the rest of Theorem 2.1, we need further bounds on u, including an improvement of (2.13). For the total energy

$$E(t) = \int\limits_M e(t,x)\, dV(x) = \frac{1}{2} \int\limits_M |\nabla u|^2\, dV(x), \tag{2.14}$$

we claim there is the identity

$$E'(t) = -\int\limits_M |u_t|^2\, dV(x). \tag{2.15}$$

Indeed, one easily obtains $E'(t) = -\int \langle u_t, \Delta u\rangle\, dV(x)$. Then replace Δu by $u_t + \Gamma(u)(\nabla u, \nabla u)$. Since u_t is tangent to N and $\Gamma(u)(\nabla u, \nabla u)$ is normal to N, (2.15) follows. The desired improvement of (2.13) will be a consequence of the following estimate:

Lemma 2.2. *Let $e(t,x) \geq 0$ satisfy the differential inequality (2.12). Assume that*

$$E(t) = \int e(t,x)\, dV(x) \leq E_0$$

is bounded. Then there is a uniform estimate

$$e(t,x) \leq e^c\ K\ E_0, \quad t \geq 1, \tag{2.16}$$

where K depends only on the geometry of M.

Proof. Writing $\partial e/\partial t - \Delta e = ce - g,\ g(t,x) \geq 0$, we have, for $0 \leq s \leq 1$,

$$\begin{aligned} e(t+s,x) &= e^{s(\Delta+c)} e(t,x) - \int_0^s e^{(s-\tau)(\Delta+c)} g(\tau,x)\, d\tau \\ &\leq e^{s(\Delta+c)} e(t,x). \end{aligned} \tag{2.17}$$

Since $e^{s(\Delta+c)}$ is uniformly bounded from $L^1(M)$ to $L^\infty(M)$ for $s \in [1/2, 1]$, the bound (2.16) for $t \in [1/2, \infty)$ follows from the hypothesized L^1-bound on $e(t)$.

We remark that a more elaborate argument, which can be found on pp. 84–86 of [J2], yields an explicit bound K depending on the injectivity radius of M and the first (nonzero) eigenvalue of the Laplace operator on M.

Since Lemma 2.2 applies to $e(t, x) = |\nabla u|^2$ when u solves (2.9), we see that solutions to (2.9) satisfy

$$\|u(t)\|_{C^1} \le K_1 \|v\|_{C^1}, \quad \text{for all } t \ge 0. \tag{2.18}$$

Hence, by the regularity estimate in Proposition 1.2, there are uniform bounds

$$\|u(t)\|_{C^\ell} \le K_\ell \|v\|_{C^1}, \quad t \ge 1, \tag{2.19}$$

for each $\ell < \infty$. Of course there are consequently also uniform Sobolev bounds.

Now, by (2.15), $E(t)$ is positive and monotone decreasing as $t \nearrow \infty$. Thus the quantity $\int_M |u_t(t, x)|^2 \, dV(x)$ is an integrable function of t, so there exists a sequence $t_j \to \infty$ such that

$$\|u_t(t_j, \cdot)\|_{L^2} \to 0. \tag{2.20}$$

From (2.19) and the PDE (2.9), we have bounds

$$\|u_t(t, \cdot)\|_{H^k} \le C_k,$$

and interpolation with (2.20) then gives, for any $\ell \in \mathbb{Z}^+$,

$$\|u_t(t_j, \cdot)\|_{H^\ell} \to 0. \tag{2.21}$$

Therefore, by the PDE (2.9), one has for $u_j(x) = u(t_j, x)$,

$$\Delta u_j - \Gamma(u_j)(\nabla u_j, \nabla u_j) \to 0 \quad \text{in } H^\ell(M), \tag{2.22}$$

as well as a uniform bound from (2.19). It easily follows that a subsequence converges in a strong norm to an element $w \in C^\infty(M, N)$ solving (2.8) and homotopic to v, which completes the proof of Theorem 2.1.

We next show that there is an *energy-minimizing* harmonic map $w : M \to N$ within each homotopy class when N has negative sectional curvature.

Proposition 2.3. *Under the hypotheses of Theorem 2.1, if we are given $v \in C^\infty(M, N)$, then there is a smooth map $w : M \to N$ that is harmonic, and homotopic to v, and such that $E(w) \le E(\tilde{v})$ for any $\tilde{v} \in C^\infty(M, N)$ homotopic to v.*

Proof. If α is the infimum of the energies of smooth maps homotopic to v, pick v_ν, homotopic to v, such that $E(v_\nu) \searrow \alpha$. Then solve (2.9), for u_ν, with initial data $u_\nu(0) = v_\nu$. We have some sequence $u_\nu(t_{\nu j}) \to w_\nu \in C^\infty(M, N)$, harmonic. The proof of Theorem 2.1 gives $E(w_\nu) \le E(v_\nu)$, hence $E(w_\nu) \to \alpha$. Also, via (2.16) and (2.19), we have uniform C^ℓ-bounds on w_ν, for all ℓ. Thus $\{w_\nu\}$ has a limit point w with the desired properties.

We record a local existence result for parabolic equations with a structure like that of (2.9), with initial data less smooth than C^1. Thus we look at equations of the form (1.1), with

$$F(x, D^1_x u) = B(u)(\nabla u, \nabla u), \tag{2.23}$$

a quadratic form in ∇u. In this case, we take

$$X = H^{1,p}, \quad Y = L^q, \quad q = \frac{p}{2}, \quad p > n, \tag{2.24}$$

and verify the conditions (1.3)–(1.6), using the Sobolev imbedding result

$$H^{s,p} \subset L^{np/(n-sp)}, \ p < \frac{n}{s}.$$

This yields the following:

Proposition 2.4. *If (2.23) is a quadratic form in ∇u, then the PDE*

$$\frac{\partial u}{\partial t} = \Delta u + B(u)(\nabla u, \nabla u), \quad u(0) = f, \tag{2.25}$$

has a solution in $C([0, T], H^{1,p}) \cap C^\infty((0, T) \times M)$, provided $f \in H^{1,p}(M)$, $p > n$.

The smoothness is established by the same sort of arguments as described before. Of course, the proof of Proposition 2.4 yields persistence of solutions as long as $\|u(t)\|_{H^{1,p}}$ is bounded for some $p > n$.

We mention further results on harmonic maps. First, in the setting of Theorem 2.1, that is, when N has negative sectional curvature, any harmonic map is energy minimizing in its homotopy class, a fact that makes Proposition 2.3 superfluous. An elegant proof of this fact can be found in [Sch]. It is followed by a proof of a uniqueness result of P. Hartman, which says that under the hypotheses of Theorem 2.1, any two homotopic harmonic maps coincide, unless both have rank ≤ 1.

Theorem 2.1 does not extend to arbitrary N. For example, it was established by Eells and Wood that if $v \in C^\infty(\mathbb{T}^2, S^2)$ has degree 1, then v is *not* homotopic to a harmonic map. Among positive results not contained in Theorem 2.1, we mention a result of Lemaire and Sacks–Uhlenbeck that if $\pi_2(N) = 0$ and dim

$M = 2$, then any $v \in C^\infty(M, N)$ is homotopic to a smooth harmonic map. If $\dim M \geq 3$, there are nonsmooth harmonic maps, and there has been considerable work on the nature of possible singularities. Details on matters mentioned in this paragraph, and further references, can be found in [Hild, J1, Str, Str2]. We also refer to [Ham] for extensions of Theorem 2.1 to cases where M and N have boundary.

In case M and N are compact Riemann surfaces of genus ≥ 2 (endowed with metrics of negative curvature, as done in § 2 of Chap. 14), harmonic maps of degree 1 are unique and are diffeomorphisms, as shown by R. Schoen and S.-T. Yau. They measure well the degree to which M and N may fail to be conformally equivalent, and they provide an excellent analytical tool for the study of Teichmuller theory, replacing the more classical use of "quasi-conformal maps." This material is treated in [Tro].

We mention some other important geometrical results attacked via parabolic equations. R. Hamilton [Ham2] obtained topological information on 3-manifolds with positive Ricci curvature and in [Ham3] provided another approach to the uniformization theorem for surfaces, an approach that works for the sphere as well as for surfaces of higher genus; see also [Chow]. S. Donaldson [Don] constructed Hermitian–Einstein metrics on stable bundles over compact algebraic surfaces; see [Siu] for an exposition. Some facets of the Yamabe problem were treated via the "Yamabe flow" in [Ye].

Hamilton's Ricci flow equation

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g)$$

is a degenerate parabolic equation, but D. DeTurk [DeT] produced a strongly parabolic modification, which fits into the framework of § 7 of this chapter, giving short time solutions. Solutions typically develop singularities, and there has been a lot of work on their behavior. Work of G. Perelman, [Per1]–[Per3], was a tremendous breakthrough, greatly refining understanding of the Ricci flow and using this to prove the Poincaré Conjecture and Thurston's Geometrization Conjecture, for compact 3-dimensional manifolds. This work has generated a large additional body of work, quite a bit of it devoted to giving more digestible presentations of Perelman's work. We refer to [CZ] and [MT] for such presentations, and other references.

Exercises

For Exercises 1–3, choose local coordinates x near a point $p \in M$ and local coordinates y near $q = u(p) \in N$. Then the energy density is given by

$$e(t, x) = \frac{1}{2} \frac{\partial u_\nu}{\partial x_k} \frac{\partial u_\mu}{\partial x_\ell} g^{k\ell}(x) h_{\mu\nu}(u(t, x)), \tag{2.26}$$

where $u(x) = \big(u_1(x), \cdots, u_n(x)\big)$ in the y-coordinate system, $n = \dim N$. Here, $g_{k\ell}$ and $h_{\mu\nu}$ define the metrics on M and N, respectively, and we use the summation convention, here and below. Assume the coordinate systems are normal at p and q, respectively.

1. Using these coordinate systems, show that the PDE (2.9) takes the form

$$\frac{\partial u_\nu}{\partial t} = g^{k\ell} \frac{\partial^2 u_\nu}{\partial x_k \partial x_\ell} - g^{k\ell}\, {}^M\Gamma^j{}_{k\ell} \frac{\partial u_\nu}{\partial x_j} + g^{k\ell}\, {}^N\Gamma^\nu{}_{\lambda\mu} \frac{\partial u_\lambda}{\partial x_k} \frac{\partial u_\mu}{\partial x_\ell}, \tag{2.27}$$

where ${}^M\Gamma^j{}_{k\ell}$ and ${}^N\Gamma^\nu{}_{\lambda\mu}$ are the connection coefficients of M and N, respectively.

2. Differentiating (2.27), show that, at p,

$$\begin{aligned} -\frac{\partial}{\partial t}\frac{\partial u_\nu}{\partial x_\ell} + \frac{\partial^3 u_\nu}{\partial x_k \partial x_k \partial x_\ell} &= \frac{1}{2}\left[\frac{\partial^2 g_{kj}}{\partial x_k \partial x_\ell} + \frac{\partial^2 g_{kj}}{\partial x_k \partial x_\ell} - \frac{\partial^2 g_{kk}}{\partial x_j \partial x_\ell}\right]\frac{\partial u_\nu}{\partial x_j} \\ &\quad - \frac{1}{2}\left[\frac{\partial^2 h_{\lambda\nu}}{\partial y_\alpha \partial y_\beta} + \frac{\partial^2 h_{\alpha\nu}}{\partial y_\lambda \partial y_\beta} - \frac{\partial^2 h_{\lambda\alpha}}{\partial y_\nu \partial y_\beta}\right]\frac{\partial u_\beta}{\partial x_\ell}\frac{\partial u_\lambda}{\partial x_k}\frac{\partial u_\alpha}{\partial x_k}. \end{aligned} \tag{2.28}$$

3. Using (2.28), show that, at p,

$$\begin{aligned} -\frac{\partial e}{\partial t} + \Delta e &= \frac{\partial^2 u_\nu}{\partial x_j \partial x_k}\frac{\partial^2 u_\nu}{\partial x_j \partial x_k} \\ &\quad - \frac{1}{2}\left[\frac{\partial^2 g_{\ell k}}{\partial x_i \partial x_i} + \frac{\partial^2 g_{ii}}{\partial x_\ell \partial x_k} - \frac{\partial^2 g_{\ell i}}{\partial x_\ell \partial x_k} - \frac{\partial^2 g_{i\ell}}{\partial x_i \partial x_k}\right]\left(\frac{\partial u_\nu}{\partial x_\ell}\frac{\partial u_\nu}{\partial x_k}\right) \\ &\quad + \frac{1}{2}\left[\frac{\partial^2 h_{\mu\nu}}{\partial y_\lambda \partial y_\rho} + \frac{\partial^2 h_{\lambda\rho}}{\partial y_\mu \partial y_\nu} - \frac{\partial^2 h_{\mu\lambda}}{\partial y_\rho \partial y_\nu} - \frac{\partial^2 h_{\rho\nu}}{\partial y_\mu \partial y_\lambda}\right]\left(\frac{\partial u_\mu}{\partial x_j}\frac{\partial u_\nu}{\partial x_j}\frac{\partial u_\lambda}{\partial x_k}\frac{\partial u_\rho}{\partial x_k}\right). \end{aligned} \tag{2.29}$$

Obtain the identity (2.11) by showing that this is equal, at p, to

$$|{}^N\nabla^2 u|^2 + \frac{1}{2}\operatorname{Ric}^M_{jk}\frac{\partial u_\nu}{\partial x_j}\frac{\partial u_\nu}{\partial x_k} - \frac{1}{2}R^N_{\mu\nu\lambda\rho}\frac{\partial u_\mu}{\partial x_j}\frac{\partial u_\lambda}{\partial x_j}\frac{\partial u_\nu}{\partial x_k}\frac{\partial u_\rho}{\partial x_k}.$$

To define ${}^N\nabla^2 u(x)$, let $E \to M$ denote the pull-back u^*TN, with its pulled-back connection $\widetilde{\nabla}$. To $Du : TM \to, TN$ we associate $\mathcal{D}u \in C^\infty(M, T^* \otimes E)$. If $\nabla^\#$ denotes the product connection on $T^*M \otimes E$, we have

$${}^N\nabla^2 u = \nabla^\# \mathcal{D}u \in C^\infty(M, T^* \otimes T^* \otimes E). \tag{2.30}$$

Compare the construction of second covariant derivatives in Chap. 2, §3, and Appendix C, §2.

If $N \subset \mathbb{R}^k$, let $F \to M$ be the pull-back $uT^*\mathbb{R}^k$, with its pulled-back (flat) connection ∇^0. We have $\mathcal{D}u \in C^\infty(M, T^* \otimes E) \subset C^\infty(M, T^* \otimes F)$ and

$$\nabla^2 u = \nabla^0 \mathcal{D}u \in C^\infty(M, T^* \otimes T^* \otimes F), \tag{2.31}$$

obtained by taking the Hessian of u componentwise.

4. Show that

$${}^N\nabla^2 u(X,Y) = P_E \nabla^2 u(X,Y), \tag{2.32}$$

where $P_E : F \to E$ is orthogonal projection on each fiber. Parallel to (2.7), produce a formula

$$\nabla^2 u = {}^N\nabla^2 u + G(u)(\nabla u, \nabla u), \quad \text{orthogonal decomposition.} \tag{2.33}$$

Relate $G(u)$ to the second fundamental form of $N \subset \mathbb{R}^k$; show that

$$G(u)(\nabla u, \nabla u)(X, Y) = II^N\big(Du(x)X, Du(x)Y\big). \tag{2.34}$$

5. Suppose N is a hypersurface of $\mathbb{R}^k$, given by $N = \{x \in \mathbb{R}^k : \varphi(x) = C\}$, with $\nabla\varphi \neq 0$ on N. Show that $\Gamma(u)(\nabla u, \nabla u)$ in (2.7) is given in this case by

$$\Gamma(u)(\nabla u, \nabla u) = -\left[\sum_j \partial_j u(x) \cdot D^2\varphi\big(u(x)\big) \cdot \partial_j u(x)\right] \frac{\nabla\varphi\big(u(x)\big)}{\big|\nabla\varphi\big(u(x)\big)\big|^2}. \tag{2.35}$$

   Compare with the geodesic equation (11.14) in Chap. 1.
6. If $\dim M = 2$, show that the energy $E(u)$ given by (2.1) of a smooth map $u : M \to N$ is invariant under a conformal change in the metric of M, that is, under replacing the metric tensor g on M by $g' = e^{2f} g$, for some real-valued $f \in C^\infty(M)$.
7. Show that any isometry $w : M \to N$ of M onto N is a harmonic map.
8. Show that if $\dim M = \dim N = 2$ and $w : M \to N$ is a conformal diffeomorphism, then it is harmonic. (*Hint*: Recall Exercise 6.)
9. If $u : M \to N$ is an isometry of M onto a submanifold $\widetilde{M} \subset N$ that is a minimal submanifold, show that u is harmonic.
10. If $\dim M = 2$ and $f : M \to N$, show that

$$E(f) \geq \operatorname{Area}\big(f(M)\big),$$

   with equality if and only if f is conformal.

3. Semilinear equations on regions with boundary

The initial-value problem

$$\frac{\partial u}{\partial t} = \Delta u + F(t, x, u, \nabla u), \quad u(0) = f, \tag{3.1}$$

for $u = u(t, x)$, was studied in § 1 for $x \in M$, a compact manifold without boundary. Here we extend many of these results to the case where $x \in \overline{M}$, a compact manifold with boundary. As in § 1, we assume F is smooth in its arguments. We will deal specifically with the Dirichlet problem:

$$u = 0 \text{ on } \mathbb{R}^+ \times \partial M. \tag{3.2}$$

There is an analogous development for other boundary conditions, such as Neumann or Robin boundary conditions.

Recall that Propositions 1.1 and 1.1A were phrased on a very general level, so a number of short-time existence results in this case follow simply by verifying the hypotheses (1.3)–(1.6), for appropriate Banach spaces X and Y on $\overline{M}$. For example, somewhat parallel to (1.10), consider $X = C_b^1(\overline{M}),\ Y = C(\overline{M})$, where, for $j \geq 0$, we set

$$C_b^j(\overline{M}) = \{f \in C^j(\overline{M}) : f = 0 \text{ on } \partial M\}. \tag{3.3}$$

In Proposition 7.4 of Chap. 13, it is shown that $e^{t\Delta}$ is a strongly continuous semigroup on $C_b^1(\overline{M})$. Also, (7.52) of Chap. 13 gives

$$\|e^{t\Delta} f\|_{C^1(\overline{M})} \leq C t^{-1/2} \|f\|_{L^\infty}, \quad \text{for } 0 < t \leq 1, \tag{3.4}$$

so we have the following:

Proposition 3.1. *If $f \in C_b^1(\overline{M})$, then (3.1)–(3.2) has a unique solution*

$$u \in C\bigl([0,T), C^1(\overline{M})\bigr), \tag{3.5}$$

for some $T > 0$, estimable from below in terms of $\|f\|_{C^1}$.

If we specialize to F independent of ∇u, hence look at

$$\frac{\partial u}{\partial t} = Lu + F(t,x,u), \quad u(0) = f, \tag{3.6}$$

we can take $X = C_b(\overline{M}),\ Y = C(\overline{M})$, and, by arguments similar to those used above, we obtain the following result:

Proposition 3.2. *If $f \in C_b(\overline{M})$, then (3.6), (3.2) has a unique solution*

$$u \in C\bigl([0,T), C(\overline{M})\bigr). \tag{3.7}$$

for some $T > 0$, estimable from below in terms of $\|f\|_{L^\infty}$.

We can obtain further regularity results on solutions to (3.1) and (3.6), with boundary condition (3.2), making use of regularity results for

$$\frac{\partial u}{\partial t} = \Delta u + g(t,x), \quad u(t,x) = 0 \ \text{ for } \ x \in \partial M, \tag{3.8}$$

established in Exercises 4–10 of Chap. 6, § 1. To recall the result, let us set, for $k \in \mathbb{Z}^+$,

$$\mathcal{H}^k(I \times M) = \{u \in L^2(I \times M) : \partial_t^j u \in L^2(I, H^{2k-2j}(M)), 0 \leq j \leq k\}. \tag{3.9}$$

The result is that if (3.8) holds on $I \times M$, with $I = [0, T_0]$, then

$$g \in \mathcal{H}^k(I \times M) \Longrightarrow u \in \mathcal{H}^{k+1}(I' \times M), \tag{3.10}$$

for $I' = [\varepsilon, T_0]$, $\varepsilon > 0$. Taking $g = F(t, x, u, \nabla u)$ for u in Proposition 3.1 and $g = F(t, x, u)$ for u in Proposition 3.2, we have in both cases $g \in \mathcal{H}^0(I \times M)$, whenever $T_0 < T$, and hence

$$u \in \mathcal{H}^1(I' \times M), \tag{3.11}$$

in both cases. One also has higher order regularity. For simplicity, we restrict attention to the setting of Proposition 3.2.

Proposition 3.3. *Assume F is smooth in its arguments. The solution (3.7) of (3.6), (3.2) has the property*

$$u \in C^\infty\big((0, T) \times \overline{M}\big). \tag{3.12}$$

Proof. In this case, we start with the implication

$$u \in C(I \times \overline{M}) \cap \mathcal{H}^1(I' \times M) \Longrightarrow F(t, x, u) \in \mathcal{H}^1(I' \times M), \tag{3.13}$$

as follows from the chain rule and Moser estimates, as in Proposition 3.9 in Chap. 13. Applying (3.10) then gives $u \in \mathcal{H}^2(I' \times M)$. More generally,

$$u \in C(I \times \overline{M}) \cap \mathcal{H}^k(I' \times M) \Longrightarrow F(t, x, u) \in \mathcal{H}^k(I' \times M). \tag{3.14}$$

Repeated applications of this plus (3.10) then yield $u \in \mathcal{H}^{k+1}(I' \times M)$ for all k, which implies (3.12).

Exercises

1. Work out results parallel to those presented in this section, when the Dirichlet boundary condition (3.2) is replaced by Neumann or Robin boundary conditions.
2. Consider the 3-D Burger equation

$$u_t + \nabla_u u = \Delta u, \quad u(0, x) = f(x), \quad u(t, x) = 0, \ \text{ for } x \in \partial\Omega, \tag{3.15}$$

where $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$, and Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary. Show that the set-up to prove local existence works, with

$$X = H_0^1(\Omega), \quad Y = L^{3/2}(\Omega).$$

(*Hint*: Show that $H_0^1(\Omega) \cdot L^2(\Omega) \subset L^{3/2}(\Omega)$ and $\mathcal{D}(\Delta^{1/4}) \subset L^3(\Omega)$, hence

$$\|e^{t\Delta} f\|_{H^1(\Omega)} \le C t^{-3/4} \|f\|_{L^{3/2}(\Omega)}, \quad \text{for } 0 < t \le 1.)$$

4. Reaction-diffusion equations

Here we study $\ell \times \ell$ systems of the form

$$\frac{\partial u}{\partial t} = Lu + X(u), \quad u(0) = f, \tag{4.1}$$

where $u = u(t,x)$ takes values in $\mathbb{R}^\ell$, X is a real vector field on $\mathbb{R}^\ell$, and L is a second-order differential operator, which we assume to be a negative-semidefinite, self-adjoint operator on $L^2(M)$. We take M to be a complete Riemannian manifold, of dimension n, often either $\mathbb{R}^n$ or compact. The numbers n and ℓ are unrelated. We do not assume L is elliptic, though that possibility is not precluded.

Such a system arises when "substances" S_ν, $1 \le \nu \le \ell$, whose concentrations are measured by u_ν, are simultaneously diffusing and interacting via a mechanism that changes these quantities. Recall from the introduction to Chap. 11 the relation between the quantity u_ν of S_ν and its flux J_ν, in case S_ν is being neither created nor destroyed. This generalizes to the identity

$$\frac{\partial}{\partial t} \int_{\mathcal{O}} u_\nu(t,x)\, dV(x) = -\int_{\partial\mathcal{O}} N \cdot J_\nu \, dS(x) + \int_{\mathcal{O}} X_\nu\big(u(t,x)\big)\, dV(x)$$

if $X_\nu(u)$ is a measure of the rate at which S_ν is created, due to interactions with the "environment," namely, with the other S_μ. Consequently, by the divergence theorem,

$$\frac{\partial u_\nu}{\partial t} = -\text{div}\, J_\nu + X_\nu(u).$$

If we assume that each S_ν obeys a diffusion law independent of the other substances, of the form considered in Chap. 11, that is,

$$J_\nu = -d_\nu \,\text{grad}\, u_\nu,$$

then we obtain the system (4.1), with $L = D\Delta$, where D is a diagonal $\ell \times \ell$ matrix with diagonal entries $d_\nu \ge 0$; we allow the possibility $d_\nu = 0$, which means S_ν is not diffusing.

An example of the sort of system that arises this way is the Fitzhugh–Nagumo system:

$$\begin{aligned} \frac{\partial v}{\partial t} &= D\frac{\partial^2 v}{\partial x^2} + f(v) - w, \\ \frac{\partial w}{\partial t} &= \varepsilon(v - \gamma w), \end{aligned} \tag{4.2}$$

with

$$f(v) = v(a - v)(v - 1).$$

In this case,

$$L = \begin{pmatrix} D\partial_x^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

Here D is a positive constant. This arose as a model for activity along the axon of a nerve, with v and w related to the voltage and the ion concentration, respectively. We will mention other examples later in this section. While we will mention what various examples model, we will not go into the mechanisms behind the models. Excellent discussions of all these models and more can be found in [Mur].

One property L in (4.3) has is the following generalization of the maximum principle:

Invariance property. *There is a compact, convex neighborhood K of the origin in $\mathbb{R}^\ell$ such that if $f \in L^2(M)$, then, for all $t \geq 0$,*

$$f(x) \in K \text{ for all } x \Longrightarrow e^{tL} f(x) \in K \text{ for all } x. \tag{4.4}$$

Thus, if $f, g \in L^2(M)$ have compact support,

$$\|e^{tL} f\|_{L^\infty} \leq \kappa \|f\|_{L^\infty}, \tag{4.5}$$

with κ independent of $t \geq 0$. If we defined a norm on $\mathbb{R}^\ell$ so that $K \cap (-K)$ was the unit ball, would have $\kappa = 1$. Note that, for such f and g, we have

$$|(e^{tL} f, g)| = |(f, e^{tL} g)| \leq \kappa \|f\|_{L^1} \|g\|_{L^\infty}, \tag{4.6}$$

so $\|e^{tL} f\|_{L^1} \leq \kappa \|f\|_{L^1}$. Thus e^{tL} has a unique extension to a linear map

$$e^{tL} : L^p(M) \longrightarrow L^p(M), \quad \|e^{tL}\| \leq \kappa_p, \tag{4.7}$$

in case $p = 1$, hence, by interpolation, for $1 \leq p \leq 2$, and, by duality, for $2 \leq p \leq \infty$, uniqueness for $p = \infty$ holding in the class of operators whose adjoints preserve $L^1(M)$.

As mentioned above, in many examples of reaction-diffusion equations, $L = DL_0$, where D is a *diagonal* $\ell \times \ell$ matrix, with constant entries $d_j \geq 0$, and L_0 is a scalar operator, generating a diffusion semigroup on $L^2(M)$; in fact, often $M = \mathbb{R}$ and $L_0 = \partial^2/\partial x^2$. For such L, any rectangular region of the form $K = \{y \in \mathbb{R}^\ell : a_j \leq y_j \leq b_j\}$ has the invariance property (4.4). If some of the diagonal entries d_j coincide, there will be a somewhat larger set of such invariant regions.

We apply the technique of § 1 to obtain solutions to (4.1), rewritten as the integral equation

$$u(t) = e^{tL} f + \int_0^t e^{(t-s)L} X\big(u(s)\big)\, ds. \tag{4.8}$$

Proposition 4.1. *Let V be a Banach space of functions on M with values in $\mathbb{R}^\ell$ such that*

$$e^{tL} : V \longrightarrow V \text{ is a strongly continuous semigroup, for } t \geq 0, \tag{4.9}$$

and

$$\mathfrak{X} : V \longrightarrow V \text{ is Lipschitz, uniformly on bounded sets,} \tag{4.10}$$

where $(\mathfrak{X} f)(x) = X(f(x))$. Then (4.8) has a unique solution

$$u \in C\big([0, T], V\big),$$

where $T > 0$ is estimable from below in terms of $\|f\|_V$.

The proof is simply a specialization of that used for Proposition 1.1. Note that (4.10) holds for a variety of spaces, such as $V = L^p(M, \mathbb{R}^\ell)$, $V = C(M, \mathbb{R}^\ell)$, when X is a vector field on $\mathbb{R}^\ell$ satisfying

$$\|X(y)\| \leq C_1, \quad \|\nabla X(y)\| \leq C_2, \quad \forall\, y \in \mathbb{R}^\ell, \tag{4.11}$$

provided M is compact. If M has infinite volume, you also need $X(0) = 0$ for $V = L^p(M, \mathbb{R}^\ell)$ to work. Whenever X has this property, and L satisfies the invariance property (4.4), it follows that Proposition 4.1 applies, for initial data $f \in L^p(M, \mathbb{R}^\ell)$, $1 \leq p < \infty$. If, in addition, $e^{tL} : C(M) \to C(M)$, we also have short-time solutions to (4.8) for $f \in C(M, \mathbb{R}^\ell)$. For example, if $M = \mathbb{R}^n$ and L has constant coefficients, then (4.7) implies

$$e^{tL} : H^{k,p}(\mathbb{R}^n) \longrightarrow H^{k,p}(\mathbb{R}^n), \quad k > 0. \tag{4.12}$$

Also

$$e^{tL} : C_o(\mathbb{R}^n) \longrightarrow C_o(\mathbb{R}^n), \tag{4.13}$$

for $t \geq 0$, since $C_o(\mathbb{R}^n)$, the space of continuous functions vanishing at infinity, is the closure of $H^{k,2}(\mathbb{R}^n)$ in $L^\infty(\mathbb{R}^n)$, for $k > n/2$.

Another useful example when $M = \mathbb{R}^n$ is the space

$$\mathcal{BC}(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : f \text{ extends continuously to } \widehat{\mathbb{R}^n}\}, \tag{4.14}$$

where $\widehat{\mathbb{R}^n}$ is the compactification of $\mathbb{R}^n$ via the sphere at infinity (approached radially). For $k \in \mathbb{Z}^+$, we say $f \in \mathcal{BC}^k(\mathbb{R}^n)$ provided $D^\alpha f \in \mathcal{BC}(\mathbb{R}^n)$ whenever $|\alpha| \leq k$.

If $M = \mathbb{R}^n$ and (4.12) holds, then Proposition 4.1 applies with $V = H^{k,p}(\mathbb{R}^n, \mathbb{R}^\ell)$ whenever the vector field X and all its derivatives of order $\leq k$

are bounded on $\mathbb{R}^\ell$ (and $X(0) = 0$). It also applies to $\mathcal{BC}(\mathbb{R}^n, \mathbb{R}^\ell)$. Now if L is not elliptic, we have no extension of the regularity result in Proposition 1.2. By a different technique we can show that under certain circumstances, if f belongs to a space like $H^{k,p}(\mathbb{R}^n, \mathbb{R}^\ell)$, then a solution $u(t)$ persists as a solution in $C\big([0,T], H^{k,p}(\mathbb{R}^n)\big)$ as long as it persists as a solution in $C\big([0,T], C_o(\mathbb{R}^n)\big)$. To get this, we reexamine the iterative formula used to solve (4.8), namely

$$u_{j+1}(t) = e^{tL} f + \int_0^t e^{(t-s)L} X\big(u_j(s)\big)\, ds. \tag{4.15}$$

As long as (4.9) holds for the Banach space V, we have

$$\begin{aligned} \|u_{j+1}(t)\|_V &\le \|e^{tL} f\|_V + C\int_0^t e^{K(t-s)} \|X(u_j(s))\|_V\, ds \\ &\le A(t) + Cte^{Kt} \sup_{0\le s\le t} \|X(u_j(s))\|_V \end{aligned} \tag{4.16}$$

and

$$\|u_{j+1}(t) - u_j(t)\|_V \le Cte^{Kt} \sup_{0\le s\le t} \|X(u_j(s)) - X(u_{j-1}(s))\|_V. \tag{4.17}$$

Now, as shown in Chap. 13, § 3, for such spaces as $V = H^{k,p}(\mathbb{R}^n)$, there are *Moser estimates*, of the form

$$\|uv\|_V \le C\|u\|_{L^\infty}\|v\|_V + C\|u\|_V\|v\|_{L^\infty} \tag{4.18}$$

and

$$\|F(u)\|_V \le C\big(\|u\|_{L^\infty}\big)\big(1 + \|u\|_V\big), \quad C(\lambda) = \sup_{|x|\le\lambda, |\mu|\le k} |F^{(\mu)}(x)|. \tag{4.19}$$

In particular, $\|X(u)\|_V$ satisfies an estimate of the form (4.19). Also, we can write

$$X(u) - X(v) = Y(u,v)(u-v), \quad Y(u,v) = \int_0^1 DX\big(\sigma u + (1-\sigma)v\big)\, d\sigma$$

and obtain the estimate

$$\begin{aligned} &\|X(u) - X(v)\|_V \\ &\quad \le C\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)\|u-v\|_V \\ &\quad + C\big(\|u\|_{L^\infty} + \|v\|_{L^\infty}\big)\big(\|u\|_V + \|v\|_V\big)\|u-v\|_{L^\infty}. \end{aligned} \tag{4.20}$$

From (4.16) we deduce

$$\|u_{j+1}(t)\|_V \le A(t) + te^{Kt} \sup_{0\le s\le t} C\big(\|u_j(s)\|_{L^\infty}\big)\big(1 + \|u_j(s)\|_V\big). \tag{4.21}$$

If $\{u_j(t) : j \in \mathbb{Z}^+\}$ is bounded in $L^\infty(M)$ for $0 \le t \le T$, this takes the form

$$\|u_{j+1}(t)\|_V \le B_1 + Bt \sup_{0\le s\le t} \|u_j(s)\|_V, \tag{4.22}$$

for $0 \le t \le T$. Also, in such a case, (4.17) and (4.20) yield

$$\begin{aligned} \|u_{j+1}(t) - u_j(t)\|_V &\le Bt \sup_{0\le s\le t} \|u_j(s) - u_{j-1}(s)\|_V \\ &+ Bt \sup_{0\le s\le t} \big(\|u_j(s)\|_V + \|u_{j-1}(s)\|_V\big)\|u_j(s) - u_{j-1}(s)\|_{L^\infty}. \end{aligned} \tag{4.23}$$

Now, in (4.22) and (4.23), B may depend on the choice of the space V, but it does not depend on the V-norm of any $u_j(s)$, only on the L^∞-norm.

Let us assume that $u_0(t) = e^{tL}f$ satisfies $\|u_0(t)\|_V \le B_1$, for $0 \le t \le T$, $B_1 \ge B$. This is the B_1 used in (4.22). Assume $T_0 \le 1/4B$, $T_0 \le 1/16BB_1$, and $T_0 \le T$. Then $\|u_j(t)\|_V \le 2B_1$ for $0 \le t \le T_0$, for all $j \in \mathbb{Z}^+$, so $\{u_j : j \in \mathbb{Z}^+\}$ is bounded in $C\big([0, T_0], V\big)$. In such a case, (4.23) yields, for $0 \le t \le T_0$,

$$\begin{aligned} \|u_{j+1}(t) - u_j(t)\|_V \le \frac{1}{4} &\sup_{0\le s\le t} \|u_j(s) - u_{j-1}(s)\|_V \\ &+ \frac{1}{4} \sup_{0\le s\le t} \|u_j(s) - u_{j-1}(s)\|_{L^\infty}, \end{aligned} \tag{4.24}$$

so $\{u_j : j \in \mathbb{Z}^+\}$ is in fact Cauchy in $C\big([0, T_0], V\big)$, having therefore a limit $u \in C([0, T_0], V)$ satisfying (4.8). The size of the interval $[0, T_0]$ on which this argument works depends on the choice of V and the size of $\|u(0)\|_{L^\infty}$, but *not* on the size of $\|u(0)\|_V$. We can iterate this argument on intervals of length T_0 as long as $\|u(t)\|_{L^\infty}$ is bounded, thus establishing the following.

Proposition 4.2. *Suppose V is a Banach space of functions such that (4.9)–(4.10) and the Moser estimates (4.18)–(4.19) hold. Let $f \in V \cap L^\infty(M)$, and suppose (4.8) has a solution $u \in L^\infty\big([0, T) \times M\big)$. Then, in fact, $u \in C\big([0, T), V\big)$. If $V = H^{k,p}(M)$, with $k \ge 2$, we thus have*

$$u \in C\big([0, T), H^{k,p}(M)\big) \cap C^1\big([0, T), H^{k-2,p}(M)\big), \tag{4.25}$$

solving (4.1).

Global existence results can be established for (4.1) when f takes values in a bounded subset of $\mathbb{R}^\ell$ shown to be invariant under the nonlinear solution operator to (4.1). An example of this is the following:

Proposition 4.3. *In (4.1), assume $Lu = D\Delta u$, where D is a diagonal $\ell \times \ell$ matrix with diagonal entries $d_j \ge 0$ and Δ acts on u componentwise, as the*

Laplace operator on a Riemannian manifold M. Assume M is compact and $f \in H^{k,p}(M,\mathbb{R}^\ell)$, $k > 2+n/p$. Or assume $M = \mathbb{R}^n$, with its Euclidean metric and $f \in \mathcal{BC}^2(\mathbb{R}^n,\mathbb{R}^\ell)$. Consider a rectangle $\mathcal{R} \subset \mathbb{R}^\ell$, of the form

$$\mathcal{R} = \{y \in \mathbb{R}^\ell : a_j \le y_j \le b_j\}. \tag{4.26}$$

Suppose that, for each $y \in \partial\mathcal{R}$,

$$X(y) \cdot N < 0, \tag{4.27}$$

where N is any outer normal to $\mathcal{R}$. If f takes values in the interior $\overset{\circ}{\mathcal{R}}$ of $\mathcal{R}$, then the solution to (4.1) exists and takes values in $\overset{\circ}{\mathcal{R}}$ for all $t \ge 0$.

Proof. First suppose M is compact. If there is an exit from $\overset{\circ}{\mathcal{R}}$, we can pick (t_0, x_0) such that

$$u_j(t_0,x_0) = a_j \text{ or } b_j, \tag{4.28}$$

for some $j = 1,\dots,\ell$, and $u(t,x) \in \overset{\circ}{\mathcal{R}}$ for all $t < t_0,\ x \in M$. Pick b_j, for example. Then

$$\partial_t u_j(t_0,x_0) \ge 0. \tag{4.29}$$

Now $\varphi(x) = u_j(t_0,x)$ must have a maximum at $x = x_0$, so

$$\partial_t u_j(t_0,x_0) = d_j \Delta u_j + X_j(u) \le X_j(u). \tag{4.30}$$

However, (4.27) implies $X_j\big(u(t_0,x_0)\big) < 0$, so (4.29) and (4.30) contradict each other.

In case $M = \mathbb{R}^n$, the existence of such $(t_0,x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ is problematic, though we can find such $(t_0,x_0) \in \mathbb{R}^+ \times \widehat{\mathbb{R}^n}$, since u has a unique continuous extension to $\mathbb{R}^+ \times \widehat{\mathbb{R}^n}$ and $\widehat{\mathbb{R}^n}$ is compact. We still have $\partial_t u(t_0,x_0) \ge 0$, and Δu is continuous on $\mathbb{R}^+ \times \widehat{\mathbb{R}^n}$, but it is not obvious in this case that $\Delta u(t_0,x_0) \le 0$, unless x_0 lies in $\mathbb{R}^n$, not at infinity. Thus we argue as follows.

Let $\widetilde{BC}^2(\mathbb{R}^n)$ denote $\{f \in \mathcal{BC}^2(\mathbb{R}^n) : D^\alpha f = 0 \text{ at } \infty, \text{ for } |\alpha| = 2\}$. This Banach space is also one for which Propositions 4.1 and 4.2 work. Furthermore, the argument above regarding $u(t_0,x_0)$ does work if we replace $f \in \mathcal{BC}^2(\mathbb{R}^n,\mathbb{R}^\ell)$ by $f_\nu \in \widetilde{BC}^2(\mathbb{R}^n,\mathbb{R}^\ell)$. Additionally, we can take a sequence of such f_ν so that $f_\nu \to f$ in $\mathcal{BC}(\mathbb{R}^n,\mathbb{R}^\ell)$, and obtain solutions u_ν such that $u_\nu(t,x) \to u(t,x)$ uniformly on $[0,T] \times \mathbb{R}^n$ for any $T < \infty$. We can replace $\mathcal{R}$ by a slightly smaller rectangle $\mathcal{R}_1$, for which (4.27) holds, and arrange that each f_ν takes values in $\overset{\circ}{\mathcal{R}}_1$. Then $u(t,x)$ always takes values in $\mathcal{R}_1 \subset \overset{\circ}{\mathcal{R}}$. This completes the proof in the case $M = \mathbb{R}^n$.

As an example of Proposition 4.3, we consider the Fitzhugh–Nagumo system (4.2), in which the vector field X on $\mathbb{R}^2$ is

$$X(v,w) = \big(f(v) - w, \varepsilon(v - \gamma w)\big), \quad f(v) = v(1-v)(v-a). \tag{4.31}$$

In Fig. 4.1 we illustrate an invariant rectangle $\mathcal{R}$ that arises from the choices

$$\gamma = 20, \quad a = 0.4, \quad \varepsilon = 0.01. \tag{4.32}$$

This invariant region contains three critical points of X, two sinks and a saddle. For this construction to work, we need the following:

The top edge of $\mathcal{R}$ lies above the line $w = v/\gamma$,
while the bottom edge of $\mathcal{R}$ lies below this line;
the left edge of $\mathcal{R}$ lies to the left of the curve $w = f(v)$,
while the right edge of $\mathcal{R}$ lies to the right of this curve.

The two curves mentioned here are the "isoclines," defining where $X_2 = 0$ and $X_1 = 0$, respectively. The condition just stated implies that X points down on the top edge of $\mathcal{R}$, up on the bottom edge, to the right on the left edge, and to the left on the right edge. In Fig. 4.1 we also depict a smaller invariant rectangle $\mathcal{R}_0$, which contains only one critical point of X, the sink at $(0,0)$. Figure 4.2 is a similar illustration, with γ changed from 20 to 10; in this case X has only one critical point.

The vector field (4.31) does not actually satisfy the hypothesis (4.11), since the coefficients blow up at infinity. But one can alter X outside $\mathcal{R}$ to produce a vector field $\widetilde{X}$ to which Propositions 4.1 and 4.2 apply. As long as the initial function $u(0) = f$ takes values inside $\mathcal{R}$, one has a solution to (4.2).

While Proposition 4.3 is an elementary consequence of the maximum principle, this result can also be seen to follow quite transparently from a "nonlinear

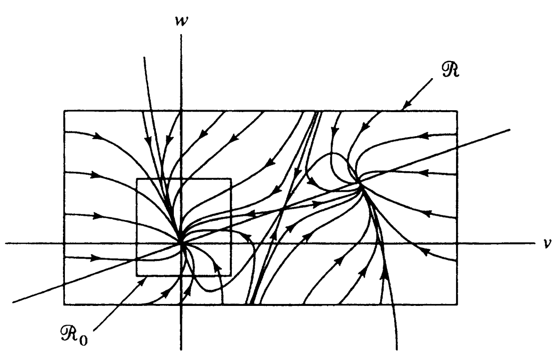

FIGURE 4.1 Invariant Rectangles for Fitzhugh-Nagumo System

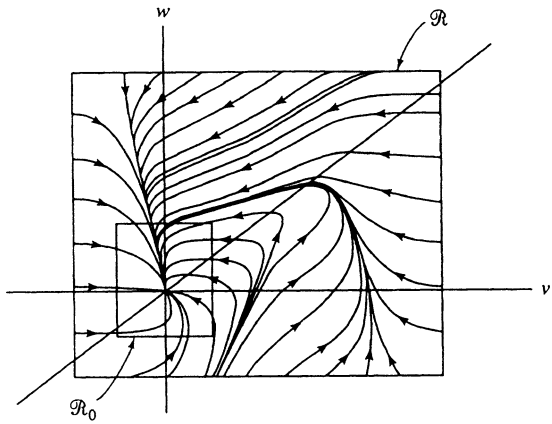

FIGURE 4.2 Invariant Rectangles with Different Parameters

Trotter product formula," namely a solution to (4.1) satisfies

$$u(t) = \lim_{n\to\infty} \left(e^{(t/n)L}\mathcal{F}^{t/n}\right)^n (f), \tag{4.33}$$

where if $\mathcal{F}^t_X$ is the flow on $\mathbb{R}^\ell$ generated by X, then

$$\mathcal{F}^t f(x) = \mathcal{F}^t_X\big(f(x)\big). \tag{4.34}$$

We will prove this in the next section. See Proposition 5.4 for a precise statement. We mention that if $f \in \mathcal{BC}^1(\mathbb{R}^n, \mathbb{R}^\ell)$, then (4.33) converges in $C\big([0,T], \mathcal{BC}^0(\mathbb{R}^n, \mathbb{R}^\ell)\big)$. We can use this result to prove the following, which is somewhat stronger than Proposition 4.3.

Proposition 4.4. *Assume $X \in C^2(\mathbb{R}^\ell)$ and $u(0) = f \in \mathcal{BC}^1(\mathbb{R}^n, \mathbb{R}^\ell)$, and let L be a second-order differential operator with constant coefficients, such that e^{tL} is a contraction on $\mathcal{BC}^0(\mathbb{R}^n, \mathbb{R}^\ell)$, for $t \geq 0$. Assume there is a family $\{K_s : 0 \leq s < \infty\}$ of compact subsets of $\mathbb{R}^\ell$ such that each K_s has the invariance property (4.4). Furthermore, assume that*

$$\mathcal{F}^t_X(K_s) \subset K_{s+t}, \quad s, t \in \mathbb{R}^+. \tag{4.35}$$

If $u(0) = f$ takes values in K_0, then (4.1) has a solution for all $t \in \mathbb{R}^+$, and $u(t,x) \in K_t$.

Proof. This is a simple consequence of the product formula (4.33).

In cases where L is diagonal and $\{K_s\}$ is a certain *shrinking* family of rectangles, this result was proved in [RaSm], by different means. An example to which their result applies arises when K_0 is the rectangle $\mathcal{R}_0$ in Fig. 4.1. Then K_s is a family of rectangles shrinking to the origin as $s \to \infty$, and one gets decay of any solution to (4.2) whose initial function $u(0) = f$ takes values in such a rectangle K_0. Of course, if there were no diffusion (i.e., $D = 0$ in (4.2)), one would get such decay whenever $u(0) = f$ took values in the region of $\mathbb{R}^2$ for which the origin is an attractor. One then has the question of whether a sufficient degree of diffusion could change the situation. We will return to this point shortly.

For the system (4.2), there are families of rectangles K_s such that K_0 contains an arbitrarily large disk centered at the origin, which contract to $K_T = \mathcal{R}$, satisfying (4.35). Hence any solution to (4.2) with initial data in $\mathcal{BC}^1(\mathbb{R}, \mathbb{R}^2)$ exists for all $t \geq 0$, and, for t large, $u(t)$ takes values in $\mathcal{R}$. However, in the cases illustrated in both Figs. 4.1 and 4.2, there is not a family of rectangles having the property (4.35) taking $\mathcal{R}$ to $\mathcal{R}_0$, and in fact not all solutions to (4.2) with initial data in $\mathcal{BC}^1(\mathbb{R}, \mathbb{R}^2)$ will decay to a constant function.

One class of nondecaying solutions to reaction-diffusion equations of particular interest is the class of "traveling wave solutions," which, in case $M = \mathbb{R}$ and L has constant coefficients, are sought in the form

$$u(t, x) = \varphi(x - ct). \tag{4.36}$$

Suppose $L = D\partial_x^2$, where D is a diagonal matrix, with entries $d_j \geq 0$. Then $\varphi(s)$ must satisfy the second-order $\ell \times \ell$ system of ODEs:

$$D\varphi'' + c\varphi' + X(\varphi) = 0. \tag{4.37}$$

Using $\psi = \varphi'$, we convert this to a first-order $(2\ell) \times (2\ell)$ system

$$\varphi' = \psi, \quad D\psi' = -c\psi - X(\varphi). \tag{4.38}$$

If some $d_j = 0$, it is best not to use ψ_j.

Let us first take a closer look at the scalar case, which we write as

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + g(v). \tag{4.39}$$

Then a traveling wave $v(t, x) = \varphi(x - ct)$ arises when $\varphi(s)$ satisfies the single ODE

$$D\varphi'' + c\varphi' + g(\varphi) = 0. \tag{4.40}$$

With $\psi = \varphi'$, we have the 2×2 system

$$\varphi' = \psi, \quad \psi' = -c\psi - g(\varphi), \tag{4.41}$$

taking $D = 1$ without essential loss of generality. This is amenable to a simple phase-plane analysis.

The vector field $Y = Y_g$ whose orbits are specified by (4.41) has critical points at $\psi = 0, g(\varphi) = 0$. For a general smooth g in (4.39)–(4.41), if $\varphi = \alpha$ is a zero of g, and if $g'(\alpha) = \sigma$, then the linearized ODE about the critical point $(\alpha, 0)$ of Y is

$$\frac{du}{dt} = Au, \quad A = \begin{pmatrix} 0 & 1 \\ -\sigma & -c \end{pmatrix}. \tag{4.42}$$

Note that

$$\operatorname{Tr} A = -c, \quad \det A = \sigma. \tag{4.43}$$

This establishes the following:

Lemma 4.5. *If $g(\alpha) = 0$, the critical point $(\alpha, 0)$ of the vector field Y defined by (4.41) is*

$$\text{a saddle if } g'(\alpha) < 0,$$
$$\text{a sink if } g'(\alpha) > 0 \text{ and } c > 0,$$
$$\text{a source if } g'(\alpha) > 0 \text{ and } c < 0.$$

Of course, when $c = 0$, (4.41) is in Hamiltonian form, with energy function

$$E(\varphi, \psi) = \frac{1}{2}\psi^2 + G(\varphi), \quad G(\varphi) = \int g(\varphi)\, d\varphi. \tag{4.44}$$

In that case, the integral curves of Y are the level curves of $E(\varphi, \psi)$, and a nondegenerate critical point for Y is either a saddle or a center. For $c = 0$, $v(t, x) = \varphi(x)$ is a stationary solution to the PDE (4.39). If $c \neq 0$, we can switch signs of s if necessary and assume $c > 0$. Then (4.40) models motion on a line, in a force field, with damping, due to friction proportional to the velocity. On any orbit of (4.41) we have

$$\frac{dE}{ds} = -c\psi(s)^2 \leq 0. \tag{4.45}$$

This implies that Y cannot have a nontrivial periodic orbit if $c > 0$.

Let us consider a case where g has three distinct zeros, $\alpha_1, \alpha_2, \alpha_3$, as depicted in Fig. 4.3. In this case, Y has saddles at $(\alpha_1, 0)$ and $(\alpha_3, 0)$, and a sink at $(\alpha_2, 0)$. Now the three points $(\alpha_j, 0)$ are also critical points of the function $E(\varphi, \psi)$, defined by (4.44), and, depending on whether the critical values at $(\alpha_1, 0)$ and $(\alpha_3, 0)$ are equal or not, the level curves of $E(\varphi, \psi)$ (orbits of the $c = 0$ case of (4.41)) are as depicted in Fig. 4.4. When we take small $c > 0$, the orbits of Y in the cases (a) and (b), respectively, are perturbed to those depicted in Fig. 4.5. In case (a), both saddles are connected to the sink, while in case (b) just one saddle is connected to the sink.

In case (b), if we let c increase, eventually the phase plane has the same behavior as (a). There will consequently be a particular value $c = c_0$ where an orbit connects the saddle $(\alpha_\mu, 0)$ to the saddle $(\alpha_\nu, 0)$, where α_μ is the zero of g for

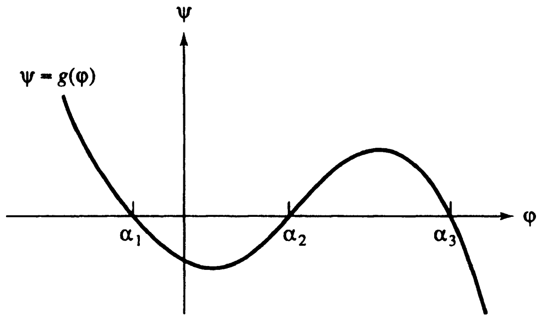

FIGURE 4.3 Function with Three Zeros

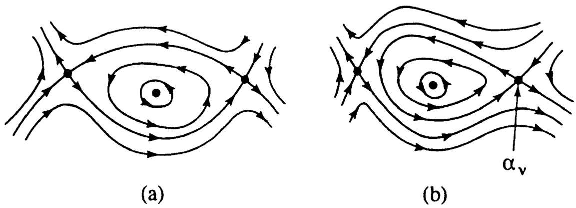

FIGURE 4.4 Vector Fields with Centers

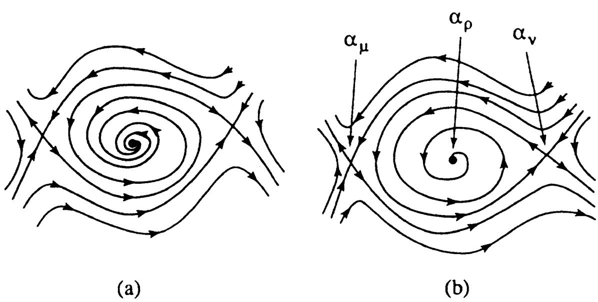

FIGURE 4.5 Vector Fields with Spiral Sinks

which $G(\varphi) = \int g(\varphi)\, d\varphi$ has the largest value. An orbit connecting two different saddles is called a "heteroclinic orbit." (Note that in case (b), at $c = 0$ there is an orbit connecting the other saddle $(\alpha_\nu, 0)$ to itself; such an orbit is called a "homoclinic orbit.") In an obvious sense, α_ν is the endpoint (either α_1 or α_3) of the "smaller" of the two "humps" of $\psi = g(\varphi)$ in Fig. 4.3, the size being measured by the area enclosed by the curve and the horizontal axis.

Such an orbit of Y connecting $(\alpha_\mu, 0)$ to $(\alpha_\nu, 0)$ then gives rise to a traveling wave solution $u(t, x) = \varphi(x - c_0 t)$, which, for each $t \geq 0$, tends to α_μ as $x \to -\infty$ and to α_ν as $x \to +\infty$. If α_ρ is the remaining zero of $g(v)$, then for each $c > 0$, there is a traveling wave $u(t, x) = \widetilde{\varphi}(x - ct)$, which tends to α_ν as $x \to -\infty$ and to α_ρ as $x \to +\infty$; and if $c > c_0$, there is a traveling wave $u(t, x) = \varphi(x - ct)$, which tends to α_μ as $x \to -\infty$ and to α_ρ as $x \to +\infty$.

Such traveling waves yield a transport of quantities much faster than straight diffusion processes, described by $\partial u/\partial t = D\Delta u$. Yet this speed is due not to any convective term in (4.1), but rather to the coupling with the nonlinear term $X(u)$. Such behavior, according to Murray [Mur], was "a major factor in starting the whole mathematical field of reaction-diffusion theory."

Note that in the limiting case of (4.2) where $\varepsilon = 0,\ w = w_0$ is independent of t, and we get a scalar equation of the form (4.39), with $g(v) = f(v) - w_0$, if w_0 is also independent of x. Another widely studied example of (4.39) is

$$g(v) = v(1-v). \tag{4.46}$$

In this case the vector field Y has two critical points: a saddle and a sink. This case of (4.39) is called the Kolmogorov–Petrovskii–Piskunov equation. It is also called the Fisher equation, when studied as a model for the spread of an advanlabeleous gene in a population; see [Mur].

If (4.1) is a 2×2 system with $L = D\Delta$, then one gets a vector field on $\mathbb{R}^4$ from (4.38), provided D is positive-definite. If $d_1 > 0$ but $d_2 = 0$, then, as noted above, one omits ψ_2 and obtains a 3×3 system. For example, for the Fitzhugh–Nagumo system (4.2), one obtains traveling waves $u = (v, w) = (\varphi_1, \varphi_2)$, provided φ_1, φ_2, and ψ_1 satisfy the system

$$\begin{aligned} \varphi_1' &= \psi_1, \\ \psi_1' &= -\frac{1}{D}\Big(c\psi_1 + f(\varphi_1) - \varphi_2\Big), \\ \varphi_2' &= -\frac{\varepsilon}{c}\Big(\varphi_1 - \gamma\varphi_2\Big). \end{aligned} \tag{4.47}$$

This has the form

$$\zeta' = Z_c(\zeta), \tag{4.48}$$

for $\zeta = (\varphi_1, \psi_1, \varphi_2)$, where Z_c is a vector field on $\mathbb{R}^3$.

Various techniques have been brought to bear to analyze orbits of such a vector field. An important role has been played by C. Conley's theory of "isolating blocks"; see [Car, Con, Smo]. It has been shown that, for small positive ε, there exist c such that Z_c has a periodic orbit, yielding periodic traveling waves for (4.2). Also, for certain $c = c(\varepsilon)$, Z_c has been shown to have a homoclinic orbit, with $(0,0,0)$ as limit point. Such homoclinic orbits have been found numerically, with the aid of computer graphics, in [Rab]. The traveling wave arising from such a homoclinic orbit is called a *pulse*. (It follows from (4.45) that such a pulse cannot arise for scalar equations of the form (4.39) if $D > 0$.) There is a phenomenological interpretation when (4.2) is taken as a model of activity along the axon of a nerve. As seen above, a sufficiently small initial condition $\big(v_0(x), w_0(x)\big)$ produces a solution decaying to $(0,0)$ at $t \to \infty$. This traveling wave then arises from a sufficiently large initial condition. One says a "threshold behavior" is involved.

For a variant of the Fitzhugh–Nagumo system proposed by H. McKean, [Wan] has established the existence of "multiple impulse" traveling wave solutions.

An interesting question is the following: For given initial data, when can you say that the solution $u(t)$ behaves for large t like a traveling wave? For the Kolmogorov–Petrovskii–Piskunov equation, work has been done on this question in [KPP] and [McK]. For other work, see [AW1, AW2, Bram, Fi].

If $M = \mathbb{R}^n, n > 1$, and $L = D\Delta$, one can seek a solution to (4.1) in the form of a traveling plane wave, $u(t,x) = \varphi(x \cdot \omega - ct)$, where $\omega \in \mathbb{R}^n$ is a unit vector. Again $\varphi(s)$ satisfies the ODE (4.37). In addition to plane waves, other interesting sorts arise in the multidimensional case, including "spiral waves" and "scroll waves." We won't go into these here; see [Grin] for an introductory account.

Let us return to the evolution of small initial data f. Recall the argument that, for $\sup |f(x)|$ sufficiently small, a solution to the Fitzhugh–Nagumo system (4.2) decays uniformly to 0. For that argument, we used more than the fact that $(0,0)$ is a sink for the vector field X in that case; we also used a family of contracting rectangles. It turns out that, for a general reaction-diffusion equation (4.1) for which X has a sink at $p \in \mathbb{R}^\ell$, specifying that $f(x)$ be uniformly close to p does *not* necessarily lead to a solution $u(t)$ tending to p as $t \to \infty$. One can have the phenomenon of "diffusion-driven instability," or a "Turing instability," which we now describe. For simplicity, let us assume $L = D\Delta$ with $D = \operatorname{diag}(d_1, \dots, d_\ell)$, where Δ is the Laplace operator (acting componentwise) on an ℓ-tuple of functions on a compact manifold M.

We first give examples of this instability when X is a *linear* vector field, $X(u) = Mu$, so that $Lu + X(u) = (L + M)u$ is a linear operator. If $\{f_j\}$ is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of Δ, satisfying $\Delta f_j = -\alpha_j^2 f_j$, then $Lu + Mu$ satisfies

$$(L + M)(yf_j) = \big(-\alpha_j^2 Dy + My\big) f_j, \quad y \in \mathbb{R}^\ell. \tag{4.49}$$

Now, under the hypothesis that $0 \in \mathbb{R}^\ell$ is a sink for X, we have that both of the $\ell \times \ell$ matrices $-\alpha_j^2 D$ and M have all their eigenvalues in the left half-plane. All there remains to the construction is the realization that if two matrices have this property but *do not commute*, then their sum need not have this property. Consider the following 2×2 case:

$$D = \begin{pmatrix} 1 & \\ & d \end{pmatrix}, \quad M = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix}. \tag{4.50}$$

Assume $0 < b < 1 + a^2$, $a > 0$. Thus $\operatorname{Tr} M = b - (1 + a^2) < 0$, while $\det M = a^2 > 0$, so M has spectrum in the left half-plane. As before, assume $d > 0$. With $\lambda = \alpha_j^2$, consider

$$N = M - \lambda D = \begin{pmatrix} b - 1 - \lambda & a^2 \\ -b & -a^2 - \lambda d \end{pmatrix}. \tag{4.51}$$

Of course, $\operatorname{Tr} N < 0$ if $\lambda > 0$; meanwhile,

$$\det N = d\lambda^2 + \big[a^2 + d(1-b)\big]\lambda + a^2 = p(\lambda). \tag{4.52}$$

The matrix N will fail to have spectrum in the left half-plane, for some $\lambda > 0$, if $p(\lambda)$ is not always > 0 for $\lambda > 0$, hence if $p(\lambda)$ has a positive root. From the quadratic formula, the roots of $p(\lambda)$ are

$$r_\pm = -\frac{a^2 + d(1-b)}{2d} \pm \frac{1}{2d}\sqrt{\big[a^2 + d(1-b)\big]^2 - 4a^2 d}. \tag{4.53}$$

Thus $p(\lambda)$ will have positive roots if and only if $a^2 + d(1-b) < 0$ and $\big[a^2 + d(1-b)\big]^2 > 4a^2 d$. Recalling the conditions on a and b to make $\operatorname{Tr} M < 0$, we have the following requirements on the positive numbers b, a, d:

$$b - 1 < a^2 < (b-1)d, \quad 2(b+1)a^2 d < (b-1)^2 d^2 + a^4, \tag{4.54}$$

which also requires $b > 1, d > 1$. For example, we could choose $b = 2, a^2 = 2$, and $d = 20$, yielding

$$D = \begin{pmatrix} 1 & \\ & 20 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}. \tag{4.55}$$

Under these circumstances, $M - \lambda D$ will have a positive eigenvalue for

$$\lambda \in (r_-, r_+) \subset (0, \infty), \tag{4.56}$$

where $r_\pm$ are given by (4.53).

Consequently, if the Laplace operator Δ on M has an eigenvalue $-\alpha_j^2$ whose negative is in the interval (4.56), arbitrarily small initial data of the form yf_j will be magnified exponentially by the solution operator to $u_t = (L + M)u$, provided y has a nonzero component in the positive eigenspace of $M - \alpha_j^2 D$, despite the fact that the origin is a stable equilibrium for the evolution if the diffusion term is omitted.

An example of a nonlinear reaction-diffusion equation that exhibits this phenomenon is the "Brusselator,"

$$\begin{aligned} \frac{\partial v}{\partial t} &= \Delta v + v^2 w - (b+1)v + a, \\ \frac{\partial w}{\partial t} &= d\Delta w - v^2 w + bv, \end{aligned} \tag{4.57}$$

governing a certain system of chemical reactions. We assume $a, b, d > 0$. The vector field X (which incidentally has flow leaving invariant the quadrant $v, w \geq 0$) has a critical point at $(a, b/a)$, and its linearization at this critical point is given by the matrix M in (4.50). Thus if $u_0 = (v_0, w_0)$ is a small perturbation of the constant state $(a, b/a)$, if the estimates (4.54) hold, and if Δ has an eigenvalue whose negative is in the range (4.56), then a vector multiple of the eigenfunction f_j will be amplified by the evolution (4.57). Of course, once this acquires appreciable size, nonlinear effects take over. In some cases, a spatial pattern emerges, reflecting the behavior of the eigenfunction $f_j(x)$. One then has the phenomenon of "pattern formation."

In light of the instability just mentioned, we see some limitations on using invariant rectangular regions to obtain estimates. Consider the following more general type of Fitzhugh–Nagumo system:

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + f(v) + a(v,w), \quad \frac{\partial w}{\partial t} = b(v,w), \tag{4.58}$$

where f, a, and b are assumed to be smooth and satisfy

$$|a(v,w)| \le A\big(|v| + |w| + 1\big), \quad |b(v,w)| \le B\big(|v| + |w| + 1\big), \tag{4.59}$$

and

$$f(v) \le C_1 v, \text{ for } v \ge C_2, \quad f(v) \ge C_1 v, \text{ for } v \le -C_2, \tag{4.60}$$

where A, B, C_1, and C_2 are positive constants. There need be no large invariant rectangles in such a case, as the example $f(v) = (2A + 2B + 1)v$ shows. Nevertheless, one will have global solutions to (4.58) with data in $\mathcal{BC}^1(\mathbb{R}, \mathbb{R}^2)$. In fact, this is a special case of the following result.

To state the result, we use the following family of rectangular solids. Let $\mathfrak{Q}(s)$ be the cube in $\mathbb{R}^\ell$ centered at 0, with volume $(2s)^\ell$, and let $\mathfrak{F}_{\pm j}(s)$ be the face of this cube whose outward normal is $\pm e_j$, where $\{e_j : 1 \le j \le \ell\}$ is the standard basis of $\mathbb{R}^\ell$.

Proposition 4.6. *Let $L = D\Delta$ with $D = diag(d_1, \dots, d_\ell)$ in (4.1). Assume the components X_j of X satisfy, for some $C_0 \in (0,\infty)$,*

$$\begin{aligned} X_j(y) &\le +C_0 s \textit{ for } y \in \mathfrak{F}_{+j}(s), \\ X_j(y) &\ge -C_0 s \textit{ for } y \in \mathfrak{F}_{-j}(s), \end{aligned} \tag{4.61}$$

for all $s \ge C_2$. Then (4.1) has a global solution, for any $f \in \mathcal{BC}^1(\mathbb{R}^n, \mathbb{R}^\ell)$.

Proof. We will obtain this as a consequence of (4.33). Use the norm $\|y\| = \max_j |y_j|$ on $\mathbb{R}^\ell$ to construct the norm on function spaces. The hypothesis implies

$$\|\mathcal{F}^t_X y\| \le e^{C_0 t}\, \|y\|, \quad t \ge 0, \tag{4.62}$$

whenever $\|y\| \geq C_2$. Consequently,

$$\left\| \left(e^{(t/n)L} \mathcal{F}^{t/n} \right)^n f \right\|_{L^\infty} \leq e^{C_0 t} \Big(\|f\|_{L^\infty} + C_2 \Big), \tag{4.63}$$

so by (4.33) we have the bound on the solution to (4.1):

$$\|u(t)\|_{L^\infty} \leq e^{C_0 t} \Big(\|f\|_{L^\infty} + C_2 \Big). \tag{4.64}$$

Note that this is an application of Proposition 4.4, in a case where $\{K_s\}$ is an *increasing* family of rectangular solids. Other proofs of global existence for (4.58) under the hypotheses (4.59)–(4.60) are given in [Rau] and [Rot].

There are some widely studied reaction-diffusion equations to which Proposition 4.6 does not apply, but for which global existence can nevertheless be established. For example, the following models the progress of an epidemic, where v is the density of individuals susceptible to a disease and w is the density of infective individuals:

$$\frac{\partial v}{\partial t} = -rvw, \quad \frac{\partial w}{\partial t} = D\Delta w + rvw - aw. \tag{4.65}$$

Assume $r, a, D > 0$. In this model, only the sick individuals wander about. Let's suppose Δ is the Laplace operator on a compact two-dimensional manifold (e.g., the surface of a planet). One can see that the domain $u, v \geq 0$ is invariant; initial data for (4.65) should take values in this domain. We might consider squares of side s, whose bottom and left sides lie on the axes, but the analogue of (4.61) fails for $X_2 = rvw - aw$, though of course $X_1 \leq 0$ is fine. To get a good estimate on a short-time solution to (4.65), taking values in the first quadrant in $\mathbb{R}^2$, note that

$$\frac{\partial}{\partial t}\big(v + w\big) = D\Delta w - aw. \tag{4.66}$$

Integrating gives

$$\frac{d}{dt} \int_M (v + w)\, dV = -a \int_M w\, dV \leq 0. \tag{4.67}$$

By positivity, $\int (v + w)\, dV = \|v(t)\|_{L^1} + \|w(t)\|_{L^1}$, which is monotonically decreasing; hence both $\|v(t)\|_{L^1}$ and $\|w(t)\|_{L^1}$ are uniformly bounded. Of course, we have already noted that $\|v(t)\|_{L^\infty} \leq \|v_0\|_{L^\infty}$. Thus, inserting these bounds into the second equation in (4.65), we have

$$\frac{\partial w}{\partial t} = D\Delta w + g(t, x), \tag{4.68}$$

where

$$\|g(t)\|_{L^1(M)} \le r\|v(t)\|_{L^\infty}\|w(t)\|_{L^1} + a\|w(t)\|_{L^1} \le C. \tag{4.69}$$

Now use of

$$w(t) = e^{tD\Delta}w_0 + \int_0^t e^{(t-s)D\Delta}g(s)\,ds \tag{4.70}$$

plus the estimate

$$\|e^{s\Delta}\|_{\mathcal{L}(L^1,L^p)} \le C s^{-(1-1/p)} \tag{4.71}$$

when dim $M = 2$, yields an L^p-estimate on $w(t)$, and another application of (4.70) then yields an L^∞-estimate on $w(t)$, hence global existence.

Actually, for a complete argument, we should replace vw in the two parts of (4.65) by $\beta_\nu(vw)$, where $\beta_\nu(s) = s$ for $|s| \le \nu$, and $\beta_\nu(s) = \nu+1$ for $s \ge \nu+1$, get global solvability for such PDE, with L^∞-estimates, and take $\nu \to \infty$. We leave the details to the reader.

The exercises below contain some other examples of global existence results. In [Rot] there are treatments of global existence for a number of interesting reaction-diffusion equations, via methods that vary from case to case.

Exercises

1. Establish the following analogue of Proposition 4.6:

   Proposition 4.6A. *Let $L = D\Delta$ with $D = \mathrm{diag}(d_1,\dots,d_\ell)$ in (4.1). Assume that the set $\mathfrak{C}^+ = \{y \in \mathbb{R}^\ell : \text{each } y_j \ge 0\}$ is invariant under the flow generated by X. If $\mathfrak{F}_{+j}(s)$ is as in Proposition 4.6, set $\mathfrak{F}_j^+(s) = \mathfrak{C}^+ \cap \mathfrak{F}_{+j}(s)$, and assume that each component X_j of X satisfies*

   $$X_j(y) \le C_0 s, \quad \text{for } y \in \mathfrak{F}_j^+(s),$$

   for all $s \ge C_2$. Then (4.1) has a global solution, for any $f \in \mathcal{BC}^1(\mathbb{R}^n,\mathbb{R}^\ell)$ taking values in the set $\mathfrak{C}^+$.

2. The following is called the *Belousov–Zhabotinski system*. It models certain chemical reactions, exhibiting remarkable properties:

   $$\begin{aligned}\frac{\partial v}{\partial t} &= \Delta v + v(1-v-rw) + Lrw,\\ \frac{\partial w}{\partial t} &= \Delta w - bvw - Mw.\end{aligned} \tag{4.72}$$

   Assume $r, L, b, M > 0$. Show that the vector field X has flow that leaves invariant the quadrant $\{v, w \ge 0\}$. Show that Proposition 4.6A applies to yield a global solvability result.

3. The following system models a predator-prey interaction:

$$\begin{aligned}\frac{\partial v}{\partial t} &= D\Delta v + v(1-v-w),\\ \frac{\partial w}{\partial t} &= \Delta w + aw(v-b),\end{aligned} \tag{4.73}$$

where v is the density of prey, w the density of predators. Assume $D \geq 0, a > 0$, and $0 < b < 1$. Show that the vector field X has a flow that leaves invariant the quadrant $\{v, w \geq 0\} = \mathfrak{C}^+$. Show that Proposition 4.6A does *not* apply to this system.
Demonstrate global existence of solutions to this system, for initial data taking values in the set $\mathfrak{C}^+$. (*Hint*: Start with the identity

$$\frac{\partial}{\partial t}\Big(v + \frac{w}{a}\Big) = D\Delta v + \frac{1}{a}\Delta w + v - v^2 - bw \tag{4.74}$$

and integrate, to obtain L^1-bounds. Also use

$$\frac{\partial v}{\partial t} - D\Delta v \leq v \tag{4.75}$$

to obtain an L^∞-bound on v. (If $D > 0$, recall Lemma 2.2.) Then pursue stronger bounds on w.)

4. If the model (4.65) of an epidemic is extended to cases where susceptible and infective populations both diffuse, we have

$$\frac{\partial v}{\partial t} = D_1\Delta v - rvw, \quad \frac{\partial w}{\partial t} = D\Delta w + rvw - aw, \tag{4.76}$$

where $D_1, D, r, a > 0$. Establish global solvability for this system, for initial data taking values in $\mathfrak{C}^+$.

5. Study global solvability for the Brusselator system (4.57), given initial data with values in $\mathfrak{C}^+$. (*Hint*: After getting an L^1-bound on $v + w$, use

$$\frac{\partial w}{\partial t} - d\Delta w \leq bv,$$

and an appropriately modified version of the argument suggested for Exercise 3, to get a stronger bound on w. Once you have this, use

$$\frac{\partial}{\partial t}(v+w) - \Delta\big(v+w\big) = (d-1)\Delta w - v + a \tag{4.77}$$

to obtain a stronger bound on $v + w$.)

6. Consider the following system, modeling a chemical reaction $A + B \leftrightharpoons C$:

$$\begin{aligned}a_t - D_1\Delta a &= c - ab,\\ b_t - D_2\Delta b &= c - ab,\\ c_t - D_3\Delta c &= ab - c.\end{aligned} \tag{4.78}$$

Note that X leaves invariant the octant $\mathfrak{C}^+ = \{a, b, c \geq 0\}$. Assume $D_j > 0$. Establish the global solvability of solutions with initial data in $\mathfrak{C}^+$. Assume Δ is the Laplace operator on M, compact, with $\dim M \le 3$. (*Hint*: First get L^1-bounds on $a + c$ and $b + c$. Then use

$$a_t - D_1 \Delta a \le c, \quad b_t - D_2 \Delta b \le c,$$

to get L^p-bounds on a and b, for some $p > 2$ (if $\dim M \le 3$). Then use

$$c_t - D_3 \Delta c \le ab$$

to get L^p-bounds on c. Continue. Alternatively, apply an argument parallel to (4.77) to $a + c$, and relax the requirement on $\dim M$.)
For a treatment that works for $\dim M \le 5$ (and $\partial M \neq \emptyset$), see [Rot].

7. Extend results of this section to the case where $L = D\Delta$, where D is a diagonal matrix, $D = \mathrm{diag}(d_1, \dots, d_\ell)$, and Δ is the Laplace operator on a compact manifold $\overline{M}$ with boundary. Consider each of the following boundary conditions:

$$\begin{aligned} &\text{(a) Dirichlet,} && u_j\big|_{\mathbb{R}^+ \times \partial M} = 0, \\ &\text{(b) Neumann,} && \partial_\nu u_j\big|_{\mathbb{R}^+ \times \partial M} = 0, \\ &\text{(c) Robin,} && \partial_\nu u_j - a_j(x) u_j\big|_{\mathbb{R}^+ \times \partial M} = 0. \end{aligned}$$

Apply such a boundary condition only if $d_j > 0$.
Also, consider nonhomogeneous boundary conditions.

5. A nonlinear Trotter product formula

In this section we discuss an approach to approximating the solutions to nonlinear parabolic equations of the form

$$\frac{\partial u}{\partial t} = Lu + X(u), \quad u(0) = f, \tag{5.1}$$

and some generalizations, to be mentioned below, by a process involving successively solving the two simpler equations

$$\frac{\partial u}{\partial t} = Lu, \quad \frac{\partial u}{\partial t} = X(u), \tag{5.2}$$

over small time intervals, and composing the resulting solution operators. If $\mathcal{F}^t$ denotes the nonlinear evolution operator solving the equation $\partial u / \partial t = X(u)$, we seek to show that the solution to (5.1) satisfies

$$u(t) = \lim_{n \to \infty} \left(e^{(t/n)L} \mathcal{F}^{t/n} \right)^n (f). \tag{5.3}$$

This is a nonlinear analogue of the Trotter product formula, discussed in Appendix A of Chap. 11. It is a popular tool in the numerical study of nonlinear evolution equations, where it is also called the "splitting method."

We will tackle this via a variant of the analysis used in (A.17)–(A.30) in our treatment of the Trotter product formula in Chap. 11. The author came to understand this approach through conversations with J. T. Beale on the work [BG]. Other approaches are discussed in [CHMM].

We begin by setting

$$v_k = \left(e^{(1/n)L}\mathcal{F}^{1/n}\right)^k(f) \tag{5.4}$$

and then set

$$v(t) = e^{sL}\mathcal{F}^s v_k, \quad \text{for } t = \frac{k}{n} + s, \quad 0 \le s < \frac{1}{n}. \tag{5.5}$$

Then (under suitable hypotheses on L, etc.) $v(t) \to v_{k+1}$ as $t \nearrow (k+1)/n$, and, for $k/n < t < (k+1)/n$,

$$\frac{\partial v}{\partial t} = Lv + e^{sL}X(\mathcal{F}^s v_k) = Lv + X(v) + R(t), \tag{5.6}$$

where, again for $t = k/n + s, \ 0 \le s < 1/n$,

$$\begin{aligned} R(t) &= e^{sL}X(\mathcal{F}^s v_k) - X(v) \\ &= \left(e^{sL} - I\right)X(\mathcal{F}^s v_k) + \left[X(\mathcal{F}^s v_k) - X(e^{sL}\mathcal{F}^s v_k)\right]. \end{aligned} \tag{5.7}$$

To compare $v(t)$ with the solution $u(t)$ to (5.1), set $w = v - u$. Subtracting (5.1) from (5.6) gives

$$\frac{\partial w}{\partial t} = Lw + X(v) - X(u) + R(t), \tag{5.8}$$

and if we write

$$X(v) - X(u) = \int_0^1 DX\left(sv + (1-s)u\right)\, ds = Y(u,v)w, \tag{5.9}$$

we have for w the linear PDE

$$\frac{\partial w}{\partial t} = Lw + A(t,x)w + R(t), \quad w(0) = 0, \tag{5.10}$$

where

$$A(t,x) = Y\left(u(t,x), v(t,x)\right), \tag{5.11}$$

which is an $\ell \times \ell$ matrix function if (5.1) is an $\ell \times \ell$ system.

We will treat this in a fashion similar to (A.24)–(A.28) in Chap. 11. To recall some points that arose there, we expect to show that $R(t)$ is small only in a *weaker* norm than that used to measure the size of $v-u$. In our first set of results, we compensate by exploiting smoothing properties of e^{tL}. Thus, for now we assume that L is a negative-semidefinite, second-order, elliptic differential operator. For the sake of definiteness, let us suppose L acts on (ℓ-tuples of) functions on $\mathbb{R}^n$, with domain

$$\mathcal{D}(L) = H^2(\mathbb{R}^n). \tag{5.12}$$

We will seek to estimate $v-u$ in some V-norm.

Now, by Duhamel's principle,

$$w(t) = \int_0^t e^{(t-\tau)L}\big[A(\tau)w(\tau) + R(\tau)\big]\, d\tau. \tag{5.13}$$

Pick $T > 0,\ \gamma \in (0,1)$, and Banach spaces of functions V, W for which we have an estimate of the form

$$\|e^{tL}g\|_V \le C\, t^{-\gamma}\, \|g\|_W, \quad 0 < t \le T. \tag{5.14}$$

The next step is to estimate the W-norm of $R(t)$, given by (5.7). We have separated this into two parts:

$$R_1(t) = (e^{sL} - I)X(\mathcal{F}^s v_k), \quad R_2(t) = X(\mathcal{F}^s v_k) - X(e^{sL}\mathcal{F}^s v_k), \tag{5.15}$$

where $t = k/n + s,\ 0 \le s < 1/n$. Parallel to (A.26), we need an estimate of the form

$$\|e^{sL} - I\|_{\mathcal{L}(V,W)} \le C\, s^\delta, \quad \delta > 0, \tag{5.16}$$

to estimate $R_1(t)$. Of course, this requires that W have a weaker topology than V. Granted this estimate, since $s \in [0, 1/n]$ in (5.15), we obtain

$$\|R_1(t)\|_W \le C\, n^{-\delta}\, \|\Phi_1(t)\|_V, \tag{5.17}$$

where

$$\Phi_1(t,x) = X\big(\Psi_1(t,x)\big), \tag{5.18}$$

with

$$\Psi_1(t,x) = \mathcal{F}^s v_k(x) = \mathcal{F}^s_X\big(v_k(x)\big), \tag{5.19}$$

where $\mathcal{F}_X^s$ is the flow on $\mathbb{R}^\ell$ generated by X, a vector field on $\mathbb{R}^\ell$. Meanwhile,

$$R_2(t) = \Phi_1(t) - \Phi_2(t), \tag{5.20}$$

where $\Phi_1(t)$ is as in (5.18)–(5.19) and

$$\Phi_2(t,x) = X\big(\Psi_2(t,x)\big), \quad \Psi_2(t) = e^{sL}\Psi_1(t). \tag{5.21}$$

Thus, with $Y(u,v)$ given by (5.9),

$$R_2(t) = Z(t)(I - e^{sL})\Psi_1(t), \quad Z(t) = Y\big(\Psi_1(t), \Psi_2(t)\big), \tag{5.22}$$

so, again using (5.16), we have

$$\|R_2(t)\|_W \le C\, n^{-\delta}\, \|Z(t)\|_{\mathcal{L}(W)}\, \|\Psi_1(t)\|_V, \tag{5.23}$$

where $Z(t)$ denotes the operation of left multiplication by the matrix-valued function $Z(t,x)$.

There remains the task of estimating the right sides of (5.17) and (5.23). This involves estimating the V-norms of

$$\begin{aligned} v_k &= \Big(e^{(1/n)L}\mathcal{F}^{1/n}\Big)^k f, \qquad \Psi_1(t) = \mathcal{F}^s v_k, \\ \Psi_2(t) &= e^{sL}\Psi_1(t), \qquad \Phi_j(t) = X\big(\Psi_j(t)\big), \end{aligned} \tag{5.24}$$

and the $\mathcal{L}(W)$-norm of $Z(t) = Y\big(\Phi_1(t), \Phi_2(t)\big)$. Thus, we want the estimates

$$\|e^{tL}\|_{\mathcal{L}(V)} \le e^{ct}, \quad \|\mathcal{F}^t(f)\|_V \le e^{ct}\|f\|_V, \quad 0 \le t \le T, \tag{5.25}$$

rather than weaker estimates in which e^{ct} is replaced by C_1e^{ct}. On most of our favorite Banach spaces V, e^{tL} is frequently a contraction semigroup, while the second estimate in (5.25) may require more work to establish. Actually, we need this second estimate only for $\|f\|_V \ge$ some constant C_1. We get a good estimate on all the quantities in (5.24) if the estimates (5.25) hold and also

$$\mathfrak{X} : V \longrightarrow V \text{ is bounded,} \tag{5.26}$$

where $\mathfrak{X}f(x) = X(f(x))$. By (5.26) we mean that $\mathfrak{X}(S)$ is bounded in V whenever S is a bounded subset of V. In most examples, $\mathfrak{X}$ will be locally Lipschitz, which is more than sufficient. Granted these hypotheses, to get a good bound on $Z(t)$ it suffices to have that

$$\mathfrak{Y} : V \times V \longrightarrow \mathcal{L}(W) \text{ is bounded,} \tag{5.27}$$

where $\mathfrak{Y}(f,g)(x) = Y(f(x), g(x))$. Let us summarize this analysis:

Proposition 5.1. *Let V and W be Banach spaces of (ℓ-tuples of) functions for which e^{tL} satisfies the estimates*

$$\|e^{tL}\|_{\mathcal{L}(V)} \le e^{ct}, \quad \|e^{tL}\|_{\mathcal{L}(W,V)} \le Ct^{-\gamma}, \quad \|e^{tL} - I\|_{\mathcal{L}(V,W)} \le Ct^{\delta}, \tag{5.28}$$

for $0 < t \le T$, with some $\delta > 0, 0 < \gamma < 1$. Let X be a vector field, generating a flow $\mathcal{F}_X^t$ on $\mathbb{R}^\ell$, satisfying

$$\|\mathcal{F}^t(f)\|_V \le C_2, \ \ \text{for } \|f\|_V \le C_1, \tag{5.29}$$

and, for $\|f\|_V \ge C_1$,

$$\|\mathcal{F}^t(f)\|_V \le e^{ct}\|f\|_V, \tag{5.30}$$

for $0 \le t \le T$, where $\mathcal{F}^t f(x) = \mathcal{F}_X^t(f(x))$. Assume also that

$$\mathfrak{X} : V \to V \text{ and } \mathfrak{Y} : V \times V \to \mathcal{L}(W) \cap \mathcal{L}(V) \text{ are bounded,} \tag{5.31}$$

where $\mathfrak{X} f(x) = X(f(x))$ and

$$\mathfrak{Y}(f, g)(x) = Y\big(f(x), g(x)\big) = \int_0^1 DX\big(sg(x) + (1-s)f(x)\big)\,ds.$$

If $f \in V$, $u \in C\big([0,T], V\big)$ is a solution to (5.1), and $v \in C\big([0,T], V\big)$ is defined by (5.4)–(5.5), then, for $0 \le t \le T$,

$$\|v(t) - u(t)\|_V \le C\big(\|f\|_V\big) \cdot n^{-\delta}. \tag{5.32}$$

Proof. The hypothesis (5.31) also yields an $\mathcal{L}(V)$-bound on $A(\tau)$ in (5.13), so we have

$$\|w(t)\|_V \le A\int_0^t e^{c(t-\tau)}\|w(\tau)\|_V\,d\tau + \|F(t)\|_V, \tag{5.33}$$

where A is a constant and

$$F(t) = \int_0^t e^{(t-\tau)L}R(\tau)\,d\tau.$$

From the hypotheses (5.28)–(5.31) and the consequent estimates in (5.17) and (5.23), we have

$$\|F(t)\|_V \le C\big(\|f\|_V\big)\int_0^t (t-\tau)^{-\gamma}\,d\tau \cdot n^{-\delta} = B\big(\|f\|_V\big) \cdot n^{-\delta}t^{1-\gamma}. \tag{5.34}$$

Thus $\beta(t) = \|w(t)\|_V$ satisfies the integral inequality

$$\beta(t) \le A \int_0^t e^{c(t-\tau)} \beta(\tau)\, d\tau + B \cdot n^{-\delta} t^{1-\gamma}, \quad \beta(0) = 0, \tag{5.35}$$

where A and B depend on $\|f\|_V$. The conclusion (5.32) follows from Gronwall's inequality.

As one simple example of useful Banach spaces, we consider

$$V = \mathcal{BC}^1(\mathbb{R}^n), \quad W = \mathcal{BC}^0(\mathbb{R}^n), \tag{5.36}$$

where, as in (4.14), $\mathcal{BC}^k(\mathbb{R}^n)$ denotes the space of functions whose derivatives of order $\le k$ are bounded and continuous on $\mathbb{R}^n$ and extend continuously to the compactification $\widehat{\mathbb{R}^n}$ via the sphere at infinity (approached radially). This is a subspace of the space $BC^k(\mathbb{R}^n)$, consisting of functions whose derivatives of order $\le k$ are bounded and continuous on $\mathbb{R}^n$. We want the functions to take values in $\mathbb{R}^\ell$, but we suppress that in the notation. Suppose L is a constant-coefficient, second-order, elliptic, self-adjoint operator. If (5.1) is an $\ell \times \ell$ system, let us hypothesize the invariance property (4.4), so, with an appropriate norm on $\mathbb{R}^\ell$, we have that e^{tL} is a contraction semigroup on V (and on W). Furthermore, the other estimates in (5.28) hold in this case, with $\gamma = \delta = 1/2$.

To investigate the estimate (5.30), we have

$$\begin{aligned} \|\mathcal{F}^t f\|_{\mathcal{BC}^1} &= \|\mathcal{F}^t f\|_{L^\infty} + \|D(\mathcal{F}^t f)\|_{L^\infty} \\ &= \sup_x \big\|\mathcal{F}^t_X\big(f(x)\big)\big\|_{\mathbb{R}^\ell} + \sup_x \big\|D\mathcal{F}^t(f) \circ Df(x)\big\|_{\mathbb{R}^\ell}. \end{aligned} \tag{5.37}$$

Thus (5.30) holds, provided

$$\|\mathcal{F}^t_X(y)\|_{\mathbb{R}^\ell} \le e^{ct} \|y\|_{\mathbb{R}^\ell}, \quad 0 \le t \le T, \tag{5.38}$$

and

$$\|D\mathcal{F}^t_X(y)\|_{\mathcal{L}(\mathbb{R}^\ell)} \le e^{ct}, \quad 0 \le t \le T. \tag{5.39}$$

As shown in § 6 of Chap. 1, $D\mathcal{F}^t_X(y) = \mathcal{G}^t(y)$ satisfies the linear ODE

$$\frac{d}{dt} \mathcal{G}^t(y) = DX\big(\mathcal{F}^t_X(y)\big)\mathcal{G}^t(y), \quad \mathcal{G}^0(y) = I. \tag{5.40}$$

Consequently, it is clear that (5.38) and (5.39) hold as long as X is a C^1-vector field on $\mathbb{R}^\ell$, satisfying

$$\|X(y)\|_{\mathbb{R}^\ell} \le c, \quad \|DX(y)\|_{\mathcal{L}(\mathbb{R}^\ell)} \le c. \tag{5.41}$$

These conditions are also enough to yield the boundedness of the maps $\mathfrak{X} : V \to V$ and $\mathfrak{Y} : V \times V \to \mathcal{L}(W)$, in (5.31), but in order to have boundedness of $\mathfrak{Y} : V \times V \to \mathcal{L}(V)$, we need C^1-bounds on $Y(f, g)$, hence C^1-bounds on DX. In other words, we need $X \in BC^2(\mathbb{R}^\ell)$. We have the following result.

Proposition 5.2. *Let $u \in C\big([0,T], \mathcal{BC}^1(\mathbb{R}^n)\big)$ solve (5.1), and let $v(t)$ be defined by (5.4)–(5.5). Assume that L is a constant-coefficient, second-order, elliptic operator, generating a contraction semigroup on $\mathcal{BC}^0(\mathbb{R}^n)$ and that X is a vector field on $\mathbb{R}^\ell$ with coefficients in $BC^2(\mathbb{R}^\ell)$. Then, for any bounded interval $t \in [0, T]$,*

$$\|u(t) - v(t)\|_{\mathcal{BC}^1} \le C\big(\|f\|_{\mathcal{BC}^1}\big) \cdot n^{-1/2}. \tag{5.42}$$

As another example of Banach spaces to which Proposition 5.1 applies, consider

$$V = H^k(\mathbb{R}^n), \quad W = H^{k-2\gamma}(\mathbb{R}^n), \quad k > \frac{n}{2}, \;\; 0 < \gamma < 1. \tag{5.43}$$

Assume $k \in \mathbb{Z}^+$. Then (5.28) holds, with $\delta = \gamma$. We have the Moser estimate

$$\|X(f)\|_{H^k} \le C_k\big(\|f\|_{L^\infty}\big) \cdot \Big(1 + \|f\|_{H^k}\Big), \tag{5.44}$$

where

$$C_k(\lambda) = C_k' \ \sup\big\{X^{(\mu)}(f) : |f| \le \lambda, |\mu| \le k\big\}. \tag{5.45}$$

Thus (5.31) is seen to hold as long as $X \in BC^k(\mathbb{R}^\ell)$. To see whether (5.30) holds, we estimate $(d/dt)\|\mathcal{F}^t f\|_{H^k}^2$, exploiting (5.44) to obtain

$$\begin{aligned} \frac{d}{dt}\|\mathcal{F}^t f\|_{H^k}^2 &= 2\big(X(\mathcal{F}^t f), \mathcal{F}^t f\big)_{H^k} \\ &\le C_k\big(\|\mathcal{F}^t f\|_{L^\infty}\big)\Big(\|\mathcal{F}^t f\|_{H^k} + \|\mathcal{F}^t f\|_{H^k}^2\Big). \end{aligned} \tag{5.46}$$

Now for $\|\mathcal{F}^t f\|_{H^k} \ge 1$, the right side is $\le 2C_k\big(\|\mathcal{F}^t f\|_{L^\infty}\big)\|\mathcal{F}^t\|_{H^k}^2$. If $X \in BC^k(\mathbb{R}^\ell)$, there is a bound on $2C_k\big(\|\mathcal{F}^t f\|_{L^\infty}\big)$ strong enough to yield (5.30). We have the following result:

Proposition 5.3. *Assume $k > n/2$ is an integer. Let $u \in C\big([0,T], H^k(\mathbb{R}^n)\big)$ solve (5.1), and let $v(t)$ be defined by (5.4)–(5.5). Assume that L is a constant-coefficient, second-order, elliptic operator, generating a contraction semigroup on $L^2(\mathbb{R}^n)$, and that X is a vector field on $\mathbb{R}^\ell$ with coefficients in $BC^k(\mathbb{R}^\ell)$. Then, for any bounded interval $t \in [0, T]$,*

$$\|u(t) - v(t)\|_{H^k} \le C\big(\|f\|_{H^k}\big) \cdot n^{-\gamma}, \tag{5.47}$$

for any $\gamma < 1$. *Furthermore, for any* $\varepsilon > 0$,

$$\|u(t) - v(t)\|_{H^{k-\varepsilon}} \le C_\varepsilon\big(\|f\|_{H^k}\big) \cdot n^{-1}. \tag{5.48}$$

It remains to establish (5.48). Indeed, if we set $\widetilde{W} = H^{k-2}(\mathbb{R}^n)$, we easily get $\|R(t)\|_{\widetilde{W}} \le Cn^{-1}$, via use of $\|e^{tL} - I\|_{\mathcal{L}(V,\widetilde{W})} \le ct$, instead of the last estimate of (5.28). Then we can use $\widetilde{V} = H^{k-\varepsilon}(\mathbb{R}^n)$, replacing the second estimate of (5.28) by $\|e^{tL}\|_{\mathcal{L}(\widetilde{W},\widetilde{V})} \le C_\varepsilon t^{-(1-\varepsilon/2)}$, and parallel the analysis in (5.33)–(5.35) to obtain (5.48).

It is desirable to have product formulas for which the existence of solutions to (5.1) is a *conclusion* rather than a hypothesis. Suppose that v, given by (5.4)–(5.5), is compared, not with the solution u to (5.1), but to the function $\widetilde{v}$, constructed by the same process as v, but using intervals of half the length. Thus, for an *integer* or *half-integer* k, define

$$\widetilde{v}_k = \Big(e^{(1/2n)L}\mathcal{F}^{1/2n}\Big)^{2k}(f), \tag{5.49}$$

and set

$$\widetilde{v}(t) = e^{sL}\mathcal{F}^s\widetilde{v}_k, \quad \text{for } t = \frac{k}{n} + s,\ 0 \le s \le \frac{1}{2n}. \tag{5.50}$$

Parallel to (5.6), we have

$$\frac{\partial \widetilde{v}}{\partial t} = L\widetilde{v} + X(\widetilde{v}) + \widetilde{R}(t), \tag{5.51}$$

where, for $t = k/n + s, 0 \le s < 1/2n$,

$$\widetilde{R}(t) = (e^{sL} - I)X(\mathcal{F}^s\widetilde{v}_k) + \big[X(\mathcal{F}^s\widetilde{v}_k) - X(e^{sL}\mathcal{F}^s\widetilde{v}_k)\big]. \tag{5.52}$$

Consequently, $\widetilde{w} = v - \widetilde{v}$ satisfies the PDE

$$\frac{\partial \widetilde{w}}{\partial t} = L\widetilde{w} + \widetilde{A}(t,x)\widetilde{w} + R(t) - \widetilde{R}(t), \quad \widetilde{w}(0) = 0, \tag{5.53}$$

where, parallel to (5.11),

$$\widetilde{A}(t,x) = Y\big(\widetilde{v}(t,x), v(t,x)\big). \tag{5.54}$$

Pick Banach spaces V and W as above, and assume $f \in V$. As long as the hypotheses (5.28)–(5.31) hold, we again have

$$\|R(t) - \widetilde{R}(t)\|_W \le Cn^{-\delta}, \quad 0 \le t \le T. \tag{5.55}$$

We also have bounds on $\|v(t)\|_V$ and $\|\widetilde{v}\|_V$, independent of n, hence bounds on $\widetilde{A}(t,x)$, so the analysis in (5.33)–(5.35) extends to yield

$$\|v(t)-\widetilde{v}(t)\|_V \le Cn^{-\delta}, \quad 0 \le t \le T. \tag{5.56}$$

Consequently, if we take $n = 2^j$ and denote v, defined by (5.4)–(5.5), by $u_{(j)}$, so $\widetilde{v}$ is logically denoted $u_{(j+1)}$, we have

$$\{u_{(j)} : j \in \mathbb{Z}^+\} \text{ is Cauchy in } C\big([0,T],V\big), \tag{5.57}$$

and the limit is seen to satisfy (5.1).

There are some unsatisfactory aspects of using the smoothing of e^{tL} that follows when L is elliptic. For example, Propositions 5.1–5.3 do not apply to the Fitzhugh–Nagumo system (4.2), since the operator L given by (4.3) is not elliptic. We now derive a convergence result that does not make use of such a hypothesis; the conclusion will be weaker, in that we get convergence in a weaker norm. We will establish the following variant of Proposition 5.1:

Proposition 5.4. *Let V and W be Banach spaces of (ℓ-tuples of) functions for which e^{tL} satisfies the estimates*

$$\|e^{tL}\|_{\mathcal{L}(V)} \le e^{ct}, \quad \|e^{tL}\|_{\mathcal{L}(W)} \le e^{ct}, \quad \|e^{tL}-I\|_{\mathcal{L}(V,W)} \le Ct^{\delta}, \tag{5.58}$$

for $0 < t \le T$, with some $\delta > 0$. Let X be a vector field on $\mathbb{R}^\ell$, generating a flow $\mathcal{F}_X^t$, whose action on functions via $\mathcal{F}^t f(x) = \mathcal{F}_X^t\big(f(x)\big)$ satisfies (5.29)–(5.31). Take $f \in V$. Then (5.1) has a solution $u \in C\big([0,T],W\big)$, and the function $v \in C\big([0,T],V\big)$ given by (5.4)–(5.5) satisfies

$$\|v(t)-u(t)\|_W \le Cn^{-\delta}, \quad 0 \le t \le T. \tag{5.59}$$

Proof. If v and $\widetilde{v}$ are defined by (5.4)–(5.5) and by (5.49)–(5.50), we will show that

$$\sup_{0\le t\le T} \|v(t)\|_V \le B, \tag{5.60}$$

with B independent of n, and that

$$\|v(t)-\widetilde{v}(t)\|_W \le Cn^{-\delta}, \quad 0 \le t \le T. \tag{5.61}$$

In fact, the hypotheses (5.58) together with (5.29)–(5.30) immediately yields (5.60). If we also have (5.31), then there is the estimate

$$\|R(t)\|_W \le Cn^{-\delta}, \quad \|\widetilde{R}(t)\|_W \le Cn^{-\delta}, \tag{5.62}$$

established just as before. Again, $\widetilde{w} = v - \widetilde{v}$ solves the PDE (5.53), and hence, parallel to (5.33)–(5.34), we have

$$\|\widetilde{w}(t)\|_W \le A \int_0^t e^{c(t-\tau)} \|\widetilde{w}(\tau)\|_W \, d\tau + C' n^{-\delta}, \quad \|\widetilde{w}(0)\|_W = 0. \tag{5.63}$$

Thus Gronwall's inequality yields (5.61), and the proposition follows.

Note that in Proposition 5.4, we can weaken the hypothesis (5.31) to

$$\mathfrak{X} : V \to V \text{ and } \mathfrak{Y} : V \times V \to \mathcal{L}(W) \text{ are bounded,} \tag{5.64}$$

omitting mention of $\mathcal{L}(V)$. Let us also note that the limit function $u \in C\big([0,T], W\big)$ also satisfies

$$u \in L^\infty\big([0,T], V\big), \tag{5.65}$$

provided V is reflexive.

Proposition 5.4 essentially applies to the Fitzhugh–Nagumo system (4.2), which we recall:

$$\begin{aligned} \frac{\partial v}{\partial t} &= D \frac{\partial^2 v}{\partial x^2} + f(v) - w, \\ \frac{\partial w}{\partial t} &= \varepsilon(v - \gamma w). \end{aligned} \tag{5.66}$$

As we did in § 4, we modify the vector field $X(v, w) = \big(f(v) - w, \varepsilon(v - \gamma w)\big)$ outside some compact set to keep its components and sufficiently many of their derivatives bounded.

Exercises

1. Investigate Strang's splitting method:

$$u(t) = \lim_{n \to \infty} \Big(\mathcal{F}^{t/2n} e^{(t/n)L} \mathcal{F}^{t/2n} \Big)^n (f).$$

   Obtain faster convergence than that given by (5.48) for the splitting method (5.3).
2. Write a computer program to solve numerically the Fitzhugh–Nagumo system (4.2), using the splitting method. Take $M = S^1$. Use (4.32) to specify the constants γ, a, and ε. Alternatively, take $\gamma = 10$. Try various values of D. Use the FFT to solve the linear PDE $\partial v/\partial t = D \partial_x^2 v$, and use a reasonable difference scheme to integrate the planar vector field X.

6. The Stefan problem

The *Stefan problem* models the melting of ice. We consider the problem in one space dimension. We assume that the point separating ice from water at time t is given by $x = s(t)$, with water, at a temperature $u(t, x) \ge 0$, on the left, and ice, at temperature 0, on the right. Let us also assume that the region $x \le 0$ is occupied by a solid maintained at temperature 0. In appropriate units, u and s satisfy the equations

$$
\begin{aligned}
u_t &= u_{xx}, & u(0,x) &= f(x), \\
u(t,0) &= 0, & u(t,s(t)) &= 0,
\end{aligned} \tag{6.1}
$$

where $0 \le x \le s(t)$ and

$$
\dot{s} = -a\, u_x(t,s(t)), \quad s(0) = 1. \tag{6.2}
$$

We suppose f is given, in $C^\infty(I)$, $I = [0,1]$, such that $f(x) \ge 0$ and $f(0) = f(1) = 0$. In (6.2), a is a positive constant.

It is convenient to change variables, setting $v(t,x) = u(t,s(t)x)$, for $0 \le x \le 1$. The equations then become

$$
\begin{gathered}
v_t = s(t)^{-2} v_{xx} + \frac{\dot{s}}{s} x\, v_x, \qquad v(0,x) = f(x), \\
v(t,0) = 0, \qquad v(t,1) = 0,
\end{gathered} \tag{6.3}
$$

and

$$
\dot{s} = -\frac{a}{s}\, v_x(t,1). \tag{6.4}
$$

Note that (6.4) is equivalent to $(d/dt)s^2 = -2a v_x(t,1)$, so if we set $\xi(t) = s(t)^2$, we can rewrite the system as

$$
v_t = \xi(t)^{-1} v_{xx} + \frac{1}{2}\frac{\dot{\xi}}{\xi} x v_x, \quad v(0,x) = f(x), \quad v(t,0) = v(t,1) = 0, \tag{6.5}
$$

$$
\dot{\xi}(t) = -2a\, v_x(t,1), \qquad \xi(0) = 1. \tag{6.6}
$$

Note that the system (6.5)–(6.6) is equivalent to the system of integral equations

$$
v(t) = e^{\alpha(t,0)\Delta} f + \int_0^t \beta(\tau) e^{\alpha(t,\tau)\Delta}\big(x v_x(\tau)\big)\, d\tau, \tag{6.7}
$$

$$
\xi(t) = 1 - 2a \int_0^t v_x(\tau,1)\, d\tau, \tag{6.8}
$$

where

$$
\alpha(t,\tau) = \int_\tau^t \frac{d\eta}{s(\eta)^2} = \int_\tau^t \frac{d\eta}{\xi(\eta)}, \quad \beta(\tau) = \frac{\dot{s}(\tau)}{s(\tau)} = \frac{1}{2}\frac{\dot{\xi}(\tau)}{\xi(\tau)}. \tag{6.9}
$$

Here, $e^{t\Delta}$ is the solution operator to the heat equation on $\mathbb{R}^+ \times I$, with Dirichlet boundary conditions at $x = 0,\ 1$.

We will construct a short-time solution as a limit of approximations as follows. Start with $\xi_0(t) = 1 - 2af'(1)t$. Solve (6.5) for $v_1(t,x)$, with $\xi = \xi_1$. Then set $\xi_1(t) = 1 - 2a\int_0^t \partial_x v_1(\tau,1)\,d\tau$. Now solve (6.5) for $v_2(t,x)$, with $\xi = \xi_1$. Then set $\xi_2(t) = 1 - 2a\int_0^t \partial_x v_2(\tau,1)\,d\tau$, and continue. Thus, when you have $\xi_j(t)$, solve for $v_{j+1}(t,x)$ the equation

$$\frac{\partial}{\partial t}v_{j+1} = \xi_j(t)^{-1}\,\partial_x^2 v_{j+1} + \frac{1}{2}\frac{\dot{\xi}_j}{\xi_j}x\,\partial_x v_{j+1}, \quad v_{j+1}(0,x) = f(x),$$
$$v_{j+1}(t,0) = v_{j+1}(t,1) = 0. \tag{6.10}$$

Then set

$$\xi_{j+1}(t) = 1 - 2a\int_0^t \partial_x v_{j+1}(\tau,1)\,d\tau. \tag{6.11}$$

Lemma 6.1. *Suppose $\xi_j(t)$ satisfies*

$$\xi_j(0) = 1, \quad \dot{\xi}_j(0) = -2af'(1), \quad \dot{\xi}_j \ge 0. \tag{6.12}$$

Then ξ_{j+1} also has these properties.

Proof. The first two properties are obvious from (6.11), which implies

$$\dot{\xi}_{j+1}(t) = -2a\,\partial_x v_{j+1}(t,1). \tag{6.13}$$

Furthermore, the maximum principle applied to (6.10) yields

$$v_{j+1}(t,x) \ge 0. \tag{6.14}$$

Since $v_{j+1}(t,1) = 0$, we must have $\partial_x v_{j+1}(t,1) \le 0$.

The PDE (6.10) for v_{j+1} is equivalent to

$$v_{j+1}(t) = e^{\alpha_j(t,0)\Delta}f + \int_0^t \beta_j(\tau)e^{\alpha_j(t,\tau)\Delta}\big(x\partial_x v_{j+1}(\tau)\big)\,d\tau, \tag{6.15}$$

where

$$\alpha_j(t,\tau) = \int_\tau^t \frac{d\eta}{\xi_j(\eta)}, \quad \beta_j(\tau) = \frac{1}{2}\frac{\dot{\xi}_j(\tau)}{\xi_j(\tau)}. \tag{6.16}$$

One way to analyze $e^{t\Delta}$ on functions on I is to construct S^1, the "double" of I, and use the identity

$$e^{t\Delta} g = \rho\, e^{t\Delta}(\mathcal{O} g), \tag{6.17}$$

where $\mathcal{O} g$ is the extension of $g \in L^2(I)$ to $\mathcal{O} g \in L^2(S^1)$, which is odd with respect to the natural involution on S^1 (i.e., the reflection across ∂I), and ρG is the restriction of $G \in L^2(S^1)$ to I. It is useful to note that

$$\mathcal{O} : C_b^r(I) \longrightarrow C^r(S^1), \quad \text{for } 0 \le r < 2, \tag{6.18}$$

where $C_b^r(I)$ is the subspace of $u \in C^r(I)$ such that $u(0) = u(1) = 0$. If $r = 1 + \mu,\ 0 < \mu < 1$, then $C^r(I) = C^{1,\mu}(I)$. Furthermore,

$$\mathcal{O} : C_b^{1,1}(I) \longrightarrow C^{1,1}(S^1). \tag{6.19}$$

It is useful to note that

$$e^{\alpha\Delta}(\partial_x g) = \partial_x e_N^{\alpha\Delta} g \quad \text{if } g \in C^1(I), \tag{6.20}$$

where $e_N^{\alpha\Delta}$ is the solution operator to the heat equation on $\mathbb{R}^+ \times I$, with Neumann boundary conditions, as can be seen by taking the even extension of g to S^1. Hence (6.15) can be written as

$$\begin{aligned} v_{j+1}(t) &= e^{\alpha_j(t,0)\Delta} f \\ &\qquad + \int_0^t \beta_j(\tau)\Big(\partial_x e_N^{\alpha_j(t,\tau)\Delta} M_x - e^{\alpha_j(t,\tau)\Delta}\Big) v_{j+1}(\tau)\, d\tau, \end{aligned} \tag{6.21}$$

where M_x is multiplication by x. In analogy with (6.17), we have

$$e_N^{t\Delta} g = \rho\, e^{t\Delta}(\mathcal{E} g), \tag{6.22}$$

where $\mathcal{E} g$ is the even extension of g to S^1. In place of (6.18)–(6.19), we have

$$\begin{aligned} \mathcal{E} &: C^r(I) \longrightarrow C^r(S^1), \qquad \text{for } 0 \le r < 1, \\ \mathcal{E} &: C^{0,1}(I) \longrightarrow C^{0,1}(S^1). \end{aligned} \tag{6.23}$$

We now look for estimates on v_{j+1} and ξ_{j+1}. The simplest is the uniform estimate

$$\|v_{j+1}(t)\|_{L^\infty} \le \|f\|_{L^\infty}, \tag{6.24}$$

which follows from the maximum principle. Other estimates can be derived using (6.15) and (6.21) together with such estimates as

$$\|e^{t\Delta} g\|_{C^r} \le C t^{-r/2} \|g\|_{L^\infty}, \quad \|e_N^{t\Delta} g\|_{C^r} \le C t^{-r/2} \|g\|_{L^\infty}, \tag{6.25}$$

valid for any $f \in L^\infty(I)$, any $r > 0$, $t \in (0,T]$, with $C = C(r,T)$. Hence, using (6.21), we obtain, for $0 < \mu < 1$, an estimate

$$\|v_{j+1}(t)\|_{C^\mu} \le \|e^{\alpha_j \Delta} f\|_{C^\mu} + A_{j\mu}(t)\|f\|_{L^\infty}, \tag{6.26}$$

where

$$A_{j\mu}(t) = A \int_0^t \beta_j(\tau)\alpha_j(t,\tau)^{-(1+\mu)/2}\, d\tau. \tag{6.27}$$

Now, by (6.16), $\alpha_j(t,\tau) \ge \xi_j(t)^{-1}(t-\tau)$, granted that $\dot{\xi}_j \ge 0$, so

$$\begin{aligned} A_{j\mu}(t) &\le A\, \xi_j(t)^{(1+\mu)/2} \int_0^t \frac{\dot{\xi}_j(\tau)}{\xi_j(\tau)}\, (t-\tau)^{-(1+\mu)/2}\, d\tau \\ &\le B\, \xi_j(t)^{(1+\mu)/2} \Big[\sup_{0\le\tau\le t} \dot{\xi}_j(\tau) \Big]\, t^{(1-\mu)/2}. \end{aligned} \tag{6.28}$$

Now we can apply (6.21) again to obtain, for $0<r<1,\ 0<\mu<1,\ \mu+r\neq 1$,

$$\|v_{j+1}(t)\|_{C^{\mu+r}} \le \|e^{\alpha_j\Delta} f\|_{C^{\mu+r}} + A_{jr}(t) \sup_{0\le\tau\le t} \|v_{j+1}(\tau)\|_{C^\mu}, \tag{6.29}$$

using

$$\begin{aligned} &\|e^{t\Delta} g\|_{C^{\mu+r}} \le C t^{-r/2} \|g\|_{C^\mu}, \quad r>0,\ \mu\in[0,2),\ g\in C_b^r(I), \\ &\|e_N^{t\Delta} g\|_{C^{\mu+r}} \le C t^{-r/2} \|g\|_{C^\mu}, \quad r>0,\ \mu\in[0,1),\ g\in C^r(I), \end{aligned} \tag{6.30}$$

for $t \in (0,T]$. If $\mu + r = 1$, it is necessary to replace C^1 by the Zygmund space C^1_*. Combining this with (6.26), we obtain, for

$$N_{jr}(t) = \sup_{0\le\tau\le t} \|v_j(\tau)\|_{C^r}, \tag{6.31}$$

the estimates

$$\begin{aligned} N_{j+1,\mu+r}(t) \le C_0\|f\|_{C^{\mu+r}} &+ C_1 A_{jr}(t)\|f\|_{C^\mu} \\ &+ C_2 A_{jr}(t) A_{j\mu}(t)\|f\|_{L^\infty}. \end{aligned} \tag{6.32}$$

Recall that

$$\dot{\xi}_j(\tau) \le 2a\|v_j(\tau)\|_{C^1}. \tag{6.33}$$

Hence

$$\xi_j(t) \le 1 + 2atN_{j1}(t), \tag{6.34}$$

where $N_{j1}(t)$ is the case $r = 1$ of (6.31). Therefore, by (6.28),

$$A_{j\mu}(t) \le 2aB\Big[1 + a(1+\mu)N_{j1}(t)t\Big]N_{j1}(t)t^{(1-\mu)/2}. \tag{6.35}$$

Consequently, taking $r = \mu \in (1/2, 1)$, so $2\mu \in (1, 2)$, we have

$$N_{j+1,2\mu}(t) \le P\big(N_{j1}(t)t, N_{j1}(t)t^{(1-\mu)/2}\big), \tag{6.36}$$

where $P(X, Y)$ is a polynomial of degree 4, with coefficients depending on such quantities as $\|f\|_{C^{2\mu}}$, but not on j. A fortiori, we have

$$N_{j+1,1}(t) \le P\big(N_{j1}(t)t, N_{j1}(t)t^{(1-\mu)/2}\big). \tag{6.37}$$

Such an estimate automatically implies a uniform bound

$$N_{j1}(t) \le K, \quad \text{for } t \in [0, T], \tag{6.38}$$

for some $T > 0$, chosen sufficiently small. Appealing again to (6.36), we conclude furthermore that there are uniform bounds

$$N_{j,2\mu}(t) \le K_\mu, \quad \text{for } t \in [0, T], \quad 2\mu < 2. \tag{6.39}$$

That is to say,

$$\{v_j : j \in \mathbb{Z}^+\} \text{ is bounded in } C\big([0, T], C^r(I)\big), \quad r < 2. \tag{6.40}$$

Taking $r = 1$, we conclude that

$$\{\xi_j : j \in \mathbb{Z}^+\} \text{ is bounded in } C^1\big([0, T]\big). \tag{6.41}$$

Of course, we know that each $\xi_j(t)$ is monotone increasing, with $\xi_j(0) = 1$.

From (6.40) and (6.18), we have

$$\{\mathcal{O}v_j : j \in \mathbb{Z}^+\} \text{ bounded in } C\big([0, T], C^r(S^1)\big), \quad r < 2. \tag{6.42}$$

Also, $\{\mathcal{E}(xv_j) : j \in \mathbb{Z}^+\}$ is bounded in $C\big([0, T], C^{0,1}(S^1)\big)$, so we deduce from (6.10) that

$$\{\partial_t(\mathcal{O}v_j) : j \in \mathbb{Z}^+\} \text{ is bounded in } C\big([0, T], C_*^{-(2-r)}(S^1)\big), \quad r < 2. \tag{6.43}$$

Interpolation with (6.42), together with Ascoli's theorem, gives

$$\{\mathcal{O}v_j :\in \mathbb{Z}^+\} \text{ compact in } C^\sigma\big([0, T], C_*^{r-2\sigma}(S^1)\big), \tag{6.44}$$

for $r < 2,\ \sigma \in (0,1)$. It follows that

$$\{v_j : j \in \mathbb{Z}^+\} \text{ is compact in } C^{1/2-\delta}\big([0,T], C^1(I)\big), \quad \text{for all } \delta > 0. \tag{6.45}$$

Consequently, (6.41) is sharpened to

$$\{\xi_j : j \in \mathbb{Z}^+\} \text{ is compact in } C^{3/2-\delta}\big([0,T]\big), \tag{6.46}$$

for all $\delta > 0$. It follows that $\{v_j\}$ has a limit point

$$v \in \bigcap_{0<\sigma<1,\delta>0} C^\sigma\big([0,T], C^{2-\delta-2\sigma}(I)\big), \tag{6.47}$$

and $\{\xi_j\}$ has a limit point

$$\xi \in \bigcap_{\delta>0} C^{3/2-\delta}\big([0,T]\big). \tag{6.48}$$

It remains to show that such v, ξ are unique and give a solution to the Stefan problem.

To investigate this, choose $\zeta_0(t)$ satisfying

$$\zeta_0(0) = 1, \quad \dot{\zeta}_0(0) = -2af'(1), \quad \dot{\zeta}_0 \geq 0; \tag{6.49}$$

define $w_1(t,x)$ to solve (6.5) with $\xi = \zeta_0$; then set

$$\zeta_1(t) = 1 - 2a\int_0^t \partial_x w_1(\tau, 1)\, d\tau,$$

and continue, obtaining a sequence $w_j, \zeta_j,\ j \in \mathbb{Z}^+$, in a fashion similar to that used to get the sequence v_j, ξ_j. As in Lemma 6.1, we see that each ζ_j satisfies (6.12). Now we want to compare the differences $v_j - w_j$ and $\xi_j - \zeta_j$ with $v_{j+1} - w_{j+1}$ and $\xi_{j+1} - \zeta_{j+1}$. Set $V = v_{j+1} - w_{j+1}$. Thus V satisfies the PDE

$$\begin{aligned} \frac{\partial V}{\partial t} = \xi_j^{-1} V_{xx} + \frac{1}{2}\frac{\dot{\xi}_j}{\xi_j}\, xV_x + \left(\frac{1}{\xi_j} - \frac{1}{\zeta_j}\right)\partial_{xx} w_{j+1} \\ + \frac{1}{2}\left(\frac{\dot{\xi}_j}{\xi_j} - \frac{\dot{\zeta}_j}{\zeta_j}\right) x\partial_x w_{j+1}, \end{aligned} \tag{6.50}$$

together with

$$V(0,x) = 0, \quad V(t,0) = V(t,1) = 0. \tag{6.51}$$

Note that the analysis above also gives uniform estimates of the form (6.40)–(6.46) on w_j and ζ_j. Now, for V we have the integral equation

$$
\begin{aligned}
V(t) &= \int_0^t \beta_j(\tau) e^{\alpha_j(t,\tau)\Delta}\big(x\partial_x V(\tau)\big)\, d\tau \\
&\quad + \int_0^t \left(\frac{1}{\xi_j(\tau)} - \frac{1}{\zeta_j(\tau)}\right) e^{\alpha_j(t,\tau)\Delta} \partial_{xx} w_{j+1}(\tau)\, d\tau \\
&\quad + \frac{1}{2}\int_0^t \left(\frac{\dot{\xi}_j(\tau)}{\xi_j(\tau)} - \frac{\dot{\zeta}_j(\tau)}{\zeta_j(\tau)}\right) e^{\alpha_j(t,\tau)\Delta}\big(x\partial_x w_{j+1}(\tau)\big)\, d\tau \\
&= S_1 + S_2 + S_3,
\end{aligned}
\tag{6.52}
$$

where $\beta_j(\tau)$ and $\alpha_j(t,\tau)$ are as in (6.16). As in (6.21), we replace the integrand in S_1 by

$$
\beta_j(\tau)\Big(\partial_x e_N^{\alpha_j(t,\tau)\Delta} M_x - e^{\alpha_j(t,\tau)\Delta}\Big)V(\tau). \tag{6.53}
$$

Thus

$$
\begin{aligned}
\|S_1\|_{C^1} &\le A \int_0^t \beta_j(\tau)\alpha_j(t,\tau)^{-1/2}\|V(\tau)\|_{C^1}\, d\tau \\
&\le B\xi_j(t)^{1/2}\left[\sup_{0\le\tau\le t} \dot{\xi}_j(\tau)\right]\int_0^t (t-\tau)^{-1/2}\|V(\tau)\|_{C^1}\, d\tau \\
&\le C\, t^{1/2} \sup_{0\le\tau\le t} \|V(\tau)\|_{C^1},
\end{aligned}
\tag{6.54}
$$

provided $0 \le t \le T$, with T small enough that the uniform estimates on ξ_j and $\dot{\xi}_j$ apply.

It does not seem feasible to get a good estimate on S_2 in terms of the C^1-norm of w_{j+1}, but we do have the following:

$$
\begin{aligned}
\|S_2\|_{C^1} &\le A \int_0^t |\xi_j(\tau) - \zeta_j(\tau)|\, \alpha_j(t,\tau)^{-(2-\mu)/2}\, \|w_{j+1}(\tau)\|_{C^{1+\mu}}\, d\tau \\
&\le C \sup_{0\le\tau\le t} |\xi_j(\tau) - \zeta_j(\tau)| \cdot \sup_{0\le\tau\le t} \|w_{j+1}(\tau)\|_{C^{1+\mu}} \cdot t^{\mu/2},
\end{aligned}
\tag{6.55}
$$

for any $\mu \in (0,1)$. Finally,

$$
\|S_3\|_{C^1} \le C \sup_{0\le\tau\le t} \left|\frac{\dot{\xi}_j(\tau)}{\xi_j(\tau)} - \frac{\dot{\zeta}_j(\tau)}{\zeta_j(\tau)}\right| \sup_{0\le\tau\le t} \|w_{j+1}(\tau)\| \cdot t^{1/2}. \tag{6.56}
$$

Consequently, for $t \le T$ sufficiently small, $0 < \mu < 1$, we have

$$\begin{aligned} &\frac{1}{2} \sup_{0\le\tau\le t} \|V(\tau)\|_{C^1} \\ &\quad \le C \sup_{0\le\tau\le t} |\xi_j(\tau) - \zeta_j(\tau)| \cdot \sup_{0\le\tau\le t} \|w_{j+1}(\tau)\|_{C^{1+\mu}} \cdot t^{\mu/2} \\ &\qquad + C \sup_{0\le\tau\le t} \left| \frac{\dot{\xi}_j(\tau)}{\xi_j(\tau)} - \frac{\dot{\zeta}_j(\tau)}{\zeta_j(\tau)} \right| \cdot \sup_{0\le\tau\le t} \|w_{j+1}(\tau)\|_{C^1} \cdot t^{1/2}. \end{aligned} \tag{6.57}$$

Therefore

$$\begin{aligned} &\sup_{0\le\tau\le t} |\dot{\xi}_{j+1}(\tau) - \dot{\zeta}_{j+1}(\tau)| \\ &\quad \le C \sup_{0\le\tau\le t} |\xi_j(\tau) - \zeta_j(\tau)| \cdot \sup_{0\le\tau\le t} \|w_{j+1}(\tau)\|_{C^{1+\mu}} \cdot t^{\mu/2} \\ &\qquad + C \sup_{0\le\tau\le t} |\dot{\xi}_j(\tau) - \dot{\zeta}_j(\tau)| \sup_{0\le\tau\le t} \|w_{j+1}(\tau)\|_{C^1} \cdot t^{1/2}. \end{aligned} \tag{6.58}$$

It follows easily that, for T small enough,

$$\|\xi_j - \zeta_j\|_{C^1([0,T])} \to 0 \ \text{ and } \ \|v_j - w_j\|_{C([0,T],C^1(I))} \to 0, \tag{6.59}$$

as $j \to \infty$. Thus we have the following short-time existence result:

Proposition 6.2. *Given $f \in C^\infty(I)$, $f \ge 0$, $f(0) = f(1) = 0$, there are a $T > 0$ and a unique solution v, ξ to (6.5)–(6.6), satisfying (6.47)–(6.48). Hence there is a unique solution u, s to (6.1)–(6.2) on $0 \le t \le T$, satisfying*

$$u \in C^\sigma\big([0,T], C^{r-2\sigma}(I)\big), \quad s \in C^{3/2-\delta}\big([0,T]\big), \quad \dot{s} \ge 0, \tag{6.60}$$

for all $\sigma \in [0,1)$, $r < 2$, $\delta > 0$.

We want to improve this to a global existence theorem. To do this, we need further estimates on the local solution v, s. First, it will be useful to have some regularity results on v not given by (6.60).

Lemma 6.3. *The solution v of Proposition 6.2 satisfies*

$$v \in C\big([0,T], H^r(I)\big) \cap C^1\big([0,T], H^{r-2}(I)\big), \ \textit{for all } r < \frac{5}{2}. \tag{6.61}$$

Proof. This follows by arguments similar to those used above, plus the following variant of (6.18):

$$\mathcal{O} : H^s(I) \longrightarrow H^s(S^1), \ \text{ for } 0 \le s < \frac{1}{2}. \tag{6.62}$$

We know from (6.60) that $xv_x(\tau)$ is continuous in $\tau \in [0,T]$ with values in $C^{1-\varepsilon}(I) \subset H^{1-2\varepsilon}(I) \subset H^{\frac{1}{2}-\varepsilon}(I)$, so use of the integral equation (6.7) gives (6.61).

We now take a look at

$$h(t) = \int_0^1 v(t,x)\,dx, \tag{6.63}$$

which, by (6.14), is $\|v(t)\|_{L^1}$. We have

$$\frac{dh}{dt} = \frac{1}{\xi}\int_0^1 \partial_x^2 v(t,x)\,dx + \frac{1}{2}\frac{\dot{\xi}}{\xi}\int_0^1 x\partial_x v(t,x)\,dx, \tag{6.64}$$

and integrations by parts give for the right side:

$$\begin{aligned} &\frac{1}{\xi_j}\left(-\frac{\dot{\xi}}{2a} - \partial_x v(t,0)\right) - \frac{1}{2}\frac{\dot{\xi}}{\xi}\int_0^1 v(t,x)\,dx \\ &\quad = -\frac{1}{a}\frac{\dot{s}}{s} - \frac{1}{s^2}\partial_x v(t,0) - \frac{\dot{s}}{s}h(t). \end{aligned} \tag{6.65}$$

Consequently,

$$\frac{d}{dt}\big(sh\big) = -\frac{\dot{s}}{a} - \frac{1}{s}\partial_x v(t,0), \tag{6.66}$$

and integration of this gives

$$s(t)h(t) + \frac{1}{a}s(t) + \int_0^t \frac{1}{s(\tau)}\partial_x v(\tau,0)\,d\tau = \frac{1}{a} + \int_0^1 f(x)\,dx. \tag{6.67}$$

From (6.14), $\partial_x v(t,0) \ge 0$, so each term on the left side of (6.67) is positive. This gives the upper bound in the two-sided bound,

$$1 \le s(t) \le 1 + a\int_0^1 f(x)\,dx = A, \tag{6.68}$$

the lower bound following from the monotonicity $\dot{s} \ge 0$. Thus $1 \le \xi(t) \le A^2$. Hence, in (6.7)–(6.9), we have

$$A^{-2}(t-\tau) \le \alpha(t,\tau) \le t-\tau. \tag{6.69}$$

We can likewise examine $(d/dt)\|v(t)\|^2$, but since that analysis won't be crucial for our existence theorem, we leave it to the exercises.

We next look at the rate of change of $\|\partial_x v(t)\|_{L^2}^2$. Since $\partial_t v = 0$ for $x = 0,\ 1$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\partial_x v(t)\right\|_{L^2}^2 &= (\partial_x\partial_t v, \partial_x v) = -(\partial_t v, \partial_x^2 v) \\ &= -\frac{1}{\xi}\left\|\partial_x^2 v\right\|_{L^2}^2 - \frac{1}{2}\frac{\dot{\xi}}{\xi}\big(x\partial_x v, \partial_x^2 v\big),\end{aligned} \tag{6.70}$$

and an integration by parts, plus use of $\partial_x v(t,1) = -\dot{\xi}(t)/2a$, gives

$$\frac{d}{dt}\left\|\partial_x v(t)\right\|_{L^2}^2 = -\frac{2}{\xi}\left\|\partial_x^2 v\right\|_{L^2}^2 - \frac{\dot{\xi}}{\xi}\cdot\frac{\dot{\xi}^2}{8a^2} + \frac{1}{2}\frac{\dot{\xi}}{\xi}\left\|\partial_x v\right\|_{L^2}^2 \tag{6.71}$$

or, equivalently,

$$s\,\frac{d}{dt}\Big(\frac{1}{s}\left\|\partial_x v\right\|_{L^2}^2\Big) = -\frac{2}{\xi}\left\|\partial_x^2 v\right\|_{L^2}^2 - \frac{\dot{\xi}}{\xi}\cdot\frac{\dot{\xi}^2}{8a^2}. \tag{6.72}$$

Since the right side of (6.72) is ≤ 0, this gives, upon integration,

$$\|\partial_x v(t)\|_{L^2}^2 \le s(t)\|\partial_x f\|_{L^2}^2. \tag{6.73}$$

Note that, by (6.68), the right side is $\le A\|\partial_x f\|_{L^2}^2$, which is independent of t.

Using (6.73) and (6.7), we have, for $r \in [1,2)$,

$$\|v(t)\|_{H^r} \le \|e^{\alpha(t,0)\Delta} f\|_{H^r} + C\|\partial_x f\|_{L^2}\int_0^t \beta(\tau)\alpha(t,\tau)^{-r/2}\,d\tau, \tag{6.74}$$

and the first term on the right is $\le \|f\|_{H^r}$, since

$$f \in H^r(I)\cap H_0^1(I) \Longrightarrow \mathcal{O}f \in H^r(S^1),$$

for $r < 5/2$. By (6.69), we deduce that, for $r \in [1,2)$,

$$\|v(t)\|_{H^r} \le C_1 + C_2\int_0^t \dot{s}(\tau)\,(t-\tau)^{-r/2}\,d\tau, \tag{6.75}$$

with $C_j = C_j(r)\|f\|_{H^r}$ independent of t. Taking $r \in (3/2,2)$, and using $\dot{s}(t) = -v_x(t,1)/2a$, which is $\le C\|v\|_{H^r}$, we deduce that

$$\dot{s}(t) \le K_1 + K_2\int_0^t \dot{s}(\tau)(t-\tau)^{-r/2}\,d\tau. \tag{6.76}$$

From this we can establish the following important estimate:

Lemma 6.4. *If v solves (6.3) for $0 \le t \le T$ and satisfies (6.61), then*

$$\sup_{0\le t\le T} \dot{s}(t) \le K_0, \tag{6.77}$$

where $K_0 = K_0(\|f\|_{H^r})$ $(r = 3/2 + \varepsilon)$ is independent of T.

Proof. Pick $\rho > 0$ small enough that $\int_0^\rho t^{-r/2}\,dt \le 1/2K_2$. Thus, writing the interval $[0, t]$ as $[0, t-\rho] \cup [t-\rho, t]$, we have

$$K_2 \int_0^t \dot{s}(\tau)\,(t-\tau)^{-r/2}\,d\tau \le \frac{1}{2}\sup_{\tau\le t} \dot{s}(\tau) + C_2\rho^{-r/2}\big[s(t-\rho) - 1\big]. \tag{6.78}$$

We conclude from (6.76) and (6.68) that

$$\sup_{0\le t\le T} \dot{s}(t) \le \frac{1}{2}\sup_{0\le t\le T} \dot{s}(t) + K_1 + K_2\rho^{-r/2}a\|f\|_{L^1}, \tag{6.79}$$

which gives (6.77).

Returning to (6.75), we deduce that the solution to (6.3) given by Proposition 6.2 satisfies, for any $r \in [1, 2)$,

$$\|v(t)\|_{H^r(I)} \le K_r, \quad 0 \le t \le T, \tag{6.80}$$

with K_r independent of T. We know that $xv_x(\tau)$ has an H^s-bound for any $s < 1$, and, via (6.62), we can use such a bound on $xv_x(\tau)$ for $s < \frac{1}{2}$, to conclude, via (6.7), that (6.80) holds for any $r \in [1, 5/2)$. Now familiar methods establish the following:

Theorem 6.5. *Given $f \in C^\infty(I)$, $f \ge 0$, $f(0) = f(1) = 0$, there is a unique solution v, ξ to (6.5)–(6.6), defined for all $t \in [0,\infty)$, satisfying*

$$v \in C\big([0,\infty), H^r(I)\big) \cap C^1\big([0,\infty), H^{r-2}(I)\big), \quad r < \frac{5}{2}, \tag{6.81}$$

and

$$\xi \in C^\mu\big([0,\infty)\big), \quad \mu < \frac{3}{2}. \tag{6.82}$$

We now tackle the task of showing that u and s, or equivalently v and ξ, are smooth for $t \in (0,\infty) = J$. It is convenient to set

$$V(T,x) = v(t,x), \quad T = \alpha(t,0) = \int_0^t \xi(\tau)^{-1}\,d\tau, \tag{6.83}$$

so $\partial_T V(T,x) = \xi(t)\partial_t v(t,x)$, and we have

$$\frac{\partial V}{\partial T} = V_{xx} + \sigma(T) x V_x, \quad \sigma(T) = \frac{1}{2}\dot{\xi}(t), \tag{6.84}$$

with

$$V(0,x) = f(x), \quad V(t,x) = 0, \ \text{ for } x \in \partial I. \tag{6.85}$$

Note that (6.6) is equivalent to

$$\sigma(T) = -a V_x(T,1). \tag{6.86}$$

In place of (6.7), we use

$$V(T) = e^{T\Delta} f + \int_0^T \sigma(\tau) e^{(T-\tau)\Delta}\big(x V_x(\tau)\big)\, d\tau. \tag{6.87}$$

The results (6.81) and (6.82) imply

$$V \in C\big([0,\infty), H^r(I)\big) \cap C^1\big([0,\infty), H^{r-2}(I)\big), \quad \sigma \in C^{1/2-\delta}\big([0,\infty)\big), \tag{6.88}$$

for any $r < 5/2, \ \delta > 0$. Consequently, $V \in C^{1/2-\delta}\big([0,\infty), H^{3/2-\delta}(I)\big)$ for all $\delta > 0$, so $\sigma(\tau) x V_x(\tau) \in C^{1/2-\delta}\big([0,\infty), H^{1/2-\delta}(I)\big)$. The following lemma is useful.

Lemma 6.6. *Suppose*

$$G(T) = \int_0^T e^{(T-\tau)\Delta} F(\tau)\, d\tau.$$

Then, for any $r > 0, \ s \in (0, 1/2)$,

$$F \in C\big([0,\infty), L^2(I)\big) \cap C^r\big((0,\infty), H^s(I)\big) \Longrightarrow G \in C^r\big((0,\infty), H^{s+2-\delta}(I)\big),$$

for all $\delta > 0$.

The proof is straightforward. Applying this to $F(\tau) = \sigma(\tau) x V_x(\tau)$, we deduce that the right side of (6.87) belongs to $C^{1/2-\delta}\big((0,\infty), H^{5/2-\delta}(I)\big)$, for all $\delta > 0$. Thus, with $J = (0,\infty)$,

$$V \in C^{1/2-\delta}\big(J, H^{5/2-\delta}(I)\big). \tag{6.89}$$

Note that this is stronger than the first inclusion in (6.88). Making use of the PDE (6.84), we deduce that $V_T \in C^{1/2-\delta}(J, H^{1/2-\delta}(I))$, so

$$V \in C^{3/2-\delta}\big(J, H^{1/2-\delta}(I)\big). \tag{6.90}$$

Interpolation of (6.89) and (6.90) gives

$$V \in C^{1-\delta}\big(J, H^{3/2-\delta}(I)\big); \tag{6.91}$$

hence, by (6.86),

$$\sigma \in C^{1-\delta}(J). \tag{6.92}$$

Now we have improved all three parts of (6.88), essentially increasing the degree of regularity in T by one half, at least for $T \in J = (0,\infty)$. Iterating this argument, we obtain

$$\begin{aligned} &V \in C^{j/2-\delta}(J, H^{5/2-\delta}(I)) \cap C^{1+j/2-\delta}(J, H^{1/2-\delta}(I)),\\ &\sigma \in C^{1/2+j/2-\delta}(J), \end{aligned} \tag{6.93}$$

for each $j \in \mathbb{Z}^+$. We are well on the way to establishing the following:

Proposition 6.7. *The solutions v, ξ of Theorem 6.5 have the property*

$$v \in C^\infty\big((0,\infty)\times I\big), \quad \xi \in C^\infty\big((0,\infty)\big). \tag{6.94}$$

Proof. That $\xi \in C^\infty(J)$ follows from $\sigma \in C^\infty(J)$. For the rest, it suffices to show that $V \in C^\infty(J \times I)$. We get this from (6.93) together with the PDE (6.84). In fact, this yields

$$V_{xx} \in C^\infty(J, H^{3/2-\delta}(I)), \tag{6.95}$$

hence

$$V \in C^\infty(J, H^{7/2-\delta}(I)). \tag{6.96}$$

Iterating this argument finishes the proof.

A number of variants of (6.1)–(6.2) are studied. For example, one often sees a nonhomogeneous boundary condition at the left boundary:

$$u(t,0) = g(t). \tag{6.97}$$

Or the boundary condition at $x = 0$ may be of Neumann type:

$$u_x(0,t) = h(t). \tag{6.98}$$

Both the PDE and the boundary conditions may have t-dependent coefficients. For example, the PDE might be

$$u_t = A(t)u_{xx}. \tag{6.99}$$

Some studies of these problems can be found in Chap. 8 of [Fr1] and in [KMP], where particular attention is paid to the nature of the dependence of the solution on the coefficient $A(t)$, assumed to be > 0.

There are also two-phase Stefan problems, where the ice is not assumed to be at temperature 0, but rather at a temperature $u_i(t, x) \leq 0$, to be determined as part of the problem. Furthermore, these problems are most interesting in higher-dimensional space. More material on this can be found in [Fr2].

Exercises

1. If v solves (6.3)–(6.4), show that

$$\frac{d}{dt}\Big(s\|v(t)\|_{L^2}^2\Big) = -\frac{2}{s}\|v_x(t)\|_{L^2}^2;$$

hence

$$s(t)\|v(t)\|_{L^2}^2 + 2\int_0^t s(\tau)^{-1}\|v_x(\tau)\|_{L^2}^2\,d\tau = \|f\|_{L^2}^2.$$

2. If u satisfies (6.1)–(6.2), show that

$$s(t)^2 = 1 + 2a\int_0^1 xf(x)\,dx - 2a\int_0^{s(t)} xu(t,x)\,dx. \tag{6.100}$$

Compare the upper bound on $s(t)$ this gives with (6.68).
Show that, conversely, (6.1) and (6.100) imply (6.1)–(6.2). This result, or rather its analogue in the more general context of the nonhomogeneous boundary condition (6.97), played a role in the analysis in [CH].

7. Quasi-linear parabolic equations I

In this section we begin to study the initial-value problem

$$\frac{\partial u}{\partial t} = \sum_{j,k} A^{jk}(t,x,D_x^1 u)\,\partial_j\partial_k u + B(t,x,D_x^1 u), \quad u(0) = f. \tag{7.1}$$

Here, u takes values in $\mathbb{R}^K$, and each A^{jk} can be a symmetric $K \times K$ matrix; we assume A^{jk} and B are smooth in their arguments. We assume for simplicity that $x \in M = \mathbb{T}^n$, the n-dimensional torus. Modifications for a more general, compact M will be contained in the stronger analysis made in § 8. We impose the following "strong parabolicity condition":

$$\sum_{j,k} A^{jk}(t,x,D_x^1 u)\xi_j\xi_k \geq C_0|\xi|^2 I, \tag{7.2}$$

where to say that a pair of symmetric $K \times K$ matrices S_j satisfies $S_1 \geq S_2$ is to say that $S_1 - S_2$ is a positive-semidefinite matrix.

We will use a "modified Galerkin method" to produce a short-time solution. We consider the approximating equation

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} &= J_\varepsilon \sum A^{jk}(t,x,D_x^1 J_\varepsilon u_\varepsilon)\partial_j\partial_k J_\varepsilon u_\varepsilon + J_\varepsilon B(t,x,D_x^1 J_\varepsilon u_\varepsilon) \\ &= J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + J_\varepsilon B_\varepsilon, \\ u_\varepsilon(0) &= J_\varepsilon f. \end{aligned} \tag{7.3}$$

Here J_ε is a Friedrichs mollifier, which we can take in the form

$$J_\varepsilon = \varphi(\varepsilon\sqrt{-\Delta}),$$

with an even function $\varphi \in \mathcal{S}(\mathbb{R})$, $\varphi(0) = 1$. Equivalently, the Fourier coefficients $\hat{f}(k)$ of $f \in \mathcal{D}'(\mathbb{T}^n)$ are related to those of $J_\varepsilon f$ by

$$(J_\varepsilon f)\hat{}\,(k) = \varphi(\varepsilon|k|)\hat{f}(k), \quad k \in \mathbb{Z}^n.$$

For any fixed $\varepsilon > 0$, the right side of (7.3) is Lipschitz in u_ε with values in practically any Banach space of functions, so the existence of short-time solutions to (7.3) follows by the material of Chap. 1. Our task will be to show that the solution u_ε exists for t in an interval independent of $\varepsilon \in (0,1]$ and has a limit as $\varepsilon \searrow 0$, solving (7.1).

To do this, we estimate the H^ℓ-norm of solutions to (7.3). We begin with

$$\frac{d}{dt}\|D^\alpha u_\varepsilon(t)\|_{L^2}^2 = 2(D^\alpha J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, D^\alpha u_\varepsilon) + 2(D^\alpha B_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon). \tag{7.4}$$

Since J_ε commutes with D^α and is self-adjoint, we can write the first term on the right as

$$2(L_\varepsilon D^\alpha J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon) + 2([D^\alpha, L_\varepsilon]J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon). \tag{7.5}$$

To analyze the first term in (7.5), write it as

$$2(L_\varepsilon v, v) = -2\sum_{j,k}(A_\varepsilon^{jk}\,\partial_j v, \partial_k v) + 2\sum([A_\varepsilon^{jk}, \partial_k]\,\partial_j v, v), \tag{7.6}$$

where $A_\varepsilon^{jk} = A^{jk}(t,x,D_x^1 J_\varepsilon u_\varepsilon)$ and $v = D^\alpha J_\varepsilon u_\varepsilon$. Note that, by the strong ellipticity hypothesis (7.2), we have

$$\sum (A_\varepsilon^{jk}\,\partial_j v, \partial_k v) \geq C_0 \|\nabla v\|_{L^2}^2. \tag{7.7}$$

The commutator $[D^\alpha, L_\varepsilon] = \sum [D^\alpha, A_\varepsilon^{jk}]\,\partial_j\partial_k$ can be treated using the Moser estimates established in Chap. 13. From Proposition 3.7 of Chap. 13 we deduce that

$$\begin{aligned} &\|[D^\alpha, L_\varepsilon]w\|_{L^2} \\ &\leq C\sum_{j,k}\Big(\|A_\varepsilon^{jk}\|_{H^\ell}\|\partial_j\partial_k w\|_{L^\infty} + \|\nabla A_\varepsilon^{jk}\|_{L^\infty}\|\partial_j\partial_k w\|_{H^{\ell-1}}\Big), \end{aligned} \tag{7.8}$$

provided $|\alpha| \leq \ell$. Since $[A_\varepsilon^{jk}, \partial_k]w = -\sum(\partial_k A_\varepsilon^{jk})w$, we have the elementary estimate

$$\|[A_\varepsilon^{jk}, \partial_k]\partial_j v\|_{L^2} \leq C\|\nabla A_\varepsilon^{jk}\|_{L^\infty}\|\partial_j v\|_{L^2}. \tag{7.9}$$

Furthermore, Proposition 3.9 of Chap. 13 implies

$$\|A_\varepsilon^{jk}\|_{H^\ell} \leq C_\ell\big(\|J_\varepsilon u_\varepsilon\|_{C^1}\big)\Big(1 + \|J_\varepsilon u_\varepsilon\|_{H^{\ell+1}}\Big), \tag{7.10}$$

and we have the elementary estimate $\|\nabla A_\varepsilon^{jk}\|_{L^\infty} \leq C\big(\|J_\varepsilon u_\varepsilon\|_{C^1}\big)\|J_\varepsilon u_\varepsilon\|_{C^2}$. Hence (7.5) is less than or equal to

$$\begin{aligned} &-2C_0\|\nabla D^\alpha J_\varepsilon u_\varepsilon(t)\|_{L^2}^2 \\ &\quad + C\big(\|J_\varepsilon u_\varepsilon\|_{C^1}\big)\|J_\varepsilon u_\varepsilon\|_{C^2}\big(1 + \|J_\varepsilon u_\varepsilon\|_{H^{\ell+1}}\big)\cdot\|D^\alpha J_\varepsilon u_\varepsilon\|_{L^2}. \end{aligned} \tag{7.11}$$

Furthermore, we have a bound

$$2(D^\alpha B_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon) \leq C\big(\|J_\varepsilon u_\varepsilon\|_{C^1}\big)\big(1 + \|J_\varepsilon u_\varepsilon\|_{H^{\ell+1}}\big)\cdot\|D^\alpha J_\varepsilon u_\varepsilon\|_{L^2}, \tag{7.12}$$

by the analogue of (7.10) for $\|B(t,x,D_x^1 J_\varepsilon u_\varepsilon)\|_{H^\ell}$. Consequently, we have an upper bound for (7.4). Summing over $|\alpha| \leq \ell$, we obtain

$$\begin{aligned} \frac{d}{dt}\|u_\varepsilon(t)\|_{H^\ell}^2 \leq &-2C_0\|J_\varepsilon u_\varepsilon\|_{H^{\ell+1}}^2 \\ &+ C_1\big(\|J_\varepsilon u_\varepsilon\|_{C^2}\big)\big(1 + \|J_\varepsilon u_\varepsilon\|_{H^{\ell+1}}\big)\|J_\varepsilon u_\varepsilon\|_{H^\ell}. \end{aligned} \tag{7.13}$$

Using $AB \leq C_0 A^2 + (1/4C_0)B^2$, with $A = \|J_\varepsilon u_\varepsilon\|_{H^{\ell+1}}$, we obtain

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{H^\ell}^2 \leq -C_0\|J_\varepsilon u_\varepsilon\|_{H^{\ell+1}}^2 + C_2\big(\|J_\varepsilon u_\varepsilon\|_{C^2}\big)\big(\|J_\varepsilon u_\varepsilon\|_{H^\ell}^2 + 1\big). \tag{7.14}$$

In particular, since $\{J_\varepsilon : 0 < \varepsilon \le 1\}$ is uniformly bounded on each space $C^j(M)$ and $H^\ell(M)$, we have an estimate

$$\frac{d}{dt}\|u_\varepsilon(t)\|^2_{H^\ell} \le C_\ell\big(\|u_\varepsilon\|_{C^2}\big)\big(\|u_\varepsilon(t)\|^2_{H^\ell}+1\big). \tag{7.15}$$

This estimate permits the following analysis of the evolution equations (7.3).

Lemma 7.1. *Given $f \in H^\ell$, $\ell > n/2+2$, the solution to (7.3) exists for t in an interval $I = [0, A)$, independent of ε, and satisfies an estimate*

$$\|u_\varepsilon(t)\|_{H^\ell} \le K(t), \quad t \in I, \tag{7.16}$$

independent of $\varepsilon \in (0, 1]$.

Proof. Using the Sobolev imbedding theorem, we can dominate the right side of (7.15) by $E\big(\|u_\varepsilon(t)\|^2_{H^\ell}\big)$, so $\|u_\varepsilon(t)\|^2_{H^\ell} = y(t)$ satisfies the differential inequality

$$\frac{dy}{dt} \le E(y), \quad y(0) = \|f\|^2_{H^\ell}. \tag{7.17}$$

Gronwall's inequality then yields a function $K(t)$, finite on some interval $I = [0, A)$, giving an upper bound for all $y(t)$ satisfying (7.17). This I and $K(t)$ work for (7.16).

We are now prepared to establish the following existence result:

Theorem 7.2. *If (7.1) satisfies the parabolicity hypothesis (7.2), and if $f \in H^\ell(M)$, with $\ell > n/2+2$, then there is a solution u, on an interval $I = [0, T)$, such that*

$$u \in L^\infty\big(I, H^\ell(M)\big) \cap Lip\big(I, H^{\ell-2}(M)\big). \tag{7.18}$$

Proof. Take the I above and shrink it slightly. The bounded family

$$u_\varepsilon \in C(I, H^\ell) \cap C^1(I, H^{\ell-2})$$

will have a weak limit point u satisfying (7.18). Furthermore, by Ascoli's theorem (in the form given in Exercise 5, in § 6 of Appendix A), there is a sequence

$$u_{\varepsilon_\nu} \longrightarrow u \ \text{ in } C\big(I, H^{\ell-2}(M)\big), \tag{7.19}$$

since the inclusion $H^\ell \hookrightarrow H^{\ell-2}$ is compact. In addition, interpolation inequalities imply that $\{u_\varepsilon : 0 < \varepsilon \le 1\}$ is bounded in $C^\sigma\big(I, H^{\ell-2\sigma}(M)\big)$ for each

$\sigma \in (0,1)$. Since the inclusion $H^{\ell-2\sigma} \hookrightarrow C^2(M)$ is compact for small $\sigma > 0$ if $\ell > n/2+2$, we can arrange that

$$u_{\varepsilon_\nu} \longrightarrow u \text{ in } C\big(I, C^2(M)\big). \tag{7.20}$$

Consequently, with $\varepsilon = \varepsilon_\nu$,

$$\begin{aligned} J_\varepsilon \sum A^{jk}(t,x,D_x^1 J_\varepsilon u_\varepsilon)\,\partial_j\partial_k J_\varepsilon u_\varepsilon &\longrightarrow \sum A^{jk}(t,x,D_x^1 u)\,\partial_j\partial_k u, \\ J_\varepsilon B(t,x,D_x^1 J_\varepsilon u_\varepsilon) &\longrightarrow B(t,x,D_x^1 u) \end{aligned} \tag{7.21}$$

in $C(I\times M)$, while clearly $\partial u_{\varepsilon_\nu}/\partial t \to \partial u/\partial t$ weakly. Thus (7.1) follows in the limit from (7.3), and the theorem is proved.

We turn now to questions of the uniqueness, stability, and rate of convergence of u_ε to u; we can treat these questions simultaneously. Thus, with $\varepsilon \in [0,1]$, we compare a solution u to (7.1) with a solution u_ε to

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} &= J_\varepsilon \sum A^{jk}(t,x,D_x^1 J_\varepsilon u_\varepsilon)\,\partial_j\partial_k J_\varepsilon u_\varepsilon + J_\varepsilon B(t,x,D_x^1 J_\varepsilon u_\varepsilon), \\ u_\varepsilon(0) &= h. \end{aligned} \tag{7.22}$$

For brevity, we suppress the (t,x)-dependence and write

$$\begin{aligned} \frac{\partial u}{\partial t} &= L(D_x^1 u, D)u + B(D_x^1 u), \\ \frac{\partial u_\varepsilon}{\partial t} &= J_\varepsilon L(D_x^1 J_\varepsilon u_\varepsilon, D)J_\varepsilon u_\varepsilon + J_\varepsilon B(D_x^1 J_\varepsilon u_\varepsilon). \end{aligned} \tag{7.23}$$

Let $v = u - u_\varepsilon$. Subtracting the two equations in (7.23), we have

$$\begin{aligned} \frac{\partial v}{\partial t} = L(D_x^1 u, D)v &+ L(D_x^1 u, D)u_\varepsilon - J_\varepsilon L(D_x^1 J_\varepsilon u_\varepsilon, D)J_\varepsilon u_\varepsilon \\ &+ B(D_x^1 u) - J_\varepsilon B(D_x^1 J_\varepsilon u_\varepsilon). \end{aligned} \tag{7.24}$$

Write

$$\begin{aligned} &L(D_x^1 u, D)u_\varepsilon - J_\varepsilon L(D_x^1 J_\varepsilon u_\varepsilon, D)J_\varepsilon u_\varepsilon \\ &= \big[L(D_x^1 u, D) - L(D_x^1 u_\varepsilon, D)\big]u_\varepsilon + (1-J_\varepsilon)L(D_x^1 u_\varepsilon, D)u_\varepsilon \\ &+ J_\varepsilon L(D_x^1 u_\varepsilon, D)(1-J_\varepsilon)u_\varepsilon + J_\varepsilon\big[L(D_x^1 u_\varepsilon, D) - L(D_x^1 J_\varepsilon u_\varepsilon, D)\big]J_\varepsilon u_\varepsilon \end{aligned} \tag{7.25}$$

and

$$\begin{aligned} B(D_x^1 u) - J_\varepsilon B(D_x^1 J_\varepsilon u_\varepsilon) = &\big[B(D_x^1 u) - B(D_x^1 u_\varepsilon)\big] \\ &+ (1-J_\varepsilon)B(D_x^1 u_\varepsilon) \\ &+ J_\varepsilon\big[B(D_x^1 u_\varepsilon) - B(D_x^1 J_\varepsilon u_\varepsilon)\big]. \end{aligned} \tag{7.26}$$

Now write

$$
\begin{aligned}
B(D_x^1 u) - B(D_x^1 w) &= G(D_x^1 u, D_x^1 w)(D_x^1 u - D_x^1 w), \\
G(D_x^1 u, D_x^1 w) &= \int_0^1 B'\big(\tau D_x^1 u + (1-\tau) D_x^1 w\big)\, d\tau,
\end{aligned}
\tag{7.27}
$$

and similarly

$$
L(D_x^1 u, D) - L(D_x^1 w, D) = (D_x^1 u - D_x^1 w) \cdot M(D_x^1 u, D_x^1 w, D). \tag{7.28}
$$

Then (7.24) yields

$$
\frac{\partial v}{\partial t} = L(D_x^1 u, D) v + A(D_x^1 u, D_x^2 u_\varepsilon) D_x^1 v + R_\varepsilon, \tag{7.29}
$$

where

$$
\begin{aligned}
&A(D_x^1 u, D_x^2 u_\varepsilon) D_x^1 v \\
&\quad = D_x^1 v \cdot M(D_x^1 u, D_x^1 u_\varepsilon, D) u_\varepsilon + G(D_x^1 u, D_x^1 u_\varepsilon) D_x^1 v
\end{aligned}
\tag{7.30}
$$

incorporates the first terms on the right sides of (7.25) and (7.26), and R_ε is the sum of the rest of the terms in (7.25) and (7.26). Note that each term making up R_ε has as a factor $I - J_\varepsilon$, acting on either $D_x^1 u_\varepsilon$, $B(D_x^1 u_\varepsilon)$, or $L(D_x^1 u_\varepsilon, D) u_\varepsilon$. Thus there is an estimate

$$
\|R_\varepsilon(t)\|_{L^2}^2 \le C_\ell\big(\|u_\varepsilon(t)\|_{C^2}\big)\big(1 + \|u_\varepsilon(t)\|_{H^\ell}^2\big) r_\ell(\varepsilon)^2, \tag{7.31}
$$

where

$$
r_\ell(\varepsilon) = \|I - J_\varepsilon\|_{\mathcal{L}(H^{\ell-2}, L^2)} \approx \|I - J_\varepsilon\|_{\mathcal{L}(H^\ell, H^2)}. \tag{7.32}
$$

Now, estimating $(d/dt)\|v(t)\|_{L^2}^2$ via techniques parallel to those used for (7.4)–(7.15) yields

$$
\frac{d}{dt}\|v(t)\|_{L^2}^2 \le C(t)\|v(t)\|_{L^2}^2 + S(t), \tag{7.33}
$$

with

$$
C(t) = C\big(\|u_\varepsilon(t)\|_{C^2}, \|u(t)\|_{C^2}\big), \quad S(t) = \|R_\varepsilon(t)\|_{L^2}^2. \tag{7.34}
$$

Consequently, by Gronwall's inequality, with $K(t) = \int_0^t C(\tau)\, d\tau$,

$$
\|v(t)\|_{L^2}^2 \le e^{K(t)} \left(\|f - h\|_{L^2}^2 + \int_0^t S(\tau) e^{-K(\tau)}\, d\tau \right), \tag{7.35}
$$

for $t \in [0, T)$. Thus we have

Proposition 7.3. *For $\ell > n/2+2$, solutions to (7.1) satisfying (7.18) are unique. They are limits of soutions u_ε to (7.3), and, for $t \in I$,*

$$\|u(t) - u_\varepsilon(t)\|_{L^2} \le K_1(t)\|I - J_\varepsilon\|_{\mathcal{L}(H^{\ell-2},L^2)}. \tag{7.36}$$

Note that if $J_\varepsilon = \varphi(\varepsilon\sqrt{-\Delta})$ and $\varphi \in \mathcal{S}(\mathbb{R})$ satisfies $\varphi(\lambda) = 1$ for $|\lambda| \le 1$, we have the operator norm estimate

$$\|I - J_\varepsilon\|_{\mathcal{L}(H^{\ell-2},L^2)} \le C\,\varepsilon^{\ell-2}. \tag{7.37}$$

We next establish smoothness of the solution u given by Theorem 7.2, away from $t = 0$.

Proposition 7.4. *The solution u of Theorem 7.2 has the property that*

$$u \in C^\infty\big((0,T)\times M\big). \tag{7.38}$$

Proof. Fix any $S < T$ and take $J = [0,S)$. If we integrate (7.14) over J, we obtain a bound on $\int_J \|J_\varepsilon u_\varepsilon(t)\|^2_{H^{\ell+1}}\,dt$, provided we assume $\ell > n/2+2$, so that we can appeal to a bound on $C_2\big(\|J_\varepsilon u_\varepsilon(t)\|_{C^2}\big)$ and on $\|J_\varepsilon u_\varepsilon(t)\|^2_{H^\ell}$, for $t \in J$. Thus

$$u \in L^2\big(J, H^{\ell+1}(M)\big). \tag{7.39}$$

Recall that we know $u \in \mathrm{Lip}\big(I, H^{\ell-2}(M)\big)$. It follows that there is a subset $\mathcal{E}$ of I such that

$$\mathrm{meas}(\mathcal{E}) = 0, \quad t_0 \in I \setminus \mathcal{E} \Longrightarrow u(t_0) \in H^{\ell+1}(M). \tag{7.40}$$

Given $t_0 \in I \setminus \mathcal{E}$, consider the initial-value problem

$$\frac{\partial U}{\partial t} = \sum A^{jk}(t,x,D^1_x U)\,\partial_j\partial_k U + B(t,x,D^1_x U), \quad U(t_0) = u(t_0). \tag{7.41}$$

By the uniqueness result of Proposition 7.3, $U(t) = u(t)$ for $t_0 \le t < T$.

Now, the proof of Theorem 7.2 gives a length $L > 0$, independent of $t_0 \in J$, such that the approximation U_ε defined by the obvious analogue of (7.3) converges to U weakly in $L^\infty([t_0,t_0+L],H^\ell(M))$. In particular, $\|U_\varepsilon(t)\|_{C^2}$ is bounded on $[t_0,t_0+L]$. On the other hand, there is also an analogue of (7.15), with ℓ replaced by $\ell+1$:

$$\frac{d}{dt}\|U_\varepsilon(t)\|^2_{H^{\ell+1}} \le C_{\ell+1}\big(\|U_\varepsilon(t)\|_{C^2}\big)\big(\|U_\varepsilon(t)\|^2_{H^{\ell+1}} + 1\big). \tag{7.42}$$

Consequently, U_ε is bounded in $C([t_0, t_0 + L], H^{\ell+1}(M))$, and we obtain

$$u \in L^\infty\big([t_0, t_0 + L], H^{\ell+1}(M)\big) \cap \operatorname{Lip}\big([t_0, t_0 + L], H^{\ell-1}(M)\big). \tag{7.43}$$

Since the exceptional set $\mathcal{E}$ has measure 0, this is enough to guarantee that

$$u \in L^\infty_{\mathrm{loc}}(J, H^{\ell+1}(M)) \cap \operatorname{Lip}_{\mathrm{loc}}(J, H^{\ell-1}(M)), \tag{7.44}$$

and since J is obtained by shrinking I as little as one likes, we have (7.44) with J replaced by I. Now we can iterate this argument, obtaining $u \in L^\infty_{\mathrm{loc}}(I, H^{\ell+j}(M))$ for each $j \in \mathbb{Z}^+$, from which (7.38) is easily deduced.

We can now sharpen the description (7.18) of the solution u in another fashion:

Proposition 7.5. *The solution u of Theorem 7.2 has the property that*

$$u \in C\big(I, H^\ell(M)\big) \cap C^1\big(I, H^{\ell-2}(M)\big). \tag{7.45}$$

Proof. It suffices to show that $u(t)$ is continuous at $t = 0$, with values in $H^\ell(M)$. We know that as $t \searrow 0$, $u(t)$ is bounded in $H^\ell(M)$ and converges to $u(0) = f$ in $H^{\ell-2}(M)$; hence $u(t) \to f$ weakly in H^ℓ as $t \searrow 0$. To deduce that $u(t) \to f$ in H^ℓ-norm, it suffices to show that

$$\limsup_{t \searrow 0} \|u(t)\|_{H^\ell} \le \|f\|_{H^\ell}. \tag{7.46}$$

However, the bounds on $\|u_\varepsilon(t)\|_{H^\ell}$ implied by (7.15) easily yield this result.

Now that we have smoothness, (7.38), an argument parallel to but a bit simpler than that used to produce (7.4)–(7.15) gives

$$\frac{d}{dt}\|u(t)\|^2_{H^\ell} \le C_\ell\big(\|u(t)\|_{C^2}\big)\big(\|u(t)\|^2_{H^\ell} + 1\big), \tag{7.47}$$

for a solution $u \in C^\infty\big((0, T)\times M\big)$ to (7.1). This implies the following persistence result:

Proposition 7.6. *Suppose $u \in C^\infty\big((0, T) \times M\big)$ is a solution to (7.1). Assume also that*

$$\|u(t)\|_{C^2} \le K < \infty, \tag{7.48}$$

for $t \in (0, T)$. Then there exists $T_1 > T$ such that u extends to a solution to (7.1), belonging to $C^\infty\big((0, T_1) \times M\big)$.

A special case of (7.1) is the class of systems of the form

$$\frac{\partial u}{\partial t} = \sum_{j,k} A^{jk}(t,x,u)\,\partial_j\partial_k u + B(t,x,D^1_x u), \quad u(0) = f. \tag{7.49}$$

We retain the strong parabolicity hypothesis (7.2). In this case, when one does estimates of the form (7.5)–(7.15), and so forth, C^2-norms can be systematically replaced by C^1-norms. In particular, for a local smooth solution to (7.49), we have the following improvement of (7.47):

$$\frac{d}{dt}\|u(t)\|^2_{H^\ell} \le C_\ell\big(\|u(t)\|_{C^1}\big)\big(\|u(t)\|^2_{H^\ell}+1\big). \tag{7.50}$$

Thus we have the following:

Proposition 7.7. *If (7.49) is strongly parabolic and $f \in H^\ell(M)$ with $\ell > n/2+1$, then there is a solution u, on an interval $I = [0,T)$, such that*

$$u \in C\big([0,T), H^\ell(M)\big) \cap C^\infty\big((0,T)\times M\big). \tag{7.51}$$

Furthermore, if

$$\|u(t)\|_{C^1} \le K < \infty, \tag{7.52}$$

for $t \in [0,T)$, then there exists $T_1 > T$ such that u extends to a solution to (7.49), belonging to $C^\infty\big((0,T_1)\times M\big)$.

We apply this to obtain a global existence result for a scalar parabolic equation, in one space variable, of the form

$$\frac{\partial u}{\partial t} = A(u)\partial_x^2 u + g(u,u_x), \quad u(0) = f. \tag{7.53}$$

Take $M = S^1$. We assume $g(u,p)$ is smooth in its arguments. We will exploit the maximum principle to obtain a bound on u_x, which satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}u_x = {} & A(u)\,\partial_x^2(u_x) + A'(u)u_x\,\partial_x(u_x)\\ & + g_p(u,u_x)\,\partial_x(u_x) + g_u(u,u_x)u_x.\end{aligned} \tag{7.54}$$

The only restriction on applying the maximum principle to estimate $|u_x|$ is that we need $g_u(u,u_x) \le 0$. We can fix this by considering $e^{-tK}u_x$, which satisfies

$$\frac{\partial}{\partial t}\big(e^{-tK}u_x\big) = e^{-tK}\big(R - Ku_x\big), \tag{7.55}$$

where R is the right side of (7.54). The maximum principle yields

$$\|u_x\|_{C^0} \le e^{tK} \|\partial_x f\|_{C^0}, \tag{7.56}$$

provided

$$g_u(u, u_x) \le K. \tag{7.57}$$

We have the following result:

Proposition 7.8. *Given $f \in H^2(S^1)$, suppose you have a global a priori bound*

$$\|u(t)\|_{L^\infty} \le K_0, \tag{7.58}$$

for a solution to the scalar equation (7.53). If there is also an a priori bound (7.57) (for $|u| \le K_0$), which follows automatically in case $g = g(u)$ is a smooth function of u alone, then (7.53) has a solution for all $t \in [0, \infty)$.

The class of equations described by (7.53) includes those of the form

$$\frac{\partial u}{\partial t} = \partial_x\big(A(u)\,\partial_x u\big) + \varphi(u), \quad u(0) = f. \tag{7.59}$$

In fact, this is of the form (7.53), with

$$g(u, u_x) = A'(u)u_x^2 + \varphi(u). \tag{7.60}$$

Thus

$$g_u(u, u_x) = A''(u)u_x^2 + \varphi'(u). \tag{7.61}$$

In such a case, (7.57) applies if and only if $A''(u) \le 0$, that is, $A(u)$ is concave in u. For example, Proposition 7.8 applies to the equation

$$\frac{\partial u}{\partial t} = \partial_x\big(u\,\partial_x u\big) + \varphi(u), \quad u(0) = f, \tag{7.62}$$

in cases where it can be shown that, for some $a, b \in (0, \infty)$, $a \le u(t, x) \le b$ for all $t \ge 0$, $x \in S^1$. This in turn holds if $f(x)$ takes values in the interval $[a, b]$ with $\varphi(a) > 0$ and $\varphi(b) < 0$, by arguments similar to the proof of Proposition 4.3. A specific example is the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u(1-u), \quad u(0) = f, \tag{7.63}$$

arising in models of population growth (see [Grin], p. 224, or [Mur], p. 289). This is similar to reaction-diffusion equations studied in §4, but this time there is a nonlinear diffusion as well as a nonlinear reaction. In this case we see that (7.63) has a global solution, given smooth f with values in a interval $I = [a, b]$, with $a > 0$; $u(t, x) \in I$ for all $(t, x) \in \mathbb{R}^+ \times S^1$ if also $b > 1$. This existence result is rather special; a much larger class of global existence results will be established in § 9.

Exercises

1. In the setting of Proposition 7.3, given $u(0) = f \in H^\ell(M)$, initial data for (7.1), work out estimates for
$$\|u(t) - u_\varepsilon(t)\|_{H^j(M)}, \quad 1 \le j \le \ell - 1.$$
2. Establish global solvability on $[0, \infty) \times S^1$ for
$$\frac{\partial u}{\partial t} = A(u)\, \partial_x^2 u + \varphi(u), \quad u(0) = f, \tag{7.64}$$
given $f \in H^2(S^1)$, under the hypotheses
$$a \le f(x) \le b, \quad \varphi(a) \ge 0, \quad \varphi(b) \le 0,$$
and
$$A(u) \ge C > 0, \ \text{ for } a \le u \le b.$$
3. Establish global solvability on $[0, \infty) \times S^1$ for
$$\frac{\partial u}{\partial t} = A(u_x)u_{xx}, \quad u(0) = f, \tag{7.65}$$
given $f \in H^3(S^1)$, under the hypothesis
$$A(p) \ge C > 0 \text{ and } A''(p) \le 0, \ \text{ for } |p| \le \sup\ |f'(x)|. \tag{7.66}$$
(*Hint*: The function $v = u_x$ satisfies
$$\frac{\partial v}{\partial t} = \partial_x\big(A(v)\, \partial_x v\big), \quad v(0) = f'(x).$$
Estimate $v_x = u_{xx}$.)
Consider the example
$$\frac{\partial u}{\partial t} = \big(1 - u_x^2\big)^{1/2} u_{xx}, \tag{7.67}$$
assuming $|f'(x)| \le a < 1$, or the example
$$\frac{\partial u}{\partial t} = (1 + u_x^2)^{-1} u_{xx}, \tag{7.68}$$
assuming $|f'(x)| \le b < \sqrt{1/3}$.
A much more general global existence result is derived in § 9. See Exercise 3 of § 9 for a better existence result for (7.68).

8. Quasi-linear parabolic equations II (sharper estimates)

While most of the analysis in § 7 was fairly straightforward, the results are not as sharp as they can be, and we obtain sharper results here, making use of paradifferential operator calculus. The improvements obtained here will be coupled with Nash–Moser estimates and applied to global existence results in the next section. Most of the material of this section follows the exposition in [Tay].

Though we intend to concentrate on the quasi-linear case, we begin with completely nonlinear equations:

$$\frac{\partial u}{\partial t} = F(t, x, D_x^2 u), \quad u(0) = f, \tag{8.1}$$

for u taking values in $\mathbb{R}^K$. We suppose $F = F(t, x, \zeta)$, $\zeta = (\zeta_{\alpha j} : |\alpha| \le 2, 1 \le j \le K)$ is smooth in its arguments, and our strong parabolicity hypothesis is

$$-\mathrm{Re} \sum_{|\alpha|=2} (\partial F/\partial \zeta_\alpha)\xi^\alpha \ge C|\xi|^2 I, \tag{8.2}$$

for $\xi \in \mathbb{R}^n$, where Re $A = (1/2)(A + A^*)$, for a $K \times K$ matrix A. Using the paradifferential operator calculus developed in Chap. 13, § 10, we write

$$F(t, x, D_x^2 v) = M(v; t, x, D)v + R(v). \tag{8.3}$$

By Proposition 10.7 of Chap. 13, we have, for $r > 0$,

$$v(t) \in C^{2+r} \Longrightarrow M(v; t, x, \xi) \in \mathcal{A}_0^r S_{1,1}^2 \subset C^r S_{1,0}^2 \cap S_{1,1}^2, \tag{8.4}$$

where the symbol class $\mathcal{A}_0^r S_{1,\delta}^m$ is defined by (10.31) of Chap. 13. The hypothesis (8.2) implies

$$-\mathrm{Re}\, M(v; t, x, \xi) \ge C|\xi|^2 I > 0, \tag{8.5}$$

for $|\xi|$ large. Note that symbol smoothing in x, as in (9.27) of Chap. 13, gives

$$M(v; t, x, \xi) = M^\#(t, x, \xi) + M^b(t, x, \xi), \tag{8.6}$$

and when (8.4) holds (for fixed t),

$$M^\#(t, x, \xi) \in \mathcal{A}_0^r S_{1,\delta}^2, \quad M^b(t, x, \xi) \in S_{1,1}^{2-r\delta}. \tag{8.7}$$

We also have

$$-\mathrm{Re}\, M^\#(t, x, \xi) \ge C|\xi|^2 I > 0, \tag{8.8}$$

for $|\xi|$ large.

We will obtain a solution to (8.1) as a limit of solutions u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon F(t, x, D_x^2 J_\varepsilon u_\varepsilon), \quad u_\varepsilon(0) = f. \tag{8.9}$$

Thus we need to show that $u_\varepsilon(t, x)$ exists on an interval $t \in [0, T)$ independent of $\varepsilon \in (0, 1]$ and has a limit as $\varepsilon \to 0$ solving (8.1). As before, all this follows from an estimate on the H^s-norm, and we begin with

$$\begin{aligned} \frac{d}{dt}\|\Lambda^s u_\varepsilon(t)\|_{L^2}^2 &= 2(\Lambda^s J_\varepsilon F(t, x, D_x^2 J_\varepsilon u_\varepsilon), \Lambda^s u_\varepsilon) \\ &= 2(\Lambda^s M_\varepsilon J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) + 2(\Lambda^s R_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon). \end{aligned} \tag{8.10}$$

The last term is easily bounded by

$$C(\|u_\varepsilon(t)\|_{L^2})\big[\|J_\varepsilon u_\varepsilon(t)\|_{H^s}^2 + 1\big].$$

Here $M_\varepsilon = M(J_\varepsilon u_\varepsilon; t, x, D)$. Writing $M_\varepsilon = M_\varepsilon^\# + M_\varepsilon^b$ as in (8.6), we see that

$$\begin{aligned} &(\Lambda^s M_\varepsilon^b J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) \\ &\quad = (\Lambda^{s-1} M_\varepsilon^b J_\varepsilon u_\varepsilon, \Lambda^{s+1} J_\varepsilon u_\varepsilon) \\ &\quad \le C(\|J_\varepsilon u_\varepsilon\|_{C^{2+r}})\|J_\varepsilon u_\varepsilon\|_{H^{s+1-r\delta}}\|J_\varepsilon u_\varepsilon\|_{H^{s+1}}, \end{aligned} \tag{8.11}$$

for $s > 1$, since by (8.7), $M_\varepsilon^b : H^{s+1-r\delta} \longrightarrow H^{s-1}$. We next estimate

$$\begin{aligned} &(\Lambda^s M_\varepsilon^\# J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) \\ &\quad = (M_\varepsilon^\# \Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) + ([\Lambda^s, M_\varepsilon^\#] J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon). \end{aligned} \tag{8.12}$$

By (8.7), plus (10.99) of Chap. 13, we have $[\Lambda^s, M_\varepsilon^\#] \in OPS_{1,\delta}^{s+2-r}$ if $0 < r < 1$, so the last term in (8.12) is bounded by

$$\begin{aligned} &(\Lambda^{-1}[\Lambda^s, M_\varepsilon^\#] J_\varepsilon u_\varepsilon, \Lambda^{s+1} J_\varepsilon u_\varepsilon) \\ &\quad \le C(\|u_\varepsilon\|_{C^{2+r}})\|J_\varepsilon u_\varepsilon\|_{H^{s+1-r}} \cdot \|J_\varepsilon u_\varepsilon\|_{H^{s+1}}. \end{aligned} \tag{8.13}$$

Finally, Gårding's inequality (Theorem 6.1 of Chap. 7) applies to $M_\varepsilon^\#$:

$$(M_\varepsilon^\# w, w) \le -C_0\|w\|_{H^1}^2 + C_1(\|u_\varepsilon\|_{C^{2+r}})\|w\|_{L^2}^2. \tag{8.14}$$

Putting together the previous estimates, we obtain

$$\frac{d}{dt}\|u_\varepsilon(t)\|_{H^s}^2 \le -\frac{1}{2}C_0\|J_\varepsilon u_\varepsilon\|_{H^{s+1}}^2 + C(\|u_\varepsilon\|_{C^{2+r}})\|J_\varepsilon u_\varepsilon\|_{H^{s+1-r\delta}}^2, \tag{8.15}$$

and using Poincaré's inequality, we can replace $-C_0/2$ by $-C_0/4$ and the $H^{s+1-r\delta}$-norm by the H^s-norm, getting

$$\begin{aligned}(8.16)\qquad \frac{d}{dt}\|u_\varepsilon(t)\|_{H^s}^2 \le &-\frac{1}{4}C_0\|J_\varepsilon u_\varepsilon(t)\|_{H^{s+1}}^2 \\ &+ C'(\|u_\varepsilon(t)\|_{C^{2+r}})\|J_\varepsilon u_\varepsilon(t)\|_{H^s}^2.\end{aligned}$$

From here, the arguments used to establish Theorem 7.2 through Proposition 7.6 yield the following result:

Proposition 8.1. *If (8.1) is strongly parabolic and $f \in H^s(M)$, with $s > n/2 + 2$, then there is a unique solution*

$$(8.17)\qquad u \in C([0,T), H^s(M)) \cap C^\infty((0,T)\times M),$$

which persists as long as $\|u(t)\|_{C^{2+r}}$ is bounded, given $r > 0$.

Note that if the method of quasi-linearization were applied to (8.1) in concert with the results of § 7, we would require $s > n/2 + 3$ and for persistence of the solution would need a bound on $\|u(t)\|_{C^3}$.

We now specialize to the quasi-linear case (7.1), that is,

$$(8.18)\qquad \frac{\partial u}{\partial t} = \sum_{j,k} A^{jk}(t,x,D_x^1 u)\,\partial_j\partial_k u + B(t,x,D_x^1 u), \quad u(0) = f.$$

This is the special case of (8.1) in which

$$(8.19)\qquad F(t,x,D_x^2 u) = \sum A^{jk}(t,x,D_x^1 u)\,\partial_j\partial_k u + B(t,x,D_x^1 u).$$

We form $M(v;t,x,D)$ as before, by (8.3). In this case, we can replace (8.4) by

$$(8.20)\qquad v \in C^{1+r} \Longrightarrow M(v;t,x,\xi) \in \mathcal{A}_0^r S_{1,1}^2 + S_{1,1}^{2-r}.$$

Thus we can produce a decomposition (8.6) such that (8.7) holds for $v \in C^{1+r}$. Hence the estimates (8.11)–(8.16) all hold with constants depending on the C^{1+r}-norm of $u_\varepsilon(t)$, rather than the C^{2+r}-norm, and we have the following improvement of Theorem 7.2 and Proposition 7.6:

Proposition 8.2. *If the quasi-linear system (8.18) is strongly parabolic and $f \in H^s(M)$, $s > n/2 + 1$, then there is a unique solution satisfying (8.17), which persists as long as $\|u(t)\|_{C^{1+r}}$ is bounded, given $r > 0$.*

We look at the parabolic equation

$$(8.21)\qquad \frac{\partial u}{\partial t} = \sum A^{jk}(t,x,u)\,\partial_j\partial_k u + B(t,x,u),$$

which is a special case of (8.1), with

$$F(t,x,D_x^2 u) = \sum A^{jk}(t,x,u)\,\partial_j\partial_k u + B(t,x,u). \tag{8.22}$$

In this case, if $r > 0$, we have

$$v \in C^r \Longrightarrow M(v;t,x,\xi) \in \mathcal{A}_0^r S^2_{1,1} + S^{2-r}_{1,1}, \tag{8.23}$$

and the following results:

Proposition 8.3. *Assume the system (8.21) is strongly parabolic. If $f \in H^s(M)$, $s > n/2+1$, then there is a unique solution satisfying (8.17), which persists as long as $\|u(t)\|_{C^r}$ is bounded, given $r > 0$.*

It is also of interest to consider the case

$$\frac{\partial u}{\partial t} = \sum \partial_j A^{jk}(t,x,u)\,\partial_k u, \quad u(0) = f. \tag{8.24}$$

Arguments similar to those done above yield the following.

Proposition 8.4. *If the system (8.24) is strongly parabolic, and if $f \in H^s(M)$, $s > n/2+1$, then there is a unique solution to (8.24), satisfying (8.17), which persists as long as $\|u(t)\|_{C^r}$ is bounded, for some $r > 0$.*

We continue to study the quasi-linear system (8.18), but we replace the strong parabolicity hypothesis (7.2) with the following more general hypothesis on

$$L_2(t,v,x,\xi) = -\sum_{j,k} A^{jk}(t,x,v)\xi_j\xi_k; \tag{8.25}$$

namely,

$$\operatorname{spec} L_2(t,v,x,\xi) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -C_0|\xi|^2\}, \tag{8.26}$$

for some $C_0 > 0$. When this holds, we say that the system (8.18) is *Petrowski-parabolic*. Again we will try to produce the solution to (8.18) as a limit of solutions u_ε to (8.9). In order to get estimates, we construct a symmetrizer.

Lemma 8.5. *Given (8.26), there exists $P_0(t,v,x,\xi)$, smooth in its arguments, for $\xi \neq 0$, homogeneous of degree 0 in ξ, positive-definite (i.e., $P_0 \ge cI > 0$), such that $-(P_0L_2 + L_2^*P_0)$ is also positive-definite, that is,*

$$-(P_0L_2 + L_2^*P_0) \ge C|\xi|^2 I > 0. \tag{8.27}$$

Such a construction is done in Chap. 5. We briefly recall the argument used there, where this result is stated as Lemma 11.5. The symmetrizer P_0, which is not unique, is constructed by establishing first that if L_2 is a *fixed* $K \times K$ matrix with spectrum in Re $z < 0$, then there exists a $K \times K$ matrix P_0 such that P_0 and $-(P_0 L_2 + L_2^* P_0)$ are positive-definite. This is an exercise in linear algebra. One then observes the following facts. One, for a given positive matrix P_0, the set of L_2 such that $-(P_0 L_2 + L_2^* P_0)$ is positive-definite is open. Next, for given L_2 with spectrum in Re $z < 0$, the set $\{P_0 : P_0 > 0, -(P_0 L_2 + L_2^* P_0) > 0\}$ is an open convex set of matrices, within the linear space of self adjoint $K \times K$ matrices. Using this and a partition-of-unity argument, one can establish the following, which then yields Lemma 8.5. (Compare with Lemma 11.4 in Chap. 5. Also compare with the construction in § 8 of Chap. 15.)

Lemma 8.6. *If $\mathcal{M}_K^-$ denotes the space of real $K \times K$ matrices with spectrum in Re $z < 0$ and $\mathcal{P}_K^+$ the space of positive-definite (complex) $K \times K$ matrices, there is a smooth map*

$$\Phi : \mathcal{M}_K^- \longrightarrow \mathcal{P}_K^+,$$

homogeneous of degree 0*, such that if $L \in \mathcal{M}_K^-$ and $P = \Phi(L)$, then $-(PL + L^*P) \in \mathcal{P}_K^+$.*

Having constructed $P_0(t, v, x, \xi)$, note that, for fixed $t,\ r \in \mathbb{R}$,

$$\begin{aligned} u \in C^{1+r} \Longrightarrow L(t, D_x^1 u, x, \xi) &\in C_*^r S_{cl}^2 \quad \text{and} \\ P_0(t, D_x^1 u, x, \xi) &\in C_*^r S_{cl}^0. \end{aligned} \tag{8.28}$$

Now apply symbol smoothing in x to $\tilde{P}_0(t, x, \xi) = P_0(t, D_x^1 u, x, \xi)$, to obtain

$$P(t) \in OP\mathcal{A}_0^r S_{1,\delta}^0; \quad P(t) - \tilde{P}(t, D_x^1 u, x, D) \in OPC^r S_{1,\delta}^{-r\delta}. \tag{8.29}$$

Then set

$$Q = \frac{1}{2}(P + P^*) + K\Lambda^{-1}, \tag{8.30}$$

with $K > 0$ chosen so that Q is positive-definite on L^2. Now, with u_ε defined as the solution to (8.9), $u_\varepsilon(0) = f$, we estimate

$$\frac{d}{dt}(\Lambda^s u_\varepsilon, Q_\varepsilon \Lambda^s u_\varepsilon) = 2(\Lambda^s \partial_t u_\varepsilon, Q_\varepsilon \Lambda^s u_\varepsilon) + (\Lambda^s u_\varepsilon, P_\varepsilon' \Lambda^s u_\varepsilon), \tag{8.31}$$

where P_ε is obtained as in (8.29)–(8.30), from symbol smoothing of the family of operators $\tilde{P}_\varepsilon = P_0(t, D_x^1 J_\varepsilon u_\varepsilon, x, D)$, and Q_ε comes from P_ε via (8.30). Note that if $u_\varepsilon(t)$ is bounded in $C^{1+r}(M)$, then $P_\varepsilon'(t)$ is bounded in $OPS_{1,\delta}^{2-r}(M)$, so

$$|(\Lambda^s u_\varepsilon, P_\varepsilon' \Lambda^s u_\varepsilon)| \le C\big(\|u_\varepsilon(t)\|_{C^{1+r}}\big)\|u_\varepsilon(t)\|_{H^{s+1-r/2}}^2. \tag{8.32}$$

We can write the first term on the right side of (8.31) as twice

$$(Q_\varepsilon \Lambda^s J_\varepsilon M_\varepsilon J_\varepsilon u_\varepsilon, \Lambda^s u_\varepsilon) + (Q_\varepsilon \Lambda^s R_\varepsilon, \Lambda^s u_\varepsilon), \tag{8.33}$$

where M_ε is as in (8.10). The last term here is easily dominated by

$$C(\|u_\varepsilon(t)\|_{C^1}) \|J_\varepsilon u_\varepsilon(t)\|_{H^{s+1}} \cdot \|u_\varepsilon(t)\|_{H^s}. \tag{8.34}$$

We write the first term in (8.33) as

$$\begin{aligned}(Q_\varepsilon M_\varepsilon \Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) &+ (Q_\varepsilon [\Lambda^s, M_\varepsilon] J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) \\ &+ ([Q_\varepsilon \Lambda^s, J_\varepsilon] M_\varepsilon J_\varepsilon u_\varepsilon, \Lambda^s u_\varepsilon).\end{aligned} \tag{8.35}$$

We have $Q_\varepsilon(t) \in OP\mathcal{A}_0^r S^0_{1,\delta}$, by (10.100) of Chap. 13, and hence, by (10.99) of Chap. 13,

$$[Q_\varepsilon \Lambda^s, J_\varepsilon] \text{ bounded in } \mathcal{L}(H^{s-1}, L^2), \tag{8.36}$$

with a bound given in terms of $\|u_\varepsilon\|_{C^{1+r}}$ if $r > 1$. Furthermore, we have

$$\|M_\varepsilon J_\varepsilon u_\varepsilon\|_{H^{s-1}} \le C(\|u_\varepsilon\|_{C^{1+r}}) \|J_\varepsilon u_\varepsilon\|_{H^{s+1}}, \tag{8.37}$$

so we can dominate the last term in (8.35) by

$$C(\|u_\varepsilon(t)\|_{C^{1+r}}) \|J_\varepsilon u_\varepsilon\|_{H^{s+1}} \cdot \|u_\varepsilon\|_{H^s}, \tag{8.38}$$

provided $r > 1$. Moving to the second term in (8.35), since

$$M_\varepsilon \in OP\mathcal{A}_0^r S^2_{1,1} + OPS^{2-r}_{1,1},$$

we have

$$\|[\Lambda^s, M_\varepsilon] v\|_{L^2} \le C\big(\|u_\varepsilon\|_{C^{1+r}}\big) \|v\|_{H^{s+1}}, \tag{8.39}$$

provided $r > 1$. Hence the second term in (8.35) is also bounded by (8.38).

This brings us to the first term in (8.35), and for this we apply the Gårding inequality to the main term arising from $M_\varepsilon = M_\varepsilon^\# + M_\varepsilon^b$, to get

$$(Q_\varepsilon M_\varepsilon v, v) \le -C_0 \|v\|^2_{H^1} + C(\|u_\varepsilon\|_{C^2}) \|v\|^2_{L^2}. \tag{8.40}$$

Substituting $v = \Lambda^s J_\varepsilon u_\varepsilon$ and using the other estimates on terms from (8.31), we have

$$\begin{aligned}\frac{d}{dt}(\Lambda^s u_\varepsilon, Q_\varepsilon \Lambda^s u_\varepsilon) &\le -C_0 \|J_\varepsilon u_\varepsilon\|^2_{H^{s+1}} \\ &\quad + C(\|u_\varepsilon\|_{C^{3+\delta}}) \|u_\varepsilon\|_{H^s} \big[\|J_\varepsilon u_\varepsilon\|_{H^{s+1}} + \|u_\varepsilon\|_{H^s}\big]\end{aligned} \tag{8.41}$$

which we can further dominate as in (8.16). Note that (8.32) is the worst term; we need $r > 2$ for it to be useful.

From here, all the other arguments yielding Propositions 8.1 and 8.2 apply, and we have the following:

Proposition 8.7. *Given the Petrowski-parabolicity hypothesis (8.25)–(8.26), if $f \in H^s(M)$ and $s > n/2 + 3$, then (8.18) has a unique solution*

$$u \in C([0,T), H^s(M)) \cap C^\infty((0,T) \times M), \tag{8.42}$$

for some $T > 0$, which persists as long as $\|u(t)\|_{C^{1+r}}$ is bounded, for some $r > 0$.

In order to check the persistence result, we run through (8.31)–(8.41) with u_ε replaced by the solution u, and with J_ε replaced by I. In such a case, the analogue of (8.32) is useful for any $r > 0$. The analogue of (8.36) is vacuous, so (8.38) works for any $r > 0$. An analogue of (8.11)–(8.13) can be applied to (8.39); recalling that this time we have (8.7) for $u \in C^{1+r}$, we also obtain a useful estimate whenever $r > 0$. This gives the persistence result stated above.

Exercises

In Exercises 1–10, we look at the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= M\Delta u - a\nabla \cdot (u\nabla v), \\ \frac{\partial v}{\partial t} &= D\Delta v + \frac{bu}{u+h} - \mu v. \end{aligned} \tag{8.43}$$

We assume that M, D, μ, a, b, and h are positive constants, and Δ is the Laplace operator on a compact Riemannian manifold. This arises in a model of *chemotaxis*, the attraction of cells to a chemical stimulus. Here, $u = u(t,x)$ represents the concentration of cells, and $v = v(t,x)$ the concentration of a certain chemical (see [Grin], p. 194, or [Mur]).

1. Show that (8.43) is a Petrowski-parabolic system.
2. If (u, v) is a sufficiently smooth solution for $t \in [0,T)$, show that

$$u(0) \geq 0,\ v(0) \geq 0 \Longrightarrow u(t) \geq 0,\ v(t) \geq 0, \quad \forall\, t \in [0,T). \tag{8.44}$$

(*Hint*: If we can deduce $u(t) \geq 0$, the result follows for

$$v(t) = e^{(D\Delta-\mu)t}v(0) + \int_0^t e^{(D\Delta-\mu)(t-\tau)}\varphi\big(u(\tau)\big)\,d\tau, \quad \varphi(u) = \frac{bu}{u+h}.$$

Temporarily strengthen the hypothesis on u to $u(0,x) > 0$, and modify the first equation in (8.43) to

$$u_t = M\Delta u - a\nabla \cdot (u\nabla v) + \varepsilon,$$

with small $\varepsilon > 0$. Show that $u(t,x) > 0$ for t in the interval of existence by considering the first t_0 at which, for some $x_0 \in M$, $u(t_0, x_0) = 0$. Derive the contradictory estimate $\partial_t u(t_0, x_0) \geq \varepsilon$. To pass from the modified problem to the original, you may find it necessary to work Exercises 3–10 for the modified problem, which will involve no extra work.)

3. Show that $\|u(t)\|_{L^1(M)}$ is constant, for $t \in [0, T)$. (*Hint*: Integrate the first equation in (8.43) over $x \in M$, and use the positivity of u.)
   Note: The desired conclusion is slightly different for the modified problem.
4. Given the conclusion of (8.44), show that, for $I = [\varepsilon, T),\ r \in (0, 1)$,

$$\sup_{t \in I} \|v(t)\|_{C^{1+r}(M)} < \infty. \tag{8.45}$$

   (*Hint*: Regard the second equation in (8.43) as a nonhomogeneous linear equation for v, with nonhomogeneous term $F(t, x) = bu/(u + h) \in L^\infty(I \times M)$.)
5. Show that, for any $\delta > 0$,

$$\sup_{t \in I} \|u(t)\|_{H^{1-\delta,1}(M)} < \infty. \tag{8.46}$$

   (*Hint*: Regard the first equation in (8.43) as a nonhomogeneous linear equation for u, with nonhomogeneous term $G(t, x) = a\nabla \cdot H(t, x)$, where $H = u\nabla v \in L^\infty(I, L^1(M))$.)
6. Given (8.45)–(8.46), deduce that $H \in L^\infty(I, H^{r,1}(M))$, for any $r \in (0, 1)$. Hence improve (8.46) to

$$\sup_{t \in I} \|u(t)\|_{H^{2-\delta,1}(M)} < \infty, \tag{8.47}$$

   for any $\delta > 0$. Consequently, for $p \in \big(1, n/(n-1)\big)$,

$$\sup_{t \in I} \|u(t)\|_{H^{1,p}(M)} < \infty. \tag{8.48}$$

7. Now deduce that $H \in L^\infty(I, H^{r,p}(M))$, for any $r \in (0, 1),\ p \in \big(1, n/(n-1)\big)$. Hence improve (8.48) to

$$\sup_{t \in I} \|u(t)\|_{H^{2-\delta,p}(M)} < \infty. \tag{8.49}$$

8. Iterate the argument above, to establish (8.49) for all $p < \infty$, hence

$$\sup_{t \in I} \|u(t)\|_{C^{1+r}(M)} < \infty, \tag{8.50}$$

   for any $r < 1$.
9. Using (8.50), improve the estimate (8.45) to

$$\sup_{t \in I} \|v(t)\|_{C^{3+r}(M)} < \infty,$$

   for any $r < 1$. Then improve (8.50) to $\sup_I \|u(t)\|_{C^{2+r}(M)} < \infty$, and then to

$$\sup_{t \in I} \|u(t)\|_{C^{3+r}(M)} < \infty.$$

10. Now deduce the solvability of (8.43) for all $t > 0$, given (8.44).

In Exercises 11–12, we look at a strongly parabolic $K \times K$ system of the form

$$u_t = A(u)u_{xx} + g(u, u_x), \quad x \in S^1. \tag{8.51}$$

11. If $u \in C^\infty\big((0,T) \times M\big)$ solves (8.51), show that

$$\begin{aligned} \frac{d}{dt}\|u_x(t)\|^2_{L^2(S^1)} &\le -C_0\|u_{xx}(t)\|^2_{L^2} + 2\big\|g\big(u(t), u_x(t)\big)\big\|_{L^2}\|u_{xx}(t)\|_{L^2} \\ &\le -\frac{C_0}{2}\|u_{xx}(t)\|^2_{L^2} + \frac{2}{C_0}\big\|g\big(u(t), u_x(t)\big)\big\|^2_{L^2}, \end{aligned} \tag{8.52}$$

where $2A(u) \ge C_0 > 0$.

12. Suppose you can establish that the solution u possesses the following property: For each $t \in (0,T)$, $\|u(t,\cdot)\|_{L^\infty} \le C_1 < \infty$. Suppose

$$|g(v,p)| \le C_2(1 + |p|), \tag{8.53}$$

for $|v| \le C_1$. Show that u extends to a solution $u \in C^\infty\big((0,T_1) \times M\big)$ of (8.51), for some $T_1 > T$. (*Hint*: Use Proposition 8.3.)

In Exercises 13–15, we look at a strongly parabolic $K \times K$ system of the form

$$\frac{\partial u}{\partial t} = \partial_x A(u)\,\partial_x u + f(u), \quad x \in S^1. \tag{8.54}$$

13. If $u \in C^\infty\big((0,T) \times M\big)$ solves (8.54), show that

$$\begin{aligned} \frac{d}{dt}\|u_x(t)\|^2_{L^2(S^1)} \le{}& (\beta\|u(t)\|_{L^\infty} - C_0)\|u_{xx}(t)\|^2_{L^2} \\ &+ 2\big\|f'\big(u(t)\big)\big\|_{L^\infty}\|u_x(t)\|^2_{L^2}, \end{aligned} \tag{8.55}$$

where

$$2A(u) \ge C_0 > 0, \quad \beta = \sup_u\ 6\|DA(u)\|.$$

(*Hint*: Use the estimate

$$\|u_x\|^2_{L^4} \le 3\|u\|_{L^\infty}\|\partial_x^2 u\|_{L^2},$$

which follows from the $p = 1,\ k = 2$ case of Proposition 3.1 in Chap. 13.)

14. Improve the estimate (8.55) to

$$\frac{d}{dt}\|u_x(t)\|^2_{L^2} \le \big(\beta\mathcal{N}(u) - C_0\big)\|u_{xx}\|^2_{L^2} + 2\|f'(u)\|_{L^\infty}\|u_x\|^2_{L^2}, \tag{8.56}$$

where

$$\mathcal{N}(g) = \inf_{\lambda\in\mathbb{R}}\ \|g - \lambda\|_{L^\infty(S^1)} = \frac{1}{2}\,\mathrm{osc}\ g. \tag{8.57}$$

15. Suppose you can establish that the solution u possesses the following property: For each $t \in (0,T)$, $u(t,\cdot)$ takes values in a region $K_t \subset \mathbb{R}^K$ so small that

$\mathcal{N}\big(u(t)\big) \le C_0/\beta$. Assume $\|v\| \le C_1 < \infty$, for $v \in K_t$. Show that u extends to a solution $u \in C^\infty\big((0, T_1) \times M\big)$ of (8.54), for some $T_1 > T$.
Compare with the treatment of (7.59). See also the treatment of (9.61).

16. Rework Exercise 12, weakening the hypothesis (8.53) to

$$|g(v, p)| \le C_2(1 + |p|) + \alpha C_0 |p|^2, \quad \alpha < \frac{1}{6C_1}, \tag{8.58}$$

for $|v| \le C_1$.

9. Quasi-linear parabolic equations III (Nash–Moser estimates)

We will be able to get global solutions to a certain class of quasi-linear parabolic equations by applying the results of § 8 together with Hölder estimates for solutions to scalar equations of the form

$$\frac{\partial u}{\partial t} - Lu = 0, \quad Lu = b^{-1} \sum_{j,k} \partial_j \big(a^{jk} b \, \partial_k u\big), \tag{9.1}$$

where $a^{jk}, b, b^{-1} \in L^\infty$. The operator L is as in (9.1) of Chap. 14, and we make the same ellipticity hypothesis as used there; thus we assume

$$\lambda_0 \sum \xi_j^2 \le \sum a^{jk}(t, x)\xi_j \xi_k \le \lambda_1 \sum \xi_j^2, \quad b_0 \le b(x) \le b_1, \tag{9.2}$$

with

$$0 < \lambda_0 \le \lambda_1 < \infty, \quad 0 < b_0 \le b_1 < \infty. \tag{9.3}$$

We take b independent of t. Hölder estimates for solutions to (9.1) under these hypotheses were first proved by Nash [Na]. Moser [Mos2] established a Harnack inequality that yielded such Hölder estimates; a simpler proof is given in [Mos3]. Another treatment of Nash's results has been given in [FS]. All these arguments are more elaborate than that used for elliptic equations in Chap. 14, partly because they produce a sharper sort of Harnack inequality. Here, we follow [Kru], who obtained a parabolic analogue of the weaker Harnack inequality discussed in Chap. 14, by methods parallel to those in Moser's first treatment of the elliptic case, in [Mos1].

As in § 9 of Chap. 14, which we will refer to as "14" for short, we use a^{jk} to define an inner product of vectors in $\mathbb{R}^n$:

$$\langle V, W \rangle = \sum V_j a^{jk} W_k; \tag{9.4}$$

we use the square norm $|V|^2 = \langle V, V\rangle$; and we use $b\ dx = dV$ to define the volume element. Parallel to (9.4) of "14," we have

$$v = f(u) \Longrightarrow (\partial_t - L)v = f'(u)(\partial_t - L)u - f''(u)|\nabla u|^2. \tag{9.5}$$

We say v is a subsolution of $\partial_t - L$ provided $(\partial_t - L)v \le 0$. Thus we see that $u \mapsto f(u)$ takes solutions to $(\partial_t - L)u = 0$ to subsolutions if f is convex, while it takes subsolutions to subsolutions if f is both convex and increasing.

Next, parallel to (9.3) of "14," we have

$$\iint\limits_{Q} w(\partial_t - L)u\ dt\ dV = \iint\limits_{Q} \langle \nabla_x u, \nabla_x w\rangle\ dt\ dV + \iint\limits_{Q} w\partial_t u\ dt\ dV, \tag{9.6}$$

where $Q = I \times \Omega = [T_1, T_2] \times \Omega$ and w vanishes near $I \times \partial\Omega$. If we set $w = \psi^2 u$, where $\psi(t, x)$ is C^∞ and vanishes for x near $\partial\Omega$, we obtain the following analogue of (9.5) of "14":

$$\begin{aligned} &\iint \psi^2 |\nabla_x u|^2\ dt\ dV \\ &= -2\iint \langle \psi\nabla_x u, u\nabla_x\psi\rangle\ dt\ dV + \iint \psi^2 g u\ dt\ dV \\ &+ \iint (\partial_t\psi^2)u^2\ dt\ dV - \frac{1}{2}\int \psi^2 u(T_2, x)\ dV + \frac{1}{2}\int \psi^2 u(T_1, x)\ dV, \end{aligned} \tag{9.7}$$

provided $(\partial_t - L)u = g$. Consequently, parallel to the estimate (9.6) of "14," we have

$$\begin{aligned} &\frac{1}{2}\iint \psi^2|\nabla_x u|^2\ dt\ dV + \frac{1}{2}\int \psi^2 u(T_2, x)^2\ dV \\ &\le 2\iint u^2\Big(|\nabla_x\psi|^2 + \frac{1}{2}\partial_t\psi^2\Big)\ dt\ dV \\ &+ \iint \psi^2 g u\ dt\ dV + \frac{1}{2}\int \psi^2 u(T_1, x)^2\ dV. \end{aligned} \tag{9.8}$$

We now proceed to a Moser iteration argument, parallel to (9.7)–(9.20) of "14." Given $Q = I \times \Omega$, consider nested sequences of regions $\Omega = \Omega_0 \supset \cdots \supset \Omega_j \supset \Omega_{j+1} \supset \cdots$ in $\mathbb{R}^n$ and intervals $I = I_0 \supset \cdots \supset I_j \supset I_{j+1} \supset \cdots$, with intersections $\mathcal{O}$ and J, respectively, so we have $Q_j = I_j \times \Omega_j \searrow \widetilde{Q} = J \times \mathcal{O}$ (see Fig. 9.1). Let us assume $I = [0, T]$ and $I_j = [\tau_j, T]$, with $J = [T/2, T]$. We suppose that the distance of any point in $\partial\Omega_{j+1}$ to $\partial\Omega_j$ is $\sim j^{-2}$ and that the length of $I_j \setminus I_{j+1}$ is $\sim j^{-2}$. We want to estimate the sup norm of a function v on $\widetilde{Q}$ in terms of its L^2-norm on Q, assuming

$$v > 0 \ \text{ and } \ (\partial_t - L)v \le 0. \tag{9.9}$$

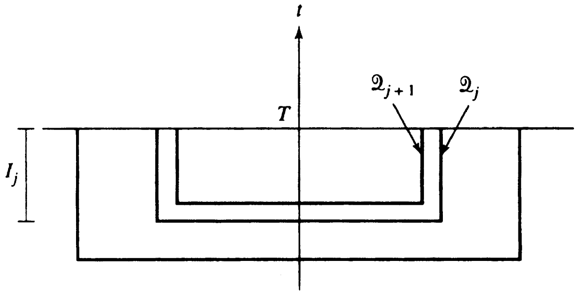

FIGURE 9.1 Setup for Moser Iteration

In view of (9.5), an example is

$$v = \left(1+u^2\right)^{1/2}, \quad Lu = 0. \tag{9.10}$$

We will obtain such an estimate in terms of certain Sobolev constants, $\gamma(Q_j)$ and C_j, arising in the following two lemmas, which are analogous to Lemmas 9.1 and 9.2 of "14."

Lemma 9.1. *For sufficiently regular v defined on Q_j, and with $\kappa \le n/(n-2)$, we have*

$$\begin{aligned} \|v^\kappa\|^2_{L^2(Q_j)} &\le \gamma(Q_j)\Big(\sigma_j(v)^{\kappa-1}\|\nabla_x v\|^{2\kappa}_{L^2(Q_j)} + \sigma_j(v)^\kappa\Big), \\ \sigma_j(v) &= \sup_{t\in I_j} \|v(t)\|^2_{L^2(\Omega_j)}. \end{aligned} \tag{9.11}$$

Proof. This is a consequence of the following slightly sharper form of (9.10) in "14":

$$\|v^\kappa\|^2_{L^2(\Omega_j)} \le \gamma(\Omega_j)\Big(\|\nabla_x v\|^2_{L^2(\Omega_j)}\|v\|^{2(\kappa-1)}_{L^2(\Omega_j)} + \|v\|^{2\kappa}_{L^2(\Omega_j)}\Big). \tag{9.12}$$

Indeed, integrating (9.12) over $t \in I_j$ gives (9.11).

Next, we have

Lemma 9.2. *If $v > 0$ is a subsolution of $\partial_t - L$, then,*

$$\|\nabla_x v\|_{L^2(Q_{j+1})} + \sup_{t\in I_{j+1}} \|v(t)\|_{L^2(\Omega_{j+1})} \le C_j\|v\|_{L^2(Q_j)}, \tag{9.13}$$

where $C_j = C(Q_j, Q_{j+1})$.

Proof. This follows from (9.8), with $u = v$, if we let $T_1 = \tau_j$, pick $\psi = \varphi_j(x)\eta_j(t)$, with $\varphi_j(x) = 0$ for x near $\partial\Omega_j$, while $\varphi_j(x) = 1$ for $x \in \Omega_{j+1}$, and $\eta_j(\tau_j) = 0$, while $\eta_j(t) = 1$ for $t \in I_{j+1}$. Then let T_2 run over $[\tau_{j+1}, T]$.

We construct the functions φ_j and η_j to go from 0 to 1 roughly linearly, over a layer of width $\sim Cj^{-2}$. As in (9.12) of "14," we can arrange that

$$\gamma(Q_j) \le \gamma_0, \quad C_j \le C_0(j^2+1). \tag{9.14}$$

Putting together these two lemmas, we see that when v satisfies (9.9),

$$\|v^\kappa\|^2_{L^2(Q_{j+1})} \le \gamma_0(C_j^{2\kappa}+1)\|v\|^{2\kappa}_{L^2(Q_j)}. \tag{9.15}$$

Now, if v satisfies (9.8), so does $v_j = v^{\kappa^j}$, by (9.5). Note that $v_{j+1} = v_j^\kappa$. From here, the estimate on

$$\|v\|_{L^\infty(\widetilde{Q})} \le \limsup_{j\to\infty} \|v_j\|^{1/\kappa^j}_{L^2(Q_j)} \tag{9.16}$$

goes precisely like the estimates on (9.16) in "14," so we have the sup-norm estimate:

Theorem 9.3. *If $v > 0$ is a subsolution of $\partial_t - L$, then*

$$\|v\|_{L^\infty(\widetilde{Q})} \le K\|v\|_{L^2(Q)}, \tag{9.17}$$

where $K = K(\gamma_0, C_0, n)$.

Next we prepare to establish a Harnack inequality. Parallel to (9.24) of "14," we take $w = \psi^2 f'(u)$ in (9.6), to get

$$\begin{aligned} &\iint \psi^2 f''(u)|\nabla_x u|^2 \,dt\,dV + \int \psi^2 f\big(u(T_2,x)\big)\,dV \\ &\quad = -2\iint \langle \psi f'(u)\nabla_x u, \nabla_x \psi\rangle \,dt\,dV + 2\iint \psi(\partial_t\psi) f(u)\,dt\,dV \\ &\qquad + \int \psi^2 f\big(u(T_1,x)\big)\,dV \end{aligned} \tag{9.18}$$

if $(\partial_t - L)u = 0$ on $[T_1,T_2]\times\Omega$, and $\psi(t,x) = 0$ for x near $\partial\Omega$. If we pick $f(u)$ to satisfy the differential inequality

$$f''(u) \ge f'(u)^2, \tag{9.19}$$

we have from (9.18) that

$$\begin{aligned} &\frac{1}{2}\iint \psi^2 f''(u)|\nabla_x u|^2\,dt\,dV + \int \psi^2 f\big(u(T_2,x)\big)\,dV \\ &\quad \le 2\iint |\nabla_x\psi|^2\,dt\,dV + 4\iint \psi\psi_t f(u)\,dt\,dV \\ &\qquad + \int \psi^2 f\big(u(T_1,x)\big)\,dV, \end{aligned} \tag{9.20}$$

provided $(\partial_t - L)u = 0$ on $[T_1, T_2] \times \Omega$. Since $f''(u)|\nabla_x u|^2 \geq f'(u)^2|\nabla_x u|^2 = |\nabla_x v|^2$, we have

$$\begin{aligned} &\frac{1}{2}\iint \psi^2|\nabla_x v|^2\,dt\,dV + \int \psi^2 v(T_2, x)\,dV \leq \\ &2\iint |\nabla_x \psi|^2\,dt\,dV + 4\iint \psi\psi_t v\,dt\,dV + \int \psi^2 v(T_1, x)\,dV. \end{aligned} \tag{9.21}$$

If we take $\psi(t, x) = \varphi(x) \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} &\frac{1}{2}\iint \varphi^2|\nabla_x v|^2\,dt\,dV + \int \varphi^2 v(T_2, x)\,dV \\ &\qquad \leq 2(T_2 - T_1)\int |\nabla_x \varphi|^2\,dV + \int \varphi^2 v(T_1, x)\,dV. \end{aligned} \tag{9.22}$$

We will apply the estimate (9.22) in the following situation. Suppose $(\partial_t - L)u = 0$ on $Q = [0, T] \times \Omega,\ u \geq 0$ on Ω, and

$$\text{meas}\{(t, x) \in Q : u(t, x) \geq 1\} \geq \frac{1}{2}\,\text{meas}\, Q. \tag{9.23}$$

Let Ω be a ball in $\mathbb{R}^n$ and $\mathcal{O}$ a concentric ball, such that

$$\text{meas}\, \mathcal{O} \geq \frac{3}{4}\,\text{meas}\, \Omega. \tag{9.24}$$

Here, $dV = b\,dx$ is used to compute the measure of a set in $\mathbb{R}^n$. Given $h > 0$, let

$$\mathcal{O}_t(h) = \{x \in \mathcal{O} : u(t, x) \geq h\}, \quad \Omega_t(h) = \{x \in \Omega : u(t, x) \geq h\}. \tag{9.25}$$

Pick $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on $\mathcal{O}$, and set

$$v = f(u) = \log^+ \frac{1}{u + h}. \tag{9.26}$$

Note that f satisfies the differential inequality (9.19), and $f(u) = 0$ for $u \geq 1$. From the hypothesis (9.23), we can pick $T_1 \in (0, T)$ such that

$$\text{meas}\, \Omega_{T_1}(1) \geq \frac{1}{2}\,\text{meas}\, \Omega. \tag{9.27}$$

We let t be any point in $(T_1, T]$ and apply (9.22), with $T_2 = t$ (discarding the first integral). Since $v \geq \log(1/2h)$ for $x \in \mathcal{O} \setminus \mathcal{O}_t(h)$ while $v \leq \log(1/h)$ on Ω and $v = 0$ on at least half of Ω, we get

$$\Bigl(\log \frac{1}{2h}\Bigr)\,\text{meas}\bigl(\mathcal{O} \setminus \mathcal{O}_t(h)\bigr) \leq K + \frac{1}{2}\Bigl(\log \frac{1}{h}\Bigr)\,\text{meas}\, \Omega, \tag{9.28}$$

with K independent of h. In view of (9.24), this implies

$$\text{meas}\,\mathcal{O}_t(h) \geq \frac{1}{4}\,\frac{1}{1-\delta(h)}\,\text{meas}\,\Omega - \frac{K}{\lambda(h)}, \tag{9.29}$$

where

$$\lambda(h) = \log\frac{1}{2h}, \quad \delta(h) = \frac{\log\ 2}{\log\frac{1}{h}}.$$

Taking h sufficiently small, we have

Lemma 9.4. *If $u \geq 0$ on $Q = [0,T]\times\Omega$ satisfies $(\partial_t - L)u = 0$, then, under the hypotheses (9.23) and (9.24), there exist $h > 0$ and $T_1 < T$ such that, for all $t \in (T_1, T]$,*

$$\textit{meas}\big\{x \in \mathcal{O} : u(t,x) \geq h\big\} \geq \frac{1}{5}\,\textit{meas}\,\mathcal{O}. \tag{9.30}$$

We are now ready to prove the following Harnack-type inequality:

Proposition 9.5. *Let $u \geq 0$ be a solution to $(\partial_t - L)u = 0$ on $Q = [0,T]\times\Omega$, where Ω is a ball in $\mathbb{R}^n$ centered at x_0. Assume that (9.23) holds. Then there is a concentric ball $\widetilde{\Omega}$, a number $\tau < T$, and $\kappa > 0$, depending only on Q and the quantities λ_j, b_j in (9.2), such that*

$$u(t,x) \geq \kappa \quad \text{on } [\tau, T]\times\widetilde{\Omega} = \widetilde{Q}. \tag{9.31}$$

Proof. Pick $\tau_0 \in (T_1, T)$, and let $Q_0 = [\tau_0, T]\times\mathcal{O}$. We will apply (9.22), with the double integral taken over Q_0, and with

$$v = f(u) = \log^+\frac{h}{u+\varepsilon}. \tag{9.32}$$

Here, h is as in (9.30), and we will take $\varepsilon \in (0, h/2]$. With $\varphi \in C_0^\infty(\mathcal{O})$, (9.22) yields

$$\frac{1}{2}\iint\limits_{Q_0} \varphi^2|\nabla_x v|^2\,dt\,dV \leq K + \int\limits_{\mathcal{O}} \varphi^2 v(\tau_0,x)\,dV \leq K + C_1\log\frac{h}{\varepsilon}. \tag{9.33}$$

Now $v = f(u) = 0$ for $u \geq h$, hence on the set $\mathcal{O}_t(h)$, whose measure was estimated from below in (9.30). Thus, for each $t \in [\tau_0, T]$,

$$\int\limits_{\widetilde{\mathcal{O}}} v(t,x)^2\,dV \leq C_2\int\limits_{\widetilde{\mathcal{O}}} \big|\nabla_x v(t,x)\big|^2\,dV \tag{9.34}$$

if we take $\widetilde{\mathcal{O}}$ to be a ball concentric with $\mathcal{O}$, such that meas $\widetilde{\mathcal{O}} \geq (9/10)$ meas $\mathcal{O}$. We make $\varphi = 1$ on $\widetilde{\mathcal{O}}$ and conclude that

$$\iint_{\mathcal{R}} v^2 \, dt \, dV \leq C_3 + C_4 \log \frac{h}{\varepsilon}, \quad \mathcal{R} = [\tau_0, T] \times \widetilde{\mathcal{O}}. \tag{9.35}$$

Since the function f in (9.32) is convex, we see that Theorem 9.3 applies to v. Hence we obtain $\widetilde{Q} \subset \mathcal{R}$ such that

$$\|v\|_{L^\infty(\widetilde{Q})} \leq C \|v\|_{L^2(\mathcal{R})} \leq C \log \frac{3h}{\varepsilon}. \tag{9.36}$$

Now, if we require that $\varepsilon \in (0, h/2]$ and take ε sufficiently small, this forces

$$u \geq \varepsilon^{1/2} - \varepsilon \quad \text{on } \widetilde{Q}, \tag{9.37}$$

and the proposition is proved.

We now deduce the Hölder continuity of a solution to $(\partial_t - L)u = 0$ on $Q = [0, T] \times \Omega$ from Proposition 9.5, by an argument parallel to that of (9.33)–(9.39) of "14." We have from (9.17) a bound $|u(t, x)| \leq K$ on any compact subset $\widetilde{Q}$ of $(0, T] \times \Omega$. Fix $(t_0, x_0) \in \widetilde{Q}$, and let

$$\omega(r) = \sup_{\mathcal{B}_r} u(t, x) - \inf_{\mathcal{B}_r} u(t, x), \tag{9.38}$$

where

$$\mathcal{B}_r = \big\{(t, x) : 0 \leq t_0 - t \leq ar^2, |x - x_0| \leq ar\big\}. \tag{9.39}$$

Say $\mathcal{B}_r \subset \widetilde{Q}$ for $r \leq \rho$. Clearly, $\omega(\rho) \leq 2K$. Adding a constant to u, we can assume

$$\sup_{\mathcal{B}_\rho} u(t, x) = -\inf_{\mathcal{B}_\rho} u(t, x) = \frac{1}{2}\omega(\rho) = M. \tag{9.40}$$

Then $u_+ = 1 + u/M$ and $u_- = 1 - u/M$ are annihilated by $\partial_t - L$. They are both ≥ 0 and at least one of them satisfies the hypotheses of Proposition 9.5 after we rescale $\mathcal{B}_\rho$, dilating x by a factor of ρ^{-1} and t by a factor of ρ^{-2}. If, for example, Proposition 9.5 applies to u_+, we have $u_+(t, x) \geq \kappa$ in $\mathcal{B}_{\sigma\rho}$, for some $\sigma \in (0, 1)$. Hence $\omega(\sigma\rho) \leq (1 - \kappa/2)\omega(\rho)$. Iterating this argument, we obtain

$$\omega(\sigma^\nu \rho) \leq \Big(1 - \frac{\kappa}{2}\Big)^\nu \omega(\rho), \tag{9.41}$$

which implies Hölder continuity:

$$\omega(r) \le C r^{\alpha}, \tag{9.42}$$

for an appropriate $\alpha > 0$. We have proved the following:

Theorem 9.6. *If u is a real-valued solution to (9.1) on $I \times \Omega$, with $I = [0, T)$, then, given $J = [T_0, T)$, $T_0 \in (0, T)$, $\mathcal{O} \subset\subset \Omega$, we have for some $\mu > 0$ an estimate*

$$\|u\|_{C^{\mu}(J \times \overline{\mathcal{O}})} \le C \|u\|_{L^2(I \times \Omega)}, \tag{9.43}$$

where C depends on the quantities λ_j, b_j in (9.2), but not on the modulus of continuity of $a^{jk}(t, x)$ or of $b(t, x)$.

Theorem 9.6 has the following implication:

Theorem 9.7. *Let M be a compact, smooth manifold. Suppose u is a bounded, real-valued function satisfying*

$$\frac{\partial u}{\partial t} = \mathit{div}\big(A(t, x)\, \mathit{grad}\, u\big) \tag{9.44}$$

on $[t_0, t_0 + a] \times M$. Assume that $A(t, x) \in \mathit{End}(T_x M)$ satisfies

$$\lambda_0 |\xi|^2 \le \big\langle A(t, x)\xi, \xi \big\rangle \le \lambda_1 |\xi|^2, \tag{9.45}$$

where the inner product and square norm are given by the metric tensor on M. Then $u(t_0 + a, x) = w(x)$ belongs to $C^r(M)$ for some $r > 0$, and there is an estimate

$$\|w\|_{C^r} \le K(M, a, \lambda_0, \lambda_1) \|u(t_0, \cdot)\|_{L^\infty}. \tag{9.46}$$

In particular, the factor $K(M, a, \lambda_0, \lambda_1)$ does not depend on the modulus of continuity of A.

We are now ready to establish some global existence results. For simplicity, we take $M = \mathbb{T}^n$.

Proposition 9.8. *Consider the equation*

$$\frac{\partial u}{\partial t} = \sum \partial_j A^{jk}(t, x, u)\, \partial_k u, \quad u(0) = f. \tag{9.47}$$

Assume this is a scalar parabolic equation, so $a^{jk} = A^{jk}(t, x, u)$ satisfies (9.2), with $\lambda_j = \lambda_j(u)$. Then the solution guaranteed by Proposition 8.4 exists for all $t > 0$.

Proof. An L^∞-bound on $u(t)$ follows from the maximum principle, and then (9.46) gives a C^r-bound on $u(t)$, for some $r > 0$. Hence global existence follows from Proposition 8.4.

Let us also consider the parabolic analogue of the PDE (10.1) of Chap. 14, namely,

$$\frac{\partial u}{\partial t} = \sum A^{jk}(\nabla u)\, \partial_j \partial_k u, \quad u(0) = f, \tag{9.48}$$

with

$$A^{jk}(p) = F_{p_j p_k}(p). \tag{9.49}$$

Again assume u is scalar. Also, for simplicity, we take $M = \mathbb{T}^n$. We make the hypothesis of uniform ellipticity:

$$A_1|\xi|^2 \le \sum F_{p_j p_k}(p)\xi_j \xi_k \le A_2|\xi|^2, \tag{9.50}$$

with $0 < A_1 < A_2 < \infty$. Then Proposition 8.2 applies, given $f \in H^s(M)$, $s > n/2 + 1$. Furthermore, $u_\ell = \partial_\ell u$ satisfies

$$\frac{\partial u_\ell}{\partial t} = \sum \partial_j A^{jk}(\nabla u)\, \partial_k u_\ell, \quad u_\ell(0) = f_\ell = \partial_\ell f. \tag{9.51}$$

The maximum principle applies to both (9.48) and (9.51). Thus, given $u \in C([0,T], H^s) \cap C^\infty((0,T) \times M)$,

$$|u(t,x)| \le \|f\|_{L^\infty}, \quad |u_\ell(t,x)| \le \|f_\ell\|_{L^\infty}, \quad 0 \le t < T. \tag{9.52}$$

Now the Nash–Moser theory applies to (9.51), to yield

$$\|u_\ell(t,\cdot)\|_{C^r(M)} \le K, \quad 0 \le t < T, \tag{9.53}$$

for some $r > 0$, as long as the ellipticity hypothesis (9.50) holds. Hence again we can apply Proposition 8.4 to obtain global solvability:

Theorem 9.9. *If $F(p)$ satisfies (9.50), then the scalar equation (9.48) has a solution for all $t > 0$, given $f \in H^s(M)$, $s > n/2 + 1$.*

Parallel to the extension of estimates for solutions of $Lu = 0$ to the case $Lu = f$ made in Theorem 9.6 of Chap. 14, there is an extension of Theorem 9.6 of this chapter to the case

$$\frac{\partial u}{\partial t} = Lu + f, \tag{9.54}$$

where L has the form (9.1).

Theorem 9.10. *Assume u is a real-valued solution to (9.54) on $I \times \Omega$, with*

$$\sup_{t\in I} \|f(t)\|_{L^p(M)} \le K_0, \quad p > \frac{n}{2}. \tag{9.55}$$

Then u continues to satisfy an estimate of the form (9.43), with C also depending on K_0.

It is possible to modify the proof of Theorem 9.6 in Chap. 14 to establish this. Other approaches can be found in [LSU] and [Kry]. We omit details.

With this, we can extend the existence theory for (9.47) to scalar equations of the form

$$\frac{\partial u}{\partial t} = \sum \partial_j A^{jk}(t,x,u)\,\partial_k u + \varphi(u), \quad u(0) = f. \tag{9.56}$$

An example is the equation

$$\frac{\partial u}{\partial t} = \sum_j \frac{\partial}{\partial x_j}\left(u\frac{\partial u}{\partial x_j}\right) + u(1-u), \quad u(0) = f, \tag{9.57}$$

the multidimensional case of the equation (7.63) for a model of population growth. We have the following result:

Proposition 9.11. *Assume the equation (9.56) satisfies the parabolicity condition (9.2), with $\lambda_j = \lambda_j(u)$. Suppose we have $a_1 < a_2$ in $\mathbb{R}$, with $\varphi(a_1) \ge 0$, $\varphi(a_2) \le 0$. If $f \in C^\infty(M)$ takes values in the interval $[a_1, a_2]$, then (9.56) has a unique solution $u \in C^\infty\big([0,\infty) \times M\big)$.*

Proof. The local solution $u \in C^\infty([0,T) \times M)$ given by Proposition 8.3 has the property that

$$u(t,x) \in [a_1, a_2], \quad \forall\, t \in [0,T), \ \ x \in M. \tag{9.58}$$

With this L^∞-bound, we deduce a C^r-bound on $u(t)$, from Theorem 9.10, and hence the continuation of u beyond $t = T$, for any $T < \infty$.

To see that (9.58) holds, we could apply a maximum-principle-type argument. Alternatively, we can extend the Trotter product formula of § 5 to treat time-dependent operators, replacing L by $L(t)$. Then, for $t \in [0,T)$,

$$u(t) = \lim_{n\to\infty} S(t, t_{n-1})\mathcal{F}^{t/n} \cdots S(t_1, 0)\mathcal{F}^{t/n} f, \tag{9.59}$$

where $t_j = (j/n)t$, $S(t,s)$ is the solution operator to

$$\frac{\partial v}{\partial t} = \sum \partial_j A^{jk}\big(t,x,u(t,x)\big)\,\partial_k v, \quad S(t,s)v(s) = v(t), \tag{9.60}$$

and $\mathcal{F}^t$ is the flow on $\mathbb{R}$ generated by φ, viewed as a vector field on $\mathbb{R}$. In this case, $\mathcal{F}^{t/n}$ and $S(t_{j+1}, t_j)$ both preserve the class of smooth functions with values in $[a_1, a_2]$.

We see that Proposition 9.11 applies to the population growth model (9.57) whenever $a_1 \in (0, 1]$ and $a_2 \in [1, \infty)$.

We now mention some *systems* for which global existence can be proved via Theorems 9.6–9.10. Keeping $M = \mathbb{T}^n$, let $u = (u_1, \dots, u_\ell)$ take values in $\mathbb{R}^\ell$, and consider

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(D(u)\frac{\partial u}{\partial x_j}\Big) + X(u), \quad u(0) = f, \tag{9.61}$$

where X is a vector field in $\mathbb{R}^\ell$ and $D(u)$ is a diagonal $\ell \times \ell$ matrix, with diagonal entries $d_k \in C^\infty(\mathbb{R}^\ell)$ satisfying

$$d_k(u) > 0, \quad \forall\, u \in \mathbb{R}^\ell. \tag{9.62}$$

We have the following; compare with Proposition 4.4.

Proposition 9.12. *Assume there is a family of rectangles*

$$K_t = \{v \in \mathbb{R}^\ell : a_j(t) \le v_j \le b_j(t), 1 \le j \le \ell\} \tag{9.63}$$

such that

$$\mathcal{F}_X^t(K_s) \subset K_{s+t}, \quad s, t \in \mathbb{R}^+, \tag{9.64}$$

where $\mathcal{F}_X^t$ is the flow on $\mathbb{R}^\ell$ generated by X. If $f \in C^\infty(M)$ takes values in K_0, then, under the hypothesis (9.62) on the diagonal matrix $D(u)$, the system (9.61) has a solution for all $t \in \mathbb{R}^+$, and $u(t, x) \in K_t$.

Proof. Using a product formula of the form (9.59), where $S(t, s)$ is the solution operator to

$$\frac{\partial v}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\Big(D(u)\frac{\partial v}{\partial x_j}\Big), \quad S(t, s)v(s) = v(t), \tag{9.65}$$

and $\mathcal{F}^t = \mathcal{F}_X^t$, we see that if u is a smooth solution to (9.61) for $t \in [0, T)$, then $u(t, x) \in K_t$ for all $(t, x) \in [0, T) \times M$, provided $f(x) \in K_0$ for all $x \in M$. This gives an L^∞-bound on $u(t)$. Now, for $1 \le k \le \ell$, regard each u_k as a solution to the nonhomogeneous scalar equation

$$\frac{\partial u_k}{\partial t} = \sum_j \frac{\partial}{\partial x_j}\Big(d_k(u)\frac{\partial u_k}{\partial x_j}\Big) + F_k, \quad F_k(t, x) = X_k\big(u(t, x)\big). \tag{9.66}$$

We can apply Theorem 9.10 to obtain Hölder estimates on each u_k. Thus the solution continues past $t = T$, for any $T < \infty$.

Exercises

1. Show that the scalar equation (9.56) has a solution for all $t \in [0, \infty)$ provided there exist $C, M \in (0, \infty)$ such that

$$u \geq M \Rightarrow \varphi(u) \leq Cu, \quad u \leq -M \Rightarrow \varphi(u) \geq -C|u|.$$

2. Formulate and establish generalizations to appropriate quasi-linear equations of results in Exercises 2–6 of § 4, on reaction-diffusion equations.
3. Reconsider (7.68), namely,

$$\frac{\partial u}{\partial t} = (1 + u_x^2)^{-1} u_{xx}, \quad u(0, x) = f(x). \tag{9.67}$$

Demonstrate global solvability, without the hypothesis $|f'(x)| \leq b < \sqrt{1/3}$. More generally, solve (7.65), under only the first of the two hypotheses in (7.66).

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16

Nonlinear Hyperbolic Equations

Introduction

Here we study nonlinear hyperbolic equations, with emphasis on quasi-linear systems arising from continuum mechanics, describing such physical phenomena as vibrating strings and membranes and the motion of a compressible fluid, such as air.

Sections 1–3 establish the local solvability for various types of nonlinear hyperbolic systems, following closely the presentation in [Tay]. At the end of § 1 we give some examples of some equations for which smooth solutions break down in finite time. In one case, there is a weak solution that persists, with a singularity. This is explored more fully later in the chapter.

In § 4 we prove the Cauchy–Kowalewsky theorem, in the nonlinear case, using the method of Garabedian [Gb2] to transform the problem to a quasi-linear, symmetric hyperbolic system.

In § 5 we derive the equations of ideal compressible fluid flow and discuss some classical results of Bernoulli, Kelvin, and Helmholtz regarding the significance of the vorticity of a fluid flow.

In § 6 we begin the study of weak solutions to quasi-linear hyperbolic systems of conservation law type, possessing singularities called shocks. Section 6 is devoted to scalar equations, for which there is a well-developed theory.

We then study $k \times k$ systems of conservation laws, with $k \geq 2$, in §§ 7–10, restricting attention to the case of one space variable. Section 7 is devoted to the "Riemann problem," in which piecewise-constant initial data are given. Section 8 discusses the role of "entropy" and of "Riemann invariants" for systems of conservation laws. These concepts are used in § 9, where we establish a result of R. DiPerna [DiP4] on the global existence of entropy-satisfying weak solutions for a class of 2×2 systems, in one space variable.

The first nonlinear hyperbolic system we derived, in § 1 of Chap. 2, was the system for vibrating strings. We return to this in § 10. Far from setting down a definitive analysis, we make note of some further subtleties that arise in the study

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of vibrating strings, giving rise to problems that have by no means been overcome. This starkly illustrates that in the study of nonlinear hyperbolic equations, a great deal remains to be done.

1. Quasi-linear, symmetric hyperbolic systems

In this section we examine existence, uniqueness, and regularity for solutions to a system of equations of the form

$$\frac{\partial u}{\partial t} = L(t, x, u, D_x)u + g(t, x, u), \quad u(0) = f. \tag{1.1}$$

We derive a short-time existence theorem under the following assumptions. We suppose that

$$L(t, x, u, D_x)v = \sum_j A_j(t, x, u)\, \partial_j v \tag{1.2}$$

and that each A_j is a $K \times K$ matrix, smooth in its arguments, and furthermore symmetric:

$$A_j = A_j^*. \tag{1.3}$$

We suppose g is smooth in its arguments, with values in $\mathbb{R}^K$; $u = u(t, x)$ takes values in $\mathbb{R}^K$. We then say (1.1) is a symmetric hyperbolic system. For simplicity, we suppose $x \in M = \mathbb{T}^n$, though any compact manifold M can be treated with minor modifications, as can the case $M = \mathbb{R}^n$. We will suppose $f \in H^k(M),\ k > n/2 + 1$.

Our strategy will be to obtain a solution to (1.1) as a limit of solutions u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + g_\varepsilon, \quad u_\varepsilon(0) = f, \tag{1.4}$$

where

$$L_\varepsilon v = \sum_j A_j(t, x, J_\varepsilon u_\varepsilon)\, \partial_j v \tag{1.5}$$

and

$$g_\varepsilon = J_\varepsilon g(t, x, J_\varepsilon u_\varepsilon). \tag{1.6}$$

In (1.4), f might also be replaced by $J_\varepsilon f$, though this is not crucial. Here, $\{J_\varepsilon : 0 < \varepsilon \le 1\}$ is a Friedrichs mollifier. For $M = \mathbb{T}^n$, we can define J_ε by a Fourier series representation:

$$(1.7) \qquad \big(J_\varepsilon v\big)\hat{}\,(\ell) = \varphi(\varepsilon\ell)\hat{v}(\ell), \quad \ell \in \mathbb{Z}^n,$$

given $\varphi \in C_0^\infty(\mathbb{R}^n)$, real-valued, $\varphi(0) = 1$. For any $\varepsilon > 0$, (1.4) can be regarded as a system of ODEs for u_ε, for which we know there is a unique solution, for t close to 0. Our task will be to show that the solution u_ε exists for t in an interval independent of $\varepsilon \in (0, 1]$ and has a limit as $\varepsilon \searrow 0$ solving (1.1).

To do this, we estimate the H^k-norm of solutions to (1.4). We begin with

$$(1.8) \qquad \frac{d}{dt}\, \|D^\alpha u_\varepsilon(t)\|_{L^2}^2 = 2(D^\alpha J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, D^\alpha u_\varepsilon) + 2(D^\alpha g_\varepsilon, D^\alpha u_\varepsilon).$$

Since J_ε commutes with D^α and is self-adjoint, we can write the first term on the right as

$$(1.9) \qquad 2(L_\varepsilon D^\alpha J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon) + 2([D^\alpha, L_\varepsilon] J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon).$$

To estimate the first term in (1.9), note that, by the symmetry hypothesis (1.3),

$$(1.10) \qquad (L_\varepsilon + L_\varepsilon^*)v = -\sum_j [\partial_j A_j(t, x, J_\varepsilon u_\varepsilon)]v,$$

so we have

$$(1.11) \qquad 2(L_\varepsilon D^\alpha J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon) \le C\big(\|J_\varepsilon u_\varepsilon(t)\|_{C^1}\big)\, \|D^\alpha J_\varepsilon u_\varepsilon\|_{L^2}^2.$$

Next, consider

$$(1.12) \qquad [D^\alpha, L_\varepsilon]v = \sum_j \Big(D^\alpha(A_{j\varepsilon}\,\partial_j v) - A_{j\varepsilon} D^\alpha(\partial_j v)\Big),$$

where $A_{j\varepsilon} = A_j(t, x, J_\varepsilon u_\varepsilon)$. By the Moser estimates from Chap. 13, §3 (see Proposition 3.7 there), we have

$$(1.13) \qquad \|[D^\alpha, L_\varepsilon]v\|_{L^2} \le C \sum_j \Big(\|A_{j\varepsilon}\|_{H^k}\, \|\partial_j v\|_{L^\infty} + \|\nabla A_{j\varepsilon}\|_{L^\infty}\, \|\partial_j v\|_{H^{k-1}}\Big),$$

provided $|\alpha| \le k$. We use this estimate with $v = J_\varepsilon u_\varepsilon$. We also use the estimate

$$(1.14) \qquad \|A_j(t, x, J_\varepsilon u_\varepsilon)\|_{H^k} \le C_k\big(\|J_\varepsilon u_\varepsilon\|_{L^\infty}\big)\big(1 + \|J_\varepsilon u_\varepsilon\|_{H^k}\big),$$

which follows from Proposition 3.9 of Chap. 13. This gives us control over the terms in (1.9), hence of the first term on the right side of (1.8). Consequently, we obtain an estimate of the form

$$(1.15) \qquad \frac{d}{dt}\, \|u_\varepsilon(t)\|_{H^k}^2 \le C_k\big(\|J_\varepsilon u_\varepsilon(t)\|_{C^1}\big)\big(1 + \|J_\varepsilon u_\varepsilon(t)\|_{H^k}^2\big).$$

This puts us in a position to prove the following:

Lemma 1.1. *Given $f \in H^k$, $k > n/2+1$, the solution to (1.4) exists for t in an interval $I = (-A, B)$, independent of ε, and satisfies an estimate*

$$\|u_\varepsilon(t)\|_{H^k} \le K(t), \quad t \in I, \tag{1.16}$$

independent of $\varepsilon \in (0,1]$.

Proof. Using the Sobolev imbedding theorem, we can dominate the right side of (1.15) by $E\big(\|u_\varepsilon(t)\|^2_{H^k}\big)$, so $\|u_\varepsilon(t)\|^2_{H^k} = y(t)$ satisfies the differential inequality

$$\frac{dy}{dt} \le E(y), \quad y(0) = \|f\|^2_{H^k}. \tag{1.17}$$

Gronwall's inequality yields a function $K(t)$, finite on some interval $[0, B)$, giving an upper bound for all $y(t)$ satisfying (1.17). Using time-reversibility of the class of symmetric hyperbolic systems, we also get a bound $K(t)$ for $y(t)$ on an interval $(-A, 0]$. This $I = (-A, B)$ and $K(t)$ work for (1.16).

We are now prepared to establish the following existence result:

Theorem 1.2. *Provided (1.1) is symmetric hyperbolic and $f \in H^k(M)$, with $k > n/2+1$, then there is a solution u, on an interval I about 0, with*

$$u \in L^\infty(I, H^k(M)) \cap Lip(I, H^{k-1}(M)). \tag{1.18}$$

Proof. Take the I above and shrink it slightly. The bounded family

$$u_\varepsilon \in C(I, H^k) \cap C^1(I, H^{k-1})$$

will have a weak limit point u satisfying (1.18). Furthermore, by Ascoli's theorem, there is a sequence

$$u_{\varepsilon_\nu} \longrightarrow u \ \text{ in } C(I, H^{k-1}(M)), \tag{1.19}$$

since the inclusion $H^k \subset H^{k-1}$ is compact. Also, by interpolation inequalities, $\{u_\varepsilon : 0 < \varepsilon \le 1\}$ is bounded in $C^\sigma(I, H^{k-\sigma}(M))$ for each $\sigma \in (0,1)$, so since the inclusion $H^{k-\sigma} \hookrightarrow C^1(M)$ is compact for small $\sigma > 0$ if $k > n/2+1$, we can arrange that

$$u_{\varepsilon_\nu} \longrightarrow u \ \text{ in } C(I, C^1(M)). \tag{1.20}$$

Consequently, with $\varepsilon = \varepsilon_\nu$,

$$\begin{aligned} J_\varepsilon L(t,x,J_\varepsilon u_\varepsilon,D)J_\varepsilon u_\varepsilon &+ J_\varepsilon g(t,x,J_\varepsilon u_\varepsilon) \\ &\to L(t,x,u,D)u + g(t,x,u) \quad \text{in } C(I\times M), \end{aligned} \tag{1.21}$$

while clearly $\partial u_{\varepsilon_\nu}/\partial t \to \partial u/\partial t$ weakly. Thus (1.1) follows in the limit from (1.4).

Let us also note that, with $y(t) = \|u_\varepsilon(t)\|^2_{H^k}$, we have, by (1.15),

$$\frac{dy}{dt} \le a(t)y + a(t), \quad a(t) = C_k\bigl(\|J_\varepsilon u_\varepsilon(t)\|_{C^1}\bigr), \tag{1.22}$$

so

$$y(t) \le e^{b(t)}\Bigl(y(0) + \int_0^t a(s)e^{-b(s)}\,ds\Bigr), \tag{1.23}$$

with $b(t) = \int_0^t a(s)\,ds$. It follows that we have $\{u_\varepsilon : 0 < \varepsilon \le 1\}$ bounded in $C(I,H^k) \cap \operatorname{Lip}(I,H^{k-1})$, as long as $k > n/2+1$, with convergence (1.19)–(1.21). A careful study of the estimates shows that I can be taken to be independent of k (provided $k > \frac{1}{2}n+1$). In fact, a stronger result will be established in Proposition 1.5.

There are questions of the uniqueness, stability, and rate of convergence of u_ε to u, which we can treat simultaneously. Thus, with $\varepsilon \in [0,1]$, we compare a solution u to (1.1) with a solution u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L(t,x,J_\varepsilon u_\varepsilon,D)J_\varepsilon u_\varepsilon + J_\varepsilon g(t,x,J_\varepsilon u_\varepsilon), \quad u_\varepsilon(0) = h. \tag{1.24}$$

Set $v = u - u_\varepsilon$, and subtract (1.24) from (1.1). Suppressing the variables (t,x), we have

$$\frac{\partial v}{\partial t} = L(u,D)v + L(u,D)u_\varepsilon - J_\varepsilon L(J_\varepsilon u_\varepsilon,D)J_\varepsilon u_\varepsilon + g(u) - J_\varepsilon g(J_\varepsilon u_\varepsilon). \tag{1.25}$$

Write

$$\begin{aligned} L(u,D)u_\varepsilon &- J_\varepsilon L(J_\varepsilon u_\varepsilon,D)J_\varepsilon u_\varepsilon \\ &= \bigl[L(u,D) - L(u_\varepsilon,D)\bigr]u_\varepsilon + (1-J_\varepsilon)L(u_\varepsilon,D)u_\varepsilon \\ &\quad + J_\varepsilon L(u_\varepsilon,D)(1-J_\varepsilon)u_\varepsilon \\ &\quad + J_\varepsilon\bigl[L(u_\varepsilon,D) - L(J_\varepsilon u_\varepsilon,D)\bigr]J_\varepsilon u_\varepsilon, \end{aligned} \tag{1.26}$$

and

$$\begin{aligned} g(u) - J_\varepsilon g(J_\varepsilon u_\varepsilon) = [g(u) - g(u_\varepsilon)] &+ (1-J_\varepsilon)g(u_\varepsilon) \\ &+ J_\varepsilon[g(u_\varepsilon) - g(J_\varepsilon u_\varepsilon)]. \end{aligned} \tag{1.27}$$

Now write

$$
\begin{aligned}
g(u) - g(w) &= G(u, w)(u - w), \\
G(u, w) &= \int_0^1 g'\big(\tau u + (1 - \tau)w\big)\, d\tau,
\end{aligned}
\tag{1.28}
$$

and similarly

$$
L(u, D) - L(w, D) = (u - w) \cdot M(u, w, D). \tag{1.29}
$$

Then (1.25) yields

$$
\frac{\partial v}{\partial t} = L(u, D)v + A(u, u_\varepsilon, \nabla u_\varepsilon)v + R_\varepsilon, \tag{1.30}
$$

where

$$
A(u, u_\varepsilon, \nabla u_\varepsilon)v = v \cdot M(u, u_\varepsilon, D)u_\varepsilon + G(u, u_\varepsilon)v \tag{1.31}
$$

incorporates the first terms on the right sides of (1.26) and (1.27), and R_ε is the sum of the rest of the terms in (1.26) and (1.27). Note that each term making up R_ε has as a factor $I - J_\varepsilon$, acting on either u_ε, $g(u_\varepsilon)$, or $L(u_\varepsilon, D)u_\varepsilon$. Thus there is an estimate

$$
\|R_\varepsilon(t)\|_{L^2}^2 \le C_k\big(\|u_\varepsilon(t)\|_{C^1}\big)\big(1 + \|u_\varepsilon(t)\|_{H^k}^2\big) r_k(\varepsilon)^2, \tag{1.32}
$$

where

$$
r_k(\varepsilon) = \|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1}, L^2)} \approx \|I - J_\varepsilon\|_{\mathcal{L}(H^k, H^1)}. \tag{1.33}
$$

Now, estimating $(d/dt)\|v(t)\|_{L^2}^2$ via the obvious analogue of (1.8)–(1.15) yields

$$
\frac{d}{dt}\, \|v(t)\|_{L^2}^2 \le C(t)\|v(t)\|_{L^2}^2 + S(t), \tag{1.34}
$$

with

$$
C(t) = C\big(\|u_\varepsilon(t)\|_{C^1}, \|u(t)\|_{C^1}\big), \quad S(t) = \|R_\varepsilon(t)\|_{L^2}^2. \tag{1.35}
$$

Consequently, by Gronwall's inequality, with $K(t) = \int_0^t C(\tau)\, d\tau$,

$$
\|v(t)\|_{L^2}^2 \le e^{K(t)}\Big(\|f - h\|_{L^2}^2 + \int_0^t S(\tau)e^{-K(\tau)}\, d\tau\Big), \tag{1.36}
$$

for $t \in [0, B)$. A similar argument with time reversed covers $t \in (-A, 0]$, and we have the following:

Proposition 1.3. *For $k > n/2+1$, solutions to (1.1) satisfying (1.18) are unique. They are limits of solutions u_ε to (1.4), and, for $t \in I$,*

$$\|u(t) - u_\varepsilon(t)\|_{L^2} \le K_1(t)\|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}. \tag{1.37}$$

Note that if J_ε is defined by (1.7) and $\varphi(\xi) = 1$ for $|\xi| \le 1$, we have the operator norm estimate

$$\|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)} \le C\ \varepsilon^{k-1}. \tag{1.38}$$

Returning to properties of solutions of (1.1), we want to establish the following small but nice improvement of (1.18):

Proposition 1.4. *Given $f \in H^k,\ k > n/2+1$, the solution u to (1.1) satisfies*

$$u \in C(I, H^k). \tag{1.39}$$

For the proof, note that (1.18) implies that $u(t)$ is a continuous function of t with values in $H^k(M)$, given the *weak* topology. To establish (1.39), it suffices to demonstrate that the norm $\|u(t)\|_{H^k}$ is a continuous function of t. We estimate the rate of change of $\|u(t)\|^2_{H^k}$ by a device similar to the analysis of (1.8). Unfortunately, it is not useful to look directly at $(d/dt)\|D^\alpha u(t)\|^2_{L^2}$ when $|\alpha| = k$, since $LD^\alpha u$ may not be in L^2. To get around this, we throw in a factor of J_ε, and for $|\alpha| \le k$ look at

$$\begin{aligned}\frac{d}{dt}\ \|D^\alpha J_\varepsilon u(t)\|^2_{L^2} = 2\big(D^\alpha J_\varepsilon L(u,D)u, D^\alpha J_\varepsilon u\big)\\ +2\big(D^\alpha J_\varepsilon g(u), D^\alpha J_\varepsilon u\big).\end{aligned} \tag{1.40}$$

As above, we have suppressed the dependence on (t,x), for notational convenience. The last term on the right is easy to estimate; we write the first term as

$$2(D^\alpha L(u,D)u, D^\alpha J_\varepsilon^2 u) = 2(LD^\alpha u, D^\alpha J_\varepsilon^2 u) + 2([D^\alpha, L]u, D^\alpha J_\varepsilon^2 u). \tag{1.41}$$

Here, for fixed t, $L(u,D)D^\alpha u \in H^{-1}(M)$, which can be paired with $D^\alpha J_\varepsilon^2 u \in C^\infty(M)$. We still have the Moser-type estimate

$$\|[D^\alpha, L]u\|_{L^2} \le C\|A_j(u)\|_{H^k}\|u\|_{C^1} + C\|A_j(u)\|_{C^1}\|u\|_{H^k}, \tag{1.42}$$

parallel to (1.13), which gives control over the last term in (1.41). We can write the first term on the right side of (1.41) as

$$\big((L+L^*)D^\alpha J_\varepsilon u, D^\alpha J_\varepsilon u\big) + 2\big([J_\varepsilon, L]D^\alpha u, D^\alpha u\big). \tag{1.43}$$

The first term is bounded just as in (1.10)–(1.11). As for the last term, we have

$$[J_\varepsilon, L]w = \sum_j [A_j(u), J_\varepsilon]\,\partial_j w. \tag{1.44}$$

Now the nature of J_ε as a Friedrichs mollifier implies the estimate

$$\|[A_j, J_\varepsilon]\partial_j w\|_{L^2} \le C\|A_j\|_{C^1}\|w\|_{L^2}; \tag{1.45}$$

see Chap. 13, § 1, Exercises 1–3.

Consequently, we have a bound

$$\frac{d}{dt}\,\|J_\varepsilon u(t)\|^2_{H^k} \le C\big(\|u(t)\|_{C^1}\big)\|u(t)\|^2_{H^k}, \tag{1.46}$$

the right side being independent of $\varepsilon \in (0,1]$. This, together with the same analysis with time reversed, shows that $\|J_\varepsilon u(t)\|^2_{H^k} = N_\varepsilon(t)$ is Lipschitz continuous in t, uniformly in ε. As $J_\varepsilon u(t) \to u(t)$ in H^k-norm for each $t \in I$, it follows that $\|u(t)\|^2_{H^k} = N_0(t) = \lim\ N_\varepsilon(t)$ has this same Lipschitz continuity. Proposition 1.4 is proved.

Unlike the linear case, nonlinear hyperbolic equations need not have smooth solutions for all t. We will give some examples at the end of this section. Here we will show, following [Mj], that in a general context, the breakdown of a classical solution must involve a blow-up of either $\sup_x |u(t,x)|$ or $\sup_x |\nabla_x u(t,x)|$.

Proposition 1.5. *Suppose $u \in C\big([0,T), H^k(M)\big)$, $k > n/2+1$ ($n = \dim M$), and assume u solves the symmetric hyperbolic system (1.1) for $t \in (0,T)$. Assume also that*

$$\|u(t)\|_{C^1(M)} \le K < \infty, \tag{1.47}$$

for $t \in [0,T)$. Then there exists $T_1 > T$ such that u extends to a solution to (1.1), belonging to $C\big([0,T_1), H^k(M)\big)$.

We remark that, if $A_j(t,x,u)$ and $g(t,x,u)$ are C^∞ in a region $\mathbb{R} \times M \times \Omega$, rather than in all of $\mathbb{R} \times M \times \mathbb{R}^K$, we also require

$$u(t,x) \subset \Omega_1 \subset\subset \Omega, \quad \text{for } t \in [0,T). \tag{1.48}$$

Proof. This follows easily from the estimate (1.46). As noted above, with $N_\varepsilon(t) = \|J_\varepsilon u(t)\|^2_{H^k}$, we have $N_\varepsilon(t) \to N_0(t) = \|u(t)\|^2_{H^k}$ pointwise as $\varepsilon \to 0$, and (1.46) takes the form $dN_\varepsilon/dt \le C_1(t)N_0(t)$. If we write this in an equivalent integral form:

$$N_\varepsilon(t+\tau) \le N_\varepsilon(t) + \int_t^{t+\tau} C_1(s)N_0(s)\,ds, \tag{1.49}$$

it is clear that we can pass to the limit $\varepsilon \to 0$, obtaining the differential inequality

$$\frac{dN_0}{dt} \le C(\|u(t)\|_{C^1}) N_0(t) \tag{1.50}$$

for the Lipschitz function $N_0(t)$. Now Gronwall's inequality implies that $N_0(t)$ cannot blow up as $t \nearrow T$ unless $\|u(t)\|_{C^1}$ does, so we are done.

One consequence of this is local existence of C^∞-solutions:

Corollary 1.6. *If (1.1) is a symmetric hyperbolic system and $f \in C^\infty(M)$, then there is a solution $u \in C^\infty(I \times M)$, for an interval I about* 0.

Proof. Pick $k > n/2 + 1$, and apply Theorem 1.2 (plus Proposition 1.4) to get a solution $u \in C(I, H^k(M))$. We can also apply these results with $f \in H^\ell(M)$, for ℓ arbitrarily large, together with uniqueness, to get $u \in C(J, H^\ell(M))$, for some interval J about 0, but possibly J is smaller than I. But then we can use Proposition 1.5 (for both forward and backward time) to obtain $u \in C(I, H^\ell(M))$. This holds, for fixed I, and for arbitrarily large ℓ. From this it easily follows that $u \in C^\infty(I \times M)$.

We make some complementary remarks on results that can be obtained from the estimates derived in this section. In particular, the arguments above hold when $A_j(t,x,u)$ and $g(t,x,u)$ have only H^k-regularity in the variables (x,u), as long as $k > n/2 + 1$. This is of interest even in the linear case, so we record the following conclusion:

Proposition 1.7. *Given $A_j(t,x)$ and $g(t,x)$ in $C(I, H^k(M))$, $k > \frac{1}{2}n + 1$, $A_j = A_j^*$, the initial-value problem*

$$\frac{\partial u}{\partial t} = \sum_j A_j(t,x)\, \partial_j u + g(t,x), \quad u(0) = f \in H^k(M), \tag{1.51}$$

has a unique solution in $C(I, H^k(M))$.

In some approaches to quasi-linear equations, this result is established first and used as a tool to solve (1.1), via an iteration of the form

$$\frac{\partial}{\partial t} u_{\nu+1} = \sum_j A_j(t,x,u_\nu)\, \partial_j u_{\nu+1} + g(t,x,u_\nu), \quad u_{\nu+1}(0) = f, \tag{1.52}$$

beginning, say, with $u_0(t) = f$. Then one's task is to show that $\{u_\nu\}$ converges, at least on some interval I about $t = 0$, to a solution to (1.1). For details on this approach, see [Mj1]; see also Exercises 3–6 below.

The approach used to prove Theorem 1.2 has connections with numerical methods used to find approximate solutions to (1.1). The approximation (1.4) is a special case of what are sometimes called *Galerkin methods*. Estimates established above, particularly Proposition 1.3, thus provide justification for some classes of Galerkin methods.

For nonlinear hyperbolic equations, short-time, smooth solutions might not extend to solutions defined and smooth for all t. We mention two simple examples of equations whose classical solutions break down in finite time. First consider

$$\frac{\partial u}{\partial t} = u^2, \quad u(0, x) = 1. \tag{1.53}$$

The solution is

$$u(t, x) = \frac{1}{1 - t}, \tag{1.54}$$

for $t < 1$, which blows up as $t \nearrow 1$.

The second example is

$$u_t + uu_x = 0, \quad u(0, x) = e^{-x^2}. \tag{1.55}$$

Writing the equation as

$$\Big(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\Big)u = 0, \tag{1.56}$$

we see that $u(t, x)$ is constant on straight lines through $(x, 0)$, with slope $u(0, x)^{-1}$, in the (x, t)-plane, as illustrated in Fig. 1.1. The line through $(0, 0)$ has slope 1, and that through $(1, 0)$ has slope e; these lines must intersect, and by that time the classical solution must break down. In this second example, $u(t, x)$ does not become unbounded, but it is clear that $\sup_x |u_x(t, x)|$ does. As we will discuss further in § 7, this provides an example of the formation of a shock wave.

A detailed study of breakdown mechanisms is given in [Al].

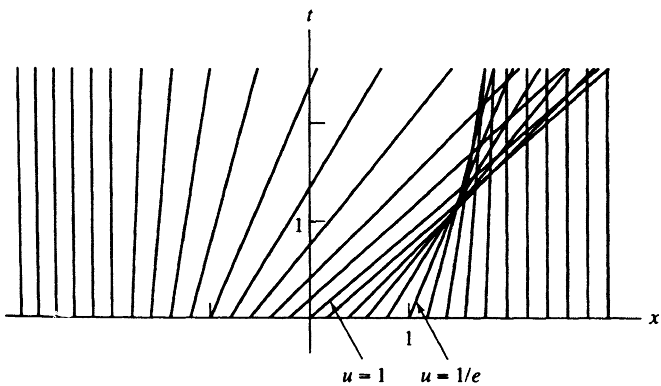

FIGURE 1.1 Crossing Characteristics

Exercises

1. Establish the results of this section when M is any compact Riemannian manifold, by the following route. Let $\{X_j : 1 \le j \le N\}$ be a finite collection of smooth vector fields that span the tangent space $T_x M$ for each x. With $J = (j_1, \dots, j_k)$, set $X^J = X_{j_1} \cdots X_{j_k}$; set $|J| = k$. Also set $X^\emptyset = I,\ |\emptyset| = 0$. Then use the square norm

$$\|u\|_{H^k}^2 = \sum_{|J| \le k} \|X^J u\|_{L^2}^2.$$

   Also, let $J_\varepsilon = e^{\varepsilon\Delta}$. To establish an analogue of (1.15), it will be useful to have

$$[X^J, J_\varepsilon] \text{ bounded in } OPS^{k-1}_{1,0}(M), \quad \text{for } |J| = k,\ \varepsilon \in (0,1].$$

2. Consider a completely nonlinear system

$$\frac{\partial u}{\partial t} = F(t, x, u, \nabla_x u), \quad u(0) = f. \tag{1.57}$$

   Suppose u takes values in $\mathbb{R}^K$. Form a first-order system for $(v_0, v_1, \dots, v_n) = (u, \partial_1 u, \dots, \partial_n u)$:

$$\begin{aligned} \frac{\partial v_0}{\partial t} &= F(t, x, v), \\ \frac{\partial v_j}{\partial t} &= \sum_{\ell=1}^n (\partial_{v_\ell} F)(t, x, v)\, \partial_\ell v_j + (\partial_{x_j} F)(t, x, v), \quad 1 \le j \le n, \end{aligned} \tag{1.58}$$

   with initial data

$$v(0) = (f, \partial_1 f, \dots, \partial_n f).$$

   We say (1.57) is symmetric hyperbolic if each $\partial_{v_\ell} F$ is a symmetric $K \times K$ matrix. Apply methods of this section to (1.58), and then show that (1.57) has a unique solution $u \in C(I, H^k(M))$, given $f \in H^k(M),\ k > n/2 + 2$.

Exercises 3–6 sketch how one can use a slight extension of Proposition 1.7 to show that the iterative method (1.52) yields u_ν converging for $|t|$ small to a solution to the quasi-linear PDE (1.1). Assume $f \in H^k(M),\ k > n/2 + 1$.

3. Extend Proposition 1.7, by taking $f \in H^\ell$, and $g \in C(I, H^\ell)$, for $\ell \in [0, k]$ (while keeping $A_j \in C(I, H^k)$ and $k > n/2 + 1$), and obtaining $u \in C(I, H^\ell)$.
4. Granted Proposition 1.7, show that $\{u_\nu\}$ is bounded in $C(I, H^k(M))$, after possibly shrinking I. (*Hint*: Produce an estimate of the form

$$\|u_{\nu+1}(t)\|_{H^k}^2 \le \Big\{\|f\|_{H^k}^2 + \int_0^t \varphi_\nu(\tau)\, d\tau\Big\} \exp\Big(\int_0^t \psi_\nu(s)\, ds\Big),$$

   where $\varphi_\nu(s) = \varphi(\|u_\nu(s)\|_{H^k})$ and $\psi_\nu(s) = \psi(\|u_\nu(s)\|_{H^k})$. Then apply Gronwall's inequality.)
5. Derive an estimate of the form

$$\|u_{\nu+1}(t) - u_\nu(t)\|_{H^{k-1}}^2 \le A|t| \sup_{s \in [0,t]} \|u_\nu(s) - u_{\nu-1}(s)\|_{H^{k-1}}^2,$$

for $t \in I$, and deduce that $\{u_\nu\}$ is Cauchy in $C(I, H^{k-1}(M))$, after possibly further shrinking I. (*Hint*: With $w = u_{\nu+1} - u_\nu$, look at a linear hyperbolic equation for w and apply the extension of Proposition 1.7 to it, with $\ell = k - 1$.)

6. Deduce that $\{u_\nu\}$ has a limit $u \in C(I, H^{k-1}(M)) \cap L^\infty(I, H^k(M))$, solving (1.1).
7. Suppose u_1 and u_2 are sufficiently smooth solutions to (1.1), with initial data $u_j(0) = f_j$. Assume (1.1) is symmetric hyperbolic. Produce a *linear*, symmetric hyperbolic equation satisfied by $u_1 - u_2$. If $f_1 = f_2$ on an open set $\mathcal{O} \subset M$, deduce that $u_1 = u_2$ on a certain subset of $\mathbb{R} \times M$, thus obtaining a finite propagation speed result, as a consequence of the finite propagation speed for solutions to linear hyperbolic systems, established via (5.26)–(5.34) of Chap. 6.
8. Obtain a smooth solution to (1.1) on a neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$ when $f \in C^\infty(M)$ and M is any open subset of $\mathbb{R}^n$. (*Hint*: To get a solution to (1.1) on a neighborhood of $(0, x_0)$, identify some neighborhood of x_0 in M with an open set in $\mathbb{T}^n$ and modify (1.1) to a PDE for functions on $\mathbb{R} \times \mathbb{T}^n$. Make use of finite propagation speed to solve the problem.)
9. Let T_* be the largest positive number such that (1.55) has a smooth solution for $0 \le t < T_*$. Show that, in this example,

$$\|u(t)\|_{C^{1/3}(\mathbb{R})} \le K < \infty, \quad \text{for } 0 \le t < T_*.$$

(*Hint*: For $s = T_* - t \nearrow 0$, consider similarities of the graph of $x \mapsto u(t, x) = y$ with the graph of $x = -y^3 - sy$.)

10. Show that the rays in Fig. 1.1 are given by

$$\Phi(x, t) = \bigl(x + te^{-x^2}, t\bigr),$$

and deduce that T_* in Exercise 9 is given by

$$T_* = \sqrt{\frac{e}{2}}.$$

11. Consider a *semilinear*, hyperbolic system

$$\frac{\partial u}{\partial t} = Lu + g(u), \quad u(0) = f. \tag{1.59}$$

Paralleling the results of Proposition 1.5, show that solutions in the space $C\bigl(I, H^k(M)\bigr)$, $k > n/2$, persist as long as one has a bound

$$\|u(t)\|_{L^\infty(M)} \le K < \infty, \quad t \in I. \tag{1.60}$$

In Exercises 12–14, we consider the semilinear system (1.59), under the following hypothesis:

$$g(0) = 0, \quad |g'(u)| \le C. \tag{1.61}$$

For simplicity, take $M = \mathbb{T}^n$.

12. Let u_ε be a solution to an approximating equation, of the form

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L J_\varepsilon u_\varepsilon + J_\varepsilon g(J_\varepsilon u_\varepsilon), \quad u_\varepsilon(0) = f. \tag{1.62}$$

Show that

$$\frac{d}{dt}\|u_\varepsilon\|_{L^2}^2 \le C\|u_\varepsilon\|_{L^2}^2, \quad \frac{d}{dt}\|\nabla u_\varepsilon\|_{L^2}^2 \le C\|\nabla u_\varepsilon\|_{L^2}^2.$$

Deduce that, for any $\varepsilon > 0$, (1.62) has a solution, defined for all $t \in \mathbb{R}$, and, for any compact $I \subset \mathbb{R}$, we have

$$u_\varepsilon \ \text{ bounded in } \ L^\infty\big(I, H^1(M)\big) \cap \ \mathrm{Lip}\big(I, L^2(M)\big).$$

13. Deduce that, passing to a subsequence u_{ε_k}, we have a limit point $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}, H^1(M)) \cap \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}, L^2(M))$, such that

$$u_{\varepsilon_k} \to u \quad \text{in } \ C\big(I, L^2(M)\big)$$

in norm, for all compact $I \subset \mathbb{R}$; hence $g(J_{\varepsilon_k} u_{\varepsilon_k}) \to g(u)$ in $C\big(\mathbb{R}, L^2(M)\big)$, and u solves (1.59). Examine the issue of uniqueness.
Remark: This result appears in [JMR]. The proof there uses the iterative method (1.52).
14. If $\dim M = 1$, combine the results of Exercises 11 and 13 to produce a global smooth solution to (1.59), under the hypothesis (1.61), given $f \in C^\infty(M)$ and g smooth.
Remark: If $\dim M$ is large, the global smoothness of u is open. For some results, see [BW].

2. Symmetrizable hyperbolic systems

The results of the previous section extend to the case

$$A_0(t,x,u)\,\frac{\partial u}{\partial t} = \sum_{j=1}^n A_j(t,x,u)\,\partial_j u + g(t,x,u), \quad u(0) = f, \tag{2.1}$$

where, as in (1.3), all A_j are symmetric, and furthermore

$$A_0(t,x,u) \ge cI > 0. \tag{2.2}$$

We have the following:

Proposition 2.1. *Given $f \in H^k(M)$, $k > n/2+1$, the existence and uniqueness results of § 1 continue to hold for (2.1).*

We obtain the solution u to (2.1) as a limit of solutions u_ε to

$$A_0(t,x,J_\varepsilon u_\varepsilon)\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon + g_\varepsilon, \quad u_\varepsilon(0) = f, \tag{2.3}$$

where L_ε and g_ε are as in (1.5)–(1.6). We need to parallel the estimates of § 1, particularly (1.8)–(1.15). The key is to replace the L^2-inner products by

$$(2.4)\qquad \big(w, A_{0\varepsilon}(t)w\big)_{L^2}, \quad A_{0\varepsilon}(t) = A_0(t, x, J_\varepsilon u_\varepsilon),$$

which by hypothesis (2.2) will define equivalent L^2 norms. We have

$$(2.5)\qquad \begin{aligned}\frac{d}{dt}\big(D^\alpha u_\varepsilon, A_{0\varepsilon}(t)D^\alpha u_\varepsilon\big) = {} & 2\big(D^\alpha(\partial u_\varepsilon/\partial t), A_{0\varepsilon}(t)D^\alpha u_\varepsilon\big) \\ & + \big(D^\alpha u_\varepsilon, A'_{0\varepsilon}(t)D^\alpha u_\varepsilon\big).\end{aligned}$$

Here and below, the L^2-inner product is understood. The first term on the right side of (2.5) can be written as

$$(2.6)\qquad 2\big(D^\alpha A_{0\varepsilon}\partial_t u_\varepsilon, D^\alpha u_\varepsilon\big) + 2\big([D^\alpha, A_{0\varepsilon}]\partial_t u_\varepsilon, D^\alpha u_\varepsilon\big);$$

in the first of these terms, we replace $A_{0\varepsilon}(\partial u_\varepsilon/\partial t)$ by the right side of (2.3), and estimate the resulting expression by the same method as was applied to the right side of (1.8) in § 1. The commutator $[D^\alpha, A_{0\varepsilon}]$ is amenable to a Moser-type estimate parallel to (1.12); then substitute for $\partial u_\varepsilon/\partial t,\ A_{0\varepsilon}^{-1}$ times the right side of (2.3), and the last term in (2.6) is easily estimated. It remains to treat the last term in (2.5). We have

$$(2.7)\qquad A'_{0\varepsilon}(t) = \frac{d}{dt}A_0(t, x, J_\varepsilon u_\varepsilon(t, x)),$$

hence

$$(2.8)\qquad \|A'_{0\varepsilon}(t)\|_{L^\infty} \le C\big(\|J_\varepsilon u_\varepsilon(t)\|_{L^\infty}, \|J_\varepsilon u'_\varepsilon(t)\|_{L^\infty}\big).$$

Of course, $\|\partial u_\varepsilon/\partial t\|_{L^\infty}$ can be estimated by $\|u_\varepsilon(t)\|_{C^1}$, due to (2.3). Consequently, we obtain an estimate parallel to (1.15), namely

$$(2.9)\qquad \frac{d}{dt}\sum_{|\alpha|\le k}(D^\alpha u_\varepsilon, A_{0\varepsilon}D^\alpha u_\varepsilon) \le C_k\big(\|u_\varepsilon(t)\|_{C^1}\big)\big(1 + \|J_\varepsilon u_\varepsilon(t)\|^2_{H^k}\big).$$

From here, the rest of the parallel with § 1 is clear.

The class of systems (2.1), with all $A_j = A_j^*$ and $A_0 \ge cI > 0$, is an extension of the class of symmetric hyperbolic systems. We call a system

$$(2.10)\qquad \frac{\partial u}{\partial t} = \sum_{j=1}^n B_j(t, x, u)\,\partial_j u + g(t, x, u), \quad u(0) = f,$$

a *symmetrizable hyperbolic system* provided there exist $A_0(t, x, u)$, positive-definite, such that $A_0(t, x, u)B_j(t, x, u) = A_j(t, x, u)$ are all symmetric. Then applying $A_0(t, x, u)$ to (2.10) yields an equation of the form (2.1) (with different g and f), so the existence and uniqueness results of § 1 hold. The factor $A_0(t, x, u)$ is called a *symmetrizer*.

An important example of such a situation is provided by the equations of compressible fluid flow:

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + \nabla_v v + \frac{1}{\rho} \operatorname{grad} p = 0, \\
&\frac{\partial \rho}{\partial t} + \nabla_v \rho \; + \; \rho \operatorname{div} v = 0.
\end{aligned}
\tag{2.11}
$$

Here v is the velocity field of a fluid of density $\rho = \rho(t, x)$. We consider the model in which p is assumed to be a function of ρ. In this situation one says the flow is *isentropic*. A particular example is

$$
p(\rho) = A\, \rho^{\gamma},
\tag{2.12}
$$

with $A > 0,\ 1 < \gamma < 2$; for air, $\gamma = 1.4$ is a good approximation. One calls (2.12) an *equation of state*. Further discussion of how (2.11) arises will be given in § 5.

The system (2.11) is not a symmetric hyperbolic system as it stands. However, one can multiply the two equations by $b(\rho) = \rho/p'(\rho)$ and ρ^{-1}, respectively, obtaining

$$
\begin{pmatrix} b(\rho) & 0 \\ 0 & \rho^{-1} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} v \\ \rho \end{pmatrix} = - \begin{pmatrix} b(\rho)\nabla_v & \operatorname{grad} \\ \operatorname{div} & \rho^{-1}\nabla_v \end{pmatrix} \begin{pmatrix} v \\ \rho \end{pmatrix}.
\tag{2.13}
$$

Now (2.13) is a symmetric hyperbolic system of the form (2.1). Recall that

$$
(\operatorname{div} v, f)_{L^2} = -(v, \operatorname{grad} f)_{L^2}.
\tag{2.14}
$$

Thus the results of § 1 apply to the equation (2.11) for compressible fluid flow, as long as ρ is bounded away from zero.

Another popular form of the equations for compressible fluid flow is obtained by rewriting (2.11) as a system for (p, v); using (2.12), one has

$$
\begin{aligned}
&\frac{\partial p}{\partial t} + \nabla_v p \; + \; (\gamma p) \operatorname{div} v = 0, \\
&\frac{\partial v}{\partial t} + \nabla_v v + \sigma(p) \operatorname{grad} p = 0,
\end{aligned}
\tag{2.15}
$$

where $\sigma(p) = 1/\rho(p) = (A/p)^{1/\gamma}$. This is also symmetrizable. Multiplying these two equations by $(\gamma p)^{-1}$ and $\rho(p)$, respectively, we can rewrite the system as

$$
\begin{pmatrix} (\gamma p)^{-1} & 0 \\ 0 & \rho(p) \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} p \\ v \end{pmatrix} = - \begin{pmatrix} (\gamma p)^{-1}\nabla_v & \operatorname{div} \\ \operatorname{grad} & \rho(p)\nabla_v \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}.
$$

See Exercises 3–4 below for another approach to symmetrizing (2.11).

We now introduce a more general notion of symmetrizer, following Lax [L1], which will bring in pseudodifferential operators. We will say that a function $R(t,u,x,\xi)$, smooth on $\mathbb{R} \times \mathbb{R}^K \times T^*M \setminus 0$, homogeneous of degree 0 in ξ, is a symmetrizer for (2.10) provided

$$R(t,u,x,\xi) \text{ is a positive-definite, } K \times K \text{ matrix} \tag{2.16}$$

and

$$R(t,u,x,\xi) \sum B_j(t,x,u)\xi_j \text{ is self-adjoint,} \tag{2.17}$$

for each (t,u,x,ξ). We then say (2.10) is symmetrizable. One reason for the importance of this notion is the following:

Proposition 2.2. *Whenever (2.10) is strictly hyperbolic, it is symmetrizable.*

Proof. If we denote the eigenvalues of $L(t,u,x,\xi) = \sum B_j(t,x,u)\xi_j$ by

$$\lambda_1(t,u,x,\xi) < \cdots < \lambda_K(t,u,x,\xi),$$

then the λ_ν are well-defined C^∞-functions of (t,u,x,ξ), homogeneous of degree 1 in ξ. If $P_\nu(t,u,x,\xi)$ are the projections onto the λ_ν-eigenspaces of L,

$$P_\nu = \frac{1}{2\pi i} \int_{\gamma_\nu} \big(\zeta - L(t,u,x,\xi)\big)^{-1} \, d\zeta, \tag{2.18}$$

then P_ν is smooth and homogeneous of degree 0 in ξ. Then

$$R(t,u,x,\xi) = \sum_j P_j(t,u,x,\xi)^* P_j(t,u,x,\xi) \tag{2.19}$$

gives the desired symmetrizer.

We will use results on pseudodifferential operators with nonregular symbols, developed in Chap. 13, § 9. Note that

$$u \in C^{1+r} \Longrightarrow R \in C^{1+r} S^0_{cl}, \tag{2.20}$$

where the symbol class on the right is defined as in (9.46) of Chap. 13. Now, with $R = R(t,u,x,D)$, set

$$Q = \frac{1}{2}(R + R^*) + K\Lambda^{-1}, \tag{2.21}$$

where $K > 0$ is chosen so that Q is a positive-definite operator on L^2.

We will work with approximate solutions u_ε to (2.10), given by (1.4), with

$$L_\varepsilon v = \sum_j B_j(t, x, J_\varepsilon u_\varepsilon)\,\partial_j v. \tag{2.22}$$

Given $|\alpha| = k$, we want to obtain estimates on $(D^\alpha u_\varepsilon(t), Q_\varepsilon D^\alpha u_\varepsilon(t))$, where Q_ε arises by the process above, from $R_\varepsilon = R(t, J_\varepsilon u_\varepsilon, x, \xi)$. We begin with

$$\begin{aligned}
&\frac{d}{dt}\big(D^\alpha u_\varepsilon, Q_\varepsilon D^\alpha u_\varepsilon\big) \\
&\quad = 2\big(D^\alpha \partial_t u_\varepsilon, Q_\varepsilon D^\alpha u_\varepsilon\big) + (D^\alpha u_\varepsilon, Q_\varepsilon' D^\alpha u_\varepsilon) \\
&\quad = 2\,\mathrm{Re}\big(D^\alpha \partial_t u_\varepsilon, R_\varepsilon D^\alpha u_\varepsilon\big) + 2K\big(D^\alpha \partial_t u_\varepsilon, \Lambda^{-1} D^\alpha u_\varepsilon\big) \\
&\qquad + 2\,\mathrm{Re}(D^\alpha u_\varepsilon, R_\varepsilon' D^\alpha u_\varepsilon).
\end{aligned} \tag{2.23}$$

For the last term, we have the estimate

$$|(D^\alpha u_\varepsilon, R_\varepsilon' D^\alpha u_\varepsilon)| \le C\big(\|u_\varepsilon(t)\|_{C^1}\big)\|u_\varepsilon(t)\|_{H^k}^2. \tag{2.24}$$

We can write the first term on the (far) right side of (2.23) as twice the real part of

$$(R_\varepsilon D^\alpha J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, D^\alpha u_\varepsilon) + (R_\varepsilon D^\alpha g_\varepsilon, D^\alpha u_\varepsilon.) \tag{2.25}$$

The last term has an easy estimate. We write the first term in (2.25) as

$$\begin{aligned}
&(R_\varepsilon L_\varepsilon D^\alpha J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon) + (R_\varepsilon [D^\alpha, L_\varepsilon] J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon) \\
&\qquad + ([R_\varepsilon D^\alpha, J_\varepsilon] L_\varepsilon J_\varepsilon u_\varepsilon, D^\alpha u_\varepsilon).
\end{aligned} \tag{2.26}$$

Note that as long as (2.20) holds, with $r > 0$, R_ε also has symbol in $C^{1+r} S^0_{cl}$, and we have, by Proposition 9.9 of Chap. 13,

$$R_\varepsilon = R_\varepsilon^\# + R_\varepsilon^b, \quad R_\varepsilon^\# \in OPS^0_{1,\delta}, \quad R_\varepsilon^b \in OPC^{1+r} S^{-(1+r)\delta}_{1,\delta}. \tag{2.27}$$

Furthermore, by (9.42) of Chap. 13,

$$D_x^\alpha R_\varepsilon^\#(x, \xi) \in S^0_{1,\delta}, \quad |\alpha| = 1 \tag{2.28}$$

if $r > 0$. In (2.27) and (2.28) we have uniform bounds for $\varepsilon \in (0, 1]$. Take δ close enough to 1 that $(1 + r)\delta \ge 1$. We then have $[R_\varepsilon, J_\varepsilon]$ bounded in $\mathcal{L}(H^{-1}, L^2)$, upon applying Proposition 9.10 of Chap. 13 to R_ε^b. Hence we have

$$[R_\varepsilon D^\alpha, J_\varepsilon] \text{ bounded in } \mathcal{L}(H^{k-1}, L^2), \tag{2.29}$$

with bound given in terms of $\|u_\varepsilon(t)\|_{C^{1+r}}$. Now Moser estimates yield

$$\|L_\varepsilon J_\varepsilon u_\varepsilon\|_{H^{k-1}} \leq C\big(\|u_\varepsilon\|_{L^\infty}\big)\|u_\varepsilon\|_{H^k} + C\big(\|u_\varepsilon\|_{C^1}\big)\|u_\varepsilon\|_{H^{k-1}}. \tag{2.30}$$

Consequently, we deduce

$$\Big|\big([R_\varepsilon D^\alpha, J_\varepsilon]L_\varepsilon J_\varepsilon u_\varepsilon, D^\alpha u_\varepsilon\big)\Big| \leq C\big(\|u_\varepsilon(t)\|_{C^{1+r}}\big)\|u_\varepsilon(t)\|^2_{H^k}. \tag{2.31}$$

Moving to the second term in (2.26), note that, for $L = \sum B_j(t,x,u)\,\partial_j$,

$$[D^\alpha, L] = \sum_j [D^\alpha, B_j(t,x,u)]\,\partial_j v. \tag{2.32}$$

By the Moser estimate, as in (1.13), we have

$$\big\|[D^\alpha, L]v\big\|_{L^2} \leq C\sum_j \Big[\|B_j\|_{\mathrm{Lip}^1}\|v\|_{H^k} + \|B_j\|_{H^k}\|v\|_{\mathrm{Lip}^1}\Big]. \tag{2.33}$$

Hence the second term in (2.26) is bounded by $C\big(\|u_\varepsilon\|_{C^1}\big)\|u_\varepsilon\|^2_{H^k}$.

It remains to estimate the first term in (2.26). We claim that

$$|(R_\varepsilon L_\varepsilon v, v)| \leq C\big(\|u_\varepsilon\|_{C^1}\big)\|v\|^2_{L^2}. \tag{2.34}$$

To see this, parallel to (2.27), we can write

$$L_\varepsilon = L_\varepsilon^\# + L_\varepsilon^b, \quad L_\varepsilon^\# \in OPS^1_{1,\delta}, \quad L_\varepsilon^b \in OPC^{1+r}S^{1-(1+r)\delta}_{1,\delta}, \tag{2.35}$$

and, parallel to (2.28),

$$D_x^\alpha L_\varepsilon^\#(x,\xi) \in S^1_{1,\delta}, \quad |\alpha| = 1. \tag{2.36}$$

Now, provided $(1+r)\delta \geq 1$,

$$R_\varepsilon L_\varepsilon = R_\varepsilon^\# L_\varepsilon^\# \mod \mathcal{L}(L^2) \tag{2.37}$$

and

$$\big(R_\varepsilon^\# L_\varepsilon^\#\big)^* = -R_\varepsilon^\# L_\varepsilon^\# \mod OPS^0_{1,\delta}, \tag{2.38}$$

so we have (2.34).

Our analysis of (2.23) is complete; we have, for any $r > 0$,

$$\frac{d}{dt}(D^\alpha u_\varepsilon, Q_\varepsilon D^\alpha u_\varepsilon) \leq C\big(\|u_\varepsilon(t)\|_{C^{1+r}}\big)\|u_\varepsilon(t)\|^2_{H^k}, \quad |\alpha| = k. \tag{2.39}$$

From here we can parallel the rest of the argument of § 1, to prove the following:

Theorem 2.3. *If (2.10) is symmetrizable, in particular if it is strictly hyperbolic, the initial-value problem, with* $u(0) = f \in H^k(M)$*, has a unique local solution* $u \in C(I, H^k(M))$ *whenever* $k > n/2 + 1$.

We have the following slightly weakened analogue of the persistence result, Proposition 1.5:

Proposition 2.4. *Suppose* $u \in C\big([0,T), H^k(M)\big)$, $k > n/2 + 1$*, and assume* u *solves the symmetrizable hyperbolic system (2.10) for* $t \in (0,T)$*. Assume also that, for some* $r > 0$,

$$\|u(t)\|_{C^{1+r}(M)} \le K < \infty, \tag{2.40}$$

for $t \in [0,T)$*. Then there exists* $T_1 > T$ *such that* u *extends to a solution to (2.10), belonging to* $C\big([0,T_1), H^k(M)\big)$.

For the proof of this (and also for the proof of the part of Theorem 2.3 asserting that whenever $f \in H^k(M)$, then u is continuous, not just bounded, in t, with values in $H^k(M)$), one estimates

$$\frac{d}{dt}\big(D^\alpha J_\varepsilon u(t), QD^\alpha J_\varepsilon u(t)\big)$$

in place of (1.40). Then estimates parallel to (2.24)–(2.39) arise, as the reader can verify, yielding the bound

$$\frac{d}{dt}\sum_{|\alpha|\le k}\big(D^\alpha J_\varepsilon u(t), QD^\alpha J_\varepsilon u(t)\big) \le C\big(\|u(t)\|_{C^{1+r}}\big)\|u(t)\|_{H^k}^2. \tag{2.41}$$

If we use this in place of (1.46), the proof of Proposition 1.5 can be parallelled to establish Proposition 2.4.

It follows that the result given in Corollary 1.6, on the local existence of C^∞-solutions, extends to the case of symmetrizable hyperbolic systems (2.10).

We mention that actually Proposition 2.4 can be sharpened to the level of Proposition 1.5. In fact, they can both be improved; the norms $C^{1+r}(M)$ and $C^1(M)$ appearing in the statements of these results can be weakened to the Zygmund norm $C^1_*(M)$. A proof, which is somewhat more complicated than the proof of the result established here, can be found in Chap. 5 of [Tay].

Exercises

1. Show that, for smooth solutions, (2.11) is equivalent to

$$\begin{aligned} \rho_t + \operatorname{div}(\rho v) &= 0, \\ v_t + \nabla_v v + \operatorname{grad} h(\rho) &= 0, \end{aligned} \tag{2.42}$$

assuming $p = p(\rho)$. Here, $h(\rho)$ satisfies

$$h'(\rho) = \rho^{-1} p'(\rho).$$

2. Assume v is a solution to (2.42) of the form $v = \nabla_x \varphi(t, x)$, for some real-valued φ. One says v defines a potential flow. Show that if φ and ρ vanish at infinity appropriately and $h(0) = 0$, then

$$\varphi_t + \frac{1}{2}|\nabla_x \varphi|^2 + h(\rho) = 0. \tag{2.43}$$

This is part of Bernoulli's law for compressible fluid flow. Compare with (5.45).

3. Set $m = \rho v$, the momentum density. Show that, for smooth solutions, (2.42) is equivalent to

$$\begin{aligned} \rho_t + \operatorname{div} m &= 0, \\ m_t + \operatorname{div}(\rho^{-1} m \otimes m) + \operatorname{grad} p(\rho) &= 0. \end{aligned} \tag{2.44}$$

(*Hint*: Make use of the identity $\operatorname{div}(u \otimes v) = (\operatorname{div} v)u + \nabla_v u$.)

4. Show that a symmetrizer for the system (2.44) is given by

$$\frac{1}{\rho} \begin{pmatrix} p'(\rho) + \rho^{-1}|v|^2 & -v \\ -v^t & I \end{pmatrix}, \quad v = \frac{m}{\rho}.$$

Reconsider this problem after doing Exercise 4 in § 8, in light of formulas (8.26)–(8.29) for one space dimension, and of formula (5.53) in general.

5. Consider the one space variable case of (2.10):

$$u_t = B(t, x, u)u_x + g(t, x, u), \quad u(0) = f. \tag{2.45}$$

Show that if this is strictly hyperbolic, that is, $B(t, x, u)$ is a $K \times K$ matrix-valued function whose eigenvalues $\lambda_\nu(t, x, u)$ are all real and distinct, then (2.45) is symmetrizable in the easy sense defined after (2.10). (*Hint*: Eliminate the ξs from the proof of Proposition 2.2.)

3. Second-order and higher-order hyperbolic systems

We begin our discussion of second-order equations with quasi-linear systems, of the form

$$\begin{aligned} u_{tt} - \sum_{j,k} A^{jk}(t, x, D^1 u)\, \partial_j \partial_k u - \sum_j B^j(t, x, D^1 u)\, \partial_j \partial_t u \\ = C(t, x, D^1 u). \end{aligned} \tag{3.1}$$

For now, we assume A^{jk} and B^j are scalar, though we allow u to take values in $\mathbb{R}^L$. Here $D^1 u$ stands for $(u, u_t, \nabla_x u)$, which we also denote $W = (u, u_0, u_1, \dots, u_n)$, so

$$u_0 = \frac{\partial u}{\partial t}, \quad u_j = \frac{\partial u}{\partial x_j}, \quad 1 \le j \le n. \tag{3.2}$$

We obtain a first-order system for W, namely

$$\begin{aligned} \frac{\partial u}{\partial t} &= u_0, \\ \frac{\partial u_0}{\partial t} &= \sum A^{jk}(t,x,W)\,\partial_j u_k + \sum B^j(t,x,W)\,\partial_j u_0 + C(t,x,W), \\ \frac{\partial u_j}{\partial t} &= \partial_j u_0, \end{aligned} \tag{3.3}$$

which is a system of the form

$$\frac{\partial W}{\partial t} = \sum_j H_j(t,x,W)\,\partial_j W + g(t,x,W). \tag{3.4}$$

We can apply to each side the matrix

$$R = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & A^{-1} \end{pmatrix} \tag{3.5}$$

(tensored with the $L \times L$ identity matrix), where A^{-1} is the inverse of the matrix $A = (A^{jk})$. The matrix R is positive-definite as long as A is, that is, as long as A is symmetric and

$$\sum A^{jk}(t,x,W)\xi_j\xi_k \ge C|\xi|^2, \quad C > 0. \tag{3.6}$$

Under this hypothesis, (3.3) is symmetrizable. Consequently we have:

Proposition 3.1. *Under the hypothesis (3.6), if we pick initial data $f \in H^{k+1}(M)$, $g \in H^k(M)$, $k > n/2 + 1$, then (3.1) has a unique local solution*

$$u \in C\big(I, H^{k+1}(M)\big) \cap C^1\big(I, H^k(M)\big) \tag{3.7}$$

satisfying $u(0) = f$, $u_t(0) = g$.

Proof. Define $W = (u, u_0, u_1, \dots, u_n)$, as the solution to (3.3), with initial data

$$u(0) = f, \quad u_0(0) = g, \quad u_j(0) = \partial_j f. \tag{3.8}$$

By Proposition 2.1, we know that there is a unique local solution $W \in C(I, H^k(M))$. It remains to show that u possesses all the stated properties. That

$u(0) = f$ is obvious, and the first line of (3.3) yields $u_t(0) = u_0(0) = g$. Also, $u_t = u_0 \in C(I, H^k)$, which gives part of (3.7). The key to completing the proof is to show that if W satisfies (3.3) with initial data (3.8), then in fact $u_j = \partial u/\partial x_j$ on $I \times M$.

To this end, set

$$v_j = u_j - \frac{\partial u}{\partial x_j}.$$

Since we know that $\partial u/\partial t = u_0$, applying $\partial/\partial t$ to each side yields

$$\frac{\partial v_j}{\partial t} = \frac{\partial u_j}{\partial t} - \frac{\partial u_0}{\partial x_j} = 0,$$

by the last line of (3.3). Since $u_j(0) = \partial_j u(0)$ by (3.8), it follows that $v_j = 0$, so indeed $u_j = \partial_j u$. Then substituting u_t for u_0 and $\partial_j u$ for u_j in the middle line of (3.3) yields the desired equation (3.1) for u.

Finally, since $u_j \in C(I, H^k)$, we have $\nabla_x u \in C(I, H^k)$, and consequently $u \in C(I, H^{k+1})$.

As in § 1, we first take $M = \mathbb{T}^n$. Parallel to Exercise 7 in § 1, we can establish a finite propagation speed result and then, as in Exercise 8 of § 1, obtain a local solution to (3.1) for other M.

We note that (3.6) is stronger than the natural hypothesis of strict hyperbolicity, which is that, for $\xi \neq 0$, the characteristic polynomial

$$\tau^2 - \sum_j B^j(t, x, W)\xi_j \tau - \sum_{j,k} A^{jk}(t, x, W)\xi_j \xi_k \tag{3.9}$$

has two distinct real roots, $\tau = \lambda_\nu(t, W, x, \xi)$. However, in the more general strictly hyperbolic case, using Cauchy data to define a Lorentz metric over the initial surface $\{t = 0\}$, we can effect a local coordinate change so that, at $t = 0$, (A^{jk}) is positive-definite, when the PDE is written in these coordinates, and then the local existence in Proposition 3.1 (and the comment following its proof) applies.

Let us reformulate this result, in a more invariant fashion. Consider a PDE of the form

$$\sum_{j,k} a^{jk}(t, x, D^1 u)\, \partial_j \partial_k u + F(t, x, D^1 u) = 0. \tag{3.10}$$

We let u take values in $\mathbb{R}^L$ but assume $a^{jk}(t, x, W)$ is real-valued. Assume the matrix (a^{jk}) has an inverse, (a_{jk}).

Proposition 3.2. *Assume $\big(a_{jk}(t, x, W)\big)$ defines a Lorentz metric on $\mathcal{O}$ and $S \subset \mathcal{O}$ is a spacelike hypersurface, on which smooth Cauchy data are given:*

$$u\big|_S = f, \quad Yu\big|_S = g, \tag{3.11}$$

where Y is a vector field transverse to S. Then the initial-value problem (3.10–3.11) has a unique smooth solution on some neighborhood of S in $\mathcal{O}$.

In Chap. 18 we will apply this result to Einstein's gravitational equations.

We now look at a second-order, quasi-linear, $L \times L$ system of the form

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j,k} A^{jk}(x, D_x^1 u)\, \partial_j \partial_k u = F(x, D_x^1 u), \tag{3.12}$$

where, for each $j, k \in \{1, \dots, n\}$, $A^{jk}(x, W)$ is a smooth, $L \times L$, matrix-valued function satisfying

$$a_0 |\xi|^2 I \le \sum_{j,k} A^{jk}(x, W) \xi_j \xi_k \le a_1 |\xi|^2 I, \tag{3.13}$$

for some $a_0, a_1 \in (0, \infty)$. This includes equations of vibrating membranes and elastic solids studied in § 1 of Chap. 2. In particular, the condition (3.13) reflects the condition (1.60) of Chap. 2. Note that the system (3.12) might not be strictly hyperbolic.

Here, using results of Chap. 13, § 10, we will write

$$\sum A^{jk}(x, D_x^1 u)\, \partial_j \partial_k u - F(x, D_x^1 u)$$

in terms of a paradifferential operator:

$$\sum_{j,k} A^{jk}(t, x, D_x^1 u)\, \partial_j \partial_k u - F(x, D_x^1 u) = -M(u; x, D)u + R(u), \tag{3.14}$$

where $R(u) \in C^\infty$ and (parallel to (8.20) of Chap. 15) if $r > 0$,

$$u \in C^{2+r} \Longrightarrow M(u; x, \xi) \in \mathcal{A}_0^{1+r} S^2_{1,1} + S^{1-r}_{1,1}. \tag{3.15}$$

Thus, given $\delta \in (0, 1)$, we can use the symbol-smoothing process as in (10.101)–(10.104) of Chap. 13 to write

$$\begin{gathered} M(u; x, \xi) = M^\#(u; x, \xi) + M^b(u; x, \xi), \\ M^\#(u; x, \xi) \in \mathcal{A}_0^{1+r} S^2_{1,\delta}, \quad M^b(u; x, \xi) \in S^{2-(1+r)\delta}_{1,1}. \end{gathered} \tag{3.16}$$

As in (3.13) we have (with perhaps different constants a_j)

$$a_0 |\xi|^2 I \le M^\#(u; x, \xi) \le a_1 |\xi|^2 I, \tag{3.17}$$

for $|\xi| \geq 1$. We can assume $M^{\#}(u;x,\xi) \geq I$, for $|\xi| \leq 1$. Thus, given (3.15),

$$M^{\#}(u;x,\xi)^{1/2} = G(u;x,\xi) \in \mathcal{A}_0^{1+r} S^1_{1,\delta}. \tag{3.18}$$

Now let us set

$$H(u;x,D) = \frac{1}{2}\big[G(u;x,D) + G(u;x,D)^*\big] + I \in OP\mathcal{A}_0^{1+r} S^1_{1,\delta}, \tag{3.19}$$

which is self-adjoint and positive-definite and satisfies

$$H(u;x,D)^2 - M(u;x,D) = B(u;x,D) \in OPS^1_{1,\delta} + OPS^{2-(1+r)\delta}_{1,1}. \tag{3.20}$$

Set $E(u;x,D) = H(u;x,D)^{-1} \in OPS^{-1}_{1,\delta}$, and set

$$v = H(u;x,D)u, \quad w = u_t. \tag{3.21}$$

We have the system

$$\begin{aligned} u_t &= w, \\ v_t &= H(u;x,D)w + C_1(u;x,D)v, \\ w_t &= -H(u;x,D)v + C_2(u;x,D)v + R(u), \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} C_1(u;x,D) &= \partial_t H(u;x,D) \cdot E(u;x,D) \in OPS^0_{1,\delta}, \\ C_2(u;x,D) &= B(u;x,D)E(u;x,D) \in OPS^0_{1,\delta} + OPS^{-\sigma}_{1,1}, \end{aligned} \tag{3.23}$$

provided δ is sufficiently close to 1 that $1-(1+r)\delta = -\sigma < 0$.

Somewhat parallel to (1.4), we obtain solutions to (3.22) as limits of solutions u_ε to

$$\begin{aligned} \partial_t u_\varepsilon &= J_\varepsilon w_\varepsilon, \\ \partial_t v_\varepsilon &= J_\varepsilon H(J_\varepsilon u_\varepsilon;x,D)J_\varepsilon w_\varepsilon + J_\varepsilon C_1(J_\varepsilon u_\varepsilon;x,D)J_\varepsilon v_\varepsilon, \\ \partial_t w_\varepsilon &= -J_\varepsilon H(J_\varepsilon u_\varepsilon;x,D)J_\varepsilon v_\varepsilon + J_\varepsilon C_2(J_\varepsilon u_\varepsilon;x,D)J_\varepsilon v_\varepsilon + R(J_\varepsilon u_\varepsilon). \end{aligned} \tag{3.24}$$

Indeed, setting $U_\varepsilon = (u_\varepsilon, v_\varepsilon, w_\varepsilon)$, one obtains an estimate

$$\frac{d}{dt}\|\Lambda^s U_\varepsilon(t)\|^2_{L^2} \leq C\big(\|U_\varepsilon\|_{C^{1+r}}\big)\Big[\|\Lambda^s U_\varepsilon(t)\|^2_{L^2} + 1\Big], \tag{3.25}$$

from which local existence follows, by arguments similar to those used in § 1. We record the result.

Proposition 3.3. *Under the hypothesis (3.13), if we pick initial data* $f \in H^{s+1}(M)$, $g \in H^s(M)$, $s > n/2 + 1$, *then (3.12) has a unique local solution*

$$u \in C\big(I, H^{s+1}(M)\big) \cap C^1\big(I, H^s(M)\big), \tag{3.26}$$

satisfying $u(0) = f,\ u_t(0) = g$.

Having considered some quasi-linear equations, we now look at a completely nonlinear, second-order equation:

$$u_{tt} = F(t, x, D^1 u, \partial_x^1 u_t, \partial_x^2 u), \quad u(0) = f,\ \ u_t(0) = g. \tag{3.27}$$

Here $F = F(t, x, \xi, \eta, \zeta)$ is smooth in its arguments; $\zeta = (\zeta_{jk}) = (\partial_j \partial_k u)$, and so on. We assume u is real-valued. As before, set $v = (v_0, v_1, \dots, v_n) = (u, \partial_1 u, \dots, \partial_n u)$. We obtain for v a quasi-linear system of the form

$$\begin{aligned} \partial_t^2 v_0 &= F(t, x, D^1 v), \\ \partial_t^2 v_i &= \sum_{j,k} (\partial_{\zeta_{jk}} F)(t, x, D^1 v)\, \partial_j \partial_k v_i \\ &\quad + \sum_j (\partial_{\eta_j} F)(t, x, D^1 v)\, \partial_j \partial_t v_i + G_i(t, x, D^1 v), \end{aligned} \tag{3.28}$$

with initial data

$$v(0) = (f, \partial_1 f, \dots, \partial_n f),\ v_t(0) = (g, \partial_1 g, \dots, \partial_n g). \tag{3.29}$$

The system (3.28) is not quite of the form (3.1), but the difference is minor. One can reduce this to a first-order system and construct a symmetrizer in the same fashion, as long as

$$\tau^2 - \sum (\partial_{\zeta_{jk}} F)(t, x, D^1 v) \xi_j \xi_k - \sum (\partial_{\eta_j} F)(t, x, D^1 v) \xi_j \tau \tag{3.30}$$

has two distinct real roots τ for each $\xi \neq 0$. This is the strict hyperbolicity condition. Proposition 3.1 holds also for (3.28), so we have the following:

Proposition 3.4. *If (3.27) is strictly hyperbolic, then given*

$$f \in H^{k+1}(M), \quad g \in H^k(M), \quad k > \frac{1}{2} n + 2,$$

there is locally a unique solution

$$u \in C(I, H^{k+1}(M)) \cap C^1(I, H^k(M)).$$

This proposition applies to the equations of prescribed Gaussian curvature, for a surface S that is the graph of $y = u(x), x \in \Omega \subset \mathbb{R}^n$, under certain circumstances. The Gauss curvature $K(x)$ is related to $u(x)$ via the PDE

$$\det H(u) - K(x)\big(1 + |\nabla u|^2\big)^{(n+2)/2} = 0, \tag{3.31}$$

where $H(u)$ is the Hessian matrix,

$$H(u) = (\partial_j \partial_k u). \tag{3.32}$$

Note that, if $F(u) = \det H(u)$, then

$$DF(u)v = \operatorname{Tr}[\mathcal{C}(u)H(v)], \tag{3.33}$$

where $\mathcal{C}(u)$ is the cofactor matrix of $H(u)$, so

$$H(u)\mathcal{C}(u) = [\det H(u)]I. \tag{3.34}$$

Of course, (3.31) is elliptic if $K > 0$. Suppose K is negative and on the hypersurface $\Sigma = \{x_n = 0\}$ Cauchy data are prescribed, $u = f(x'),\ \partial_n u = g(x'),\ x' = (x_1, \dots, x_{n-1})$. Then $\partial_k \partial_j u = \partial_k \partial_j f$ on Σ for $1 \le j, k \le n-1,\ \partial_n \partial_j u = \partial_j g$ on Σ for $1 \le j \le n-1$, and then (3.31) uniquely specifies $\partial_n^2 u$, hence $H(u)$, on Σ, provided $\det H(f) \neq 0$. If the matrix $H(u)$ has signature $(n-1, 1)$, and if Σ is spacelike for its quadratic form, then (3.31) is a hyperbolic Monge–Ampere equation, and Proposition 3.4 applies.

We next treat quasi-linear equations of degree m,

$$\partial_t^m u = \sum_{j=0}^{m-1} A_j(t, x, D^{m-1}u, D_x)\, \partial_t^j u + C(t, x, D^{m-1}u), \tag{3.35}$$

with initial conditions

$$u(0) = f_0,\ \partial_t u(0) = f_1, \dots, \partial_t^{m-1} u(0) = f_{m-1}. \tag{3.36}$$

Here, $A_j(t, x, w, D_x)$ is a differential operator, homogeneous of degree $m - j$. Assume u takes values in $\mathbb{R}^K$, but for simplicity we suppose the operators A_j have scalar coefficients. We will produce a first-order system for $v = (v_0, \dots, v_{m-1})$ with

$$v_0 = \Lambda^{m-1} u, \dots, v_j = \Lambda^{m-j-1}\, \partial_t^j u, \dots, v_{m-1} = \partial_t^{m-1} u. \tag{3.37}$$

We have

$$
\begin{aligned}
\partial_t v_0 &= \Lambda v_1, \\
&\vdots \\
\partial_t v_{m-2} &= \Lambda v_{m-1}, \\
\partial_t v_{m-1} &= \sum A_j(t,x,Pv,D_x)\Lambda^{1+j-m} v_j + C(t,x,Pv),
\end{aligned}
\tag{3.38}
$$

where $Pv = D^{m-1}u$ (i.e., $\partial_x^\beta \partial_t^j u = \partial_x^\beta \Lambda^{j+1-m} v_j$), so $P \in OPS^0$. Note that the operator $A_j(t,x,Pv,D_x)\Lambda^{1+j-m}$ is of order 1. The initial condition for (3.38) is

$$
v_0(0) = \Lambda^{m-1} f_0, \dots, v_j(0) = \Lambda^{m-j-1} f_j, \dots, v_{m-1}(0) = f_{m-1}. \tag{3.39}
$$

The system (3.38) has the form

$$
\partial_t v = L(t,x,Pv,D)v + G(t,x,Pv), \tag{3.40}
$$

where L is an $m \times m$ matrix of pseudodifferential operators, which are scalar (though each entry acts on K-vectors). Note that the eigenvalues of the principal symbol of L are $i\lambda_\nu(t,x,v,\xi)$, where $\tau = \lambda_\nu$ are the roots of the characteristic equation

$$
\tau^m - \sum_{j=0}^{m-1} A_j(t,x,Pv,\xi)\tau^j = 0. \tag{3.41}
$$

We will make the hypothesis of strict hyperbolicity, that for $\xi \neq 0$ this equation has m distinct real roots, so $L(t,x,Pv,\xi)$ has m distinct purely imaginary eigenvalues. Consequently, as in Proposition 2.2, there exists a symmetrizer, an $m \times m$, matrix-valued function $R(t,x,w,\xi)$, homogeneous of degree 0 in ξ and smooth in its arguments, such that, for $\xi \neq 0$,

$$
\begin{aligned}
&R(t,x,w,\xi) \quad \text{is positive-definite}, \\
&R(t,x,w,\xi)L(t,x,w,\xi) \quad \text{is skew-adjoint}.
\end{aligned}
\tag{3.42}
$$

Note that, given $r \in (0,\infty) \setminus \mathbb{Z}^+$,

$$
\begin{aligned}
v \in C^{1+r} \Longrightarrow L(t,x,Pv,\xi) &\in C^{1+r}S^1 \quad \text{and} \\
R(t,x,Pv,\xi) &\in C^{1+r}S^0.
\end{aligned}
\tag{3.43}
$$

From here, an argument directly parallel to (2.21)–(2.39) establishes the solvability of (3.38)–(3.39). We have the following result:

Theorem 3.5. *If (3.35) is strictly hyperbolic, and we prescribe initial data* $f_j \in H^{s+m-1-j}(M), s > n/2 + 1$*, then there is a unique local solution*

$$u \in C(I, H^{s+m-1}(M)) \cap C^{m-1}(I, H^s(M)),$$

which persists as long as, for some $r > 0$*,*

$$\|u(t)\|_{C^{m+r}} + \cdots + \|\partial_t^{m-1} u(t)\|_{C^{1+r}}$$

is bounded.

In [Tay] it is shown that the solution persists as long as

$$\|u(t)\|_{C^m_*} + \cdots + \|\partial_t^{m-1} u(t)\|_{C^1_*}$$

is bounded.

While there is a relative abundance of second-order hyperbolic equations and systems arising in various situations, particularly in mathematical physics, compared to the higher-order case, nevertheless there is value in studying higher-order equations, in addition to the fact that such study arises as a “natural” extension of the second-order case. We mention as an example the appearance of a third-order, quasi-linear hyperbolic equation, arising from the study of relativistic fluid motion; this will be discussed in §§ 6 and 8 of Chap. 18.

Exercises

1. Formulate and prove a finite propagation speed result for solutions to (3.1).
2. Recall Exercise 2 of § 2, dealing with the equation (2.42) for compressible fluid flow when v has the special form $v = \nabla_x \varphi(t, x)$. Show that φ satisfies the second-order PDE

$$\partial_t H_0(\nabla\varphi) + \sum_{j \ge 1} \partial_j H_j(\nabla\varphi) = 0, \tag{3.44}$$

where $\nabla\varphi = (\varphi_t, \nabla_x \varphi)$ and the functions H_j are given by

$$\begin{aligned} H_0(\nabla\varphi) &= -K\left(\varphi_t + \frac{1}{2}|\nabla_x\varphi|^2\right), \\ H_j(\nabla\varphi) &= (\partial_j\varphi) H_0(\nabla\varphi), \qquad j \ge 1. \end{aligned} \tag{3.45}$$

Here, K is the inverse function of h, defined by:

$$y = h(\rho) \Longleftrightarrow \rho = K(y).$$

Examine the hyperbolicity of this PDE.

3. Consider three-dimensional Minkowski space $\mathbb{R}^{1,2} = \{(t, x, y)\}$, with metric $ds^2 = -dt^2 + dx^2 + dy^2$. Let S be a surface in $\mathbb{R}^{1,2}$, given by

$$y = u(t, x).$$

Show that the condition for S to be a minimal surface in $\mathbb{R}^{1,2}$ is that

$$(1+u_x^2)u_{tt} - 2(u_t \cdot u_x)u_{xt} - (1-u_t^2)u_{xx} = 0. \tag{3.46}$$

Show that this is hyperbolic provided $u_t^2 < 1 + u_x^2$, and that this holds provided the induced metric tensor on S has signature $(1,1)$. (*Hint*: To get (3.46), adapt the calculations used to produce the minimal surface equation (7.6) in Chap. 14.)

Exercises on nonlinear Klein–Gordon equations, and variants

In these exercises we consider the initial-value problem for semilinear hyperbolic equations of the form

$$u_{tt} - \Delta u + m^2 u = f(u), \tag{3.47}$$

for real-valued u. Here, Δ is the Laplace operator on a compact Riemannian manifold M, or on $\mathbb{R}^n$. We assume $m > 0$, and set $\Lambda = \sqrt{-\Delta + m^2}$.

1. Show that, for $s > 0$, sufficiently smooth solutions to (3.47) satisfy

$$\frac{d}{dt}\big[\|\Lambda^{s+1}u\|_{L^2}^2 + \|\Lambda^s u_t\|_{L^2}^2\big] = 2\big(\Lambda^s f(u), \Lambda^s u_t\big)_{L^2}. \tag{3.48}$$

2. Using arguments such as those that arose in proving Proposition 1.5, show that smooth solutions to (3.47) persist as long as $\|u(t)\|_{L^\infty}$ can be bounded.
3. Note that the $s = 0$ case of (3.48) can be written as

$$\frac{d}{dt}\big[\|\nabla u\|_{L^2}^2 + m^2\|u\|_{L^2}^2 + \|u_t\|_{L^2}^2\big] = 2\int_M u_t f(u)\, dV. \tag{3.49}$$

Thus, if $f(u) = g'(u)$, we have

$$\|\nabla u(t)\|_{L^2}^2 + m^2\|u(t)\|_{L^2}^2 + \|u_t(t)\|_{L^2}^2 - \int g\big(u(t)\big)\, dV = \text{ const.} \tag{3.50}$$

Deduce that

$$g \le 0 \Longrightarrow \|u(t)\|_{H^1}^2 + \|u_t(t)\|_{L^2}^2 \le \text{ const.} \tag{3.51}$$

4. Deduce that (3.47) is globally solvable for nice initial data, provided that $f(u) = g'(u)$ with $g(u) \le 0$ and that $\dim M = n = 1$.
5. Note that the $s = 1$ case of (3.48) can be written as

$$\begin{aligned}\frac{d}{dt}\Big[\|Lu\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + m^2\|u_t\|_{L^2}^2\Big] \\ = 2\big(\nabla f(u), \nabla u_t\big)_{L^2} + 2m^2\big(f(u), u_t\big)_{L^2},\end{aligned} \tag{3.52}$$

where $L = -\Delta + m^2$. Assume $\dim M = n = 3$, so that, by Proposition 2.2 of Chap. 13,

$$H^1(M) \subset L^6(M).$$

Deduce that the right side of (3.52) is then

$$\begin{aligned}&\le \|\nabla f(u)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + m^2\|f(u)\|_{L^2}^2 + m^2\|u_t\|_{L^2}^2\\ &\le C\|f'(u)\|_{L^3}^2\|Lu\|_{L^2}^2 + m^2\|f(u)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + m^2\|u_t\|_{L^2}^2.\end{aligned} \tag{3.53}$$

(*Hint*: To estimate $\|f'(u)\nabla u\|_{L^2}^2$, use $\|vw\|_{L^2}^2 \le \|v\|_{L^{2p}}^2\|w\|_{L^{2p'}}^2$ with $2p' = 6$, so $2p = 3$.)

6. If $f(u) = -u^3$, then $\|f'(u)\|_{L^3} \le 9\|u\|_{L^6}^2 \le C\|u\|_{H^1}^2$ and $\|f(u)\|_{L^2} \le \|u\|_{L^6}^3$. Making use of (3.51) to estimate $\|u\|_{H^1}$, demonstrate global solvability for

$$u_{tt} - \Delta u + m^2 u = -u^3, \tag{3.54}$$

with nice initial data, given that $\dim M = n = 3$.

For further material on nonlinear Klein–Gordon equations, including treatments of (3.54) with u^3 replaced by u^5, see [Gril, Ra, Re, Seg, St, Str].

In Exercises 7–12, we consider the equation (3.47) under the hypotheses

$$f(0) = 0, \qquad |f^{(\ell)}(u)| \le C_\ell, \quad \ell \ge 1. \tag{3.55}$$

An example is $f(u) = \sin u$; then (3.47) is called the *sine–Gordon equation*.

7. Show that if u is a sufficiently smooth solution to (3.47), and we take the "energy" $E(t) = \|\Lambda u(t)\|_{L^2}^2 + \|u_t(t)\|_{L^2}^2$, then

$$\frac{dE}{dt} \le C + CE(t),$$

and hence

$$\|u(t)\|_{H^1} \le C(t). \tag{3.56}$$

This partially extends Exercise 3, in that $f(u) = g'(u)$, with $g(u) \le C_1|u|^2$.

8. Deduce that (3.47) is globally solvable for nice initial data (given (3.55)), provided that $n = 1$.

In Exercises 9–11, assume that $n \ge 2$ and that $u(0) = u_0 \in H^s(\mathbb{R}^n),\ u_t(0) = u_1 \in H^{s-1}(\mathbb{R}^n),\ s > n/2 + 1$.

9. Establish an estimate of the form

$$\|u(t)\|_{H^2} \le C(t), \tag{3.57}$$

and deduce that (3.47) is globally solvable (given (3.55)), provided $n = 2$ or 3. (*Hint*: Write $u(t) = v(t) + w(t)$, where $v(t)$ solves

$$v_{tt} - (\Delta - m^2)v = 0, \quad v(0) = u_0, \ \ v_t(0) = u_1,$$

and

$$w(t) = \int_0^t \frac{\sin(t-s)\Lambda}{\Lambda}\, f\big(u(s)\big)\, ds. \tag{3.58}$$

To get (3.57) from this, establish the estimate

$$\|f(u(t))\|_{H^1} \le C(t), \tag{3.59}$$

from (3.56).)

10. Suppose $n = 4$. Show that

$$\|u(t)\|_{H^3} \le C(t), \tag{3.60}$$

and deduce that (3.47) is globally solvable (given (3.55)), provided $n = 4$. (*Hint*: Start with $\|\partial_j \partial_k f(u)\|_{L^2} \le C_1 \|\partial_j \partial_k u\|_{L^2} + C_2 \|\nabla u\|^2_{L^4}$, and use the Sobolev estimate

$$\|\varphi\|_{L^{2p}(\mathbb{R}^n)} \le C\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}, \quad p = \frac{n}{n-2}, \tag{3.61}$$

to deduce that

$$n = 4 \Longrightarrow \|\partial_j \partial_k f(u)\|_{L^2} \le C\|u\|_{H^2} + C\|u\|^2_{H^2}. \tag{3.62}$$

Then use (3.58) to estimate $\|w(t)\|_{H^3}$.)

11. Show that (3.60) also holds when $n = 5$. Deduce that (3.47) is globally solvable (given (3.55)), provided $n = 5$. (*Hint*: Start with $\|\partial_j \partial_k f(u)\|_{L^p} \le C_1 \|\partial_j \partial_k u\|_{L^p} + C_2 \|\nabla u\|^2_{L^{2p}}$, and apply (3.61), with $p = 5/3$, to get

$$n = 5 \Longrightarrow \|f(u)\|_{H^{2,5/3}} \le C(t). \tag{3.63}$$

Use the Sobolev imbedding result $H^{\sigma,p}(\mathbb{R}^n) \subset L^{np/(n-\sigma p)}(\mathbb{R}^n)$ to deduce

$$\|f(u(t))\|_{H^{2-1/2}} \le C(t). \tag{3.64}$$

Use (3.58) to deduce

$$n = 5 \Longrightarrow \|u(t)\|_{H^{2+1/2}} \le C(t). \tag{3.65}$$

Iterate this argument, to get (3.60).)

12. Derive results on the global existence of *weak* solutions to (3.47), under the hypothesis (3.55), analogous to those in Exercises 12 and 13 of § 1.

For further results on the equation (3.47), under hypotheses like (3.55), but more general, see [BW] and [Str].

Exercises on wave maps

In these exercises, we consider the initial-value problem for semilinear hyperbolic systems of the form

$$u_{tt} - \Delta u = B(x, u, \nabla u), \tag{3.66}$$

where $B(x, u, p)$ is smooth in (x, u) and a quadratic form in p. Here, Δ is the Laplace operator on a compact Riemannian manifold X, $u(t, x)$ takes values in $\mathbb{R}^\ell$, and $\nabla u = \nabla_{t,x} u$.

1. Show that, for $s \geq 0$, sufficiently smooth solutions to (3.66) satisfy

$$\frac{d}{dt}\big(\|\nabla_x \Lambda^s u(t)\|_{L^2}^2 + \|\partial_t \Lambda^s u(t)\|^2\big) = 2\big(\Lambda^s u_t, \Lambda^s B(x, u, \nabla u)\big)_{L^2}. \tag{3.67}$$

2. Using arguments such as those that arose in proving Proposition 1.5, show that smooth solutions to (3.66) persist as long as $\|u(t)\|_{C^1} + \|\partial_t u(t)\|_{L^\infty}$ can be bounded.
3. Suppose that, for $t \in I$, $u(t, x)$ solves (3.66) and $u : I \times X \to N$, where N is a submanifold of $\mathbb{R}^\ell$. Suppose also that, for all $(t, x)\imath I \times X$,

$$B(x, u, \nabla u) \perp T_u N. \tag{3.68}$$

Show that

$$e(t, x) = \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla_x u|^2, \quad E(t) = \int_X e(t, x)\, dV(x), \tag{3.69}$$

satisfies

$$\frac{dE}{dt} = 0. \tag{3.70}$$

(*Hint*: The hypothesis (3.68) implies $u_t \cdot B(x, u, \nabla u) = 0$. Then use (3.67), with $s = 0$.)

In Exercises 4–6, suppose X is the flat torus $\mathbb{T}^n$, or perhaps $X = \mathbb{R}^n$. Assume (3.68) holds. Define

$$m_j(t, x) = u_t \cdot \partial_j u. \tag{3.71}$$

4. Show that

$$\frac{\partial e}{\partial t} - \sum_j \frac{\partial m_j}{\partial x_j} = 0. \tag{3.72}$$

(*Hint*: Start with $\partial_t e = u_t \cdot u_{tt} + \nabla_x u \cdot \nabla_x u_t$, and use the equation (3.66); then use $u_t \cdot B(x, u, \nabla u) = 0$.)
5. Show that, for each $j = 1, \dots, n$,

$$\frac{\partial m_j}{\partial t} - \frac{\partial e}{\partial x_j} = \sum_i \big\{\partial_i(\partial_i u \cdot \partial_j u) - \partial_j(\partial_i u \cdot \partial_i u)\big\}. \tag{3.73}$$

(*Hint*: Use $\partial_j u \cdot B(x, u, \nabla u)$ to get $\partial_t m_j = \Delta u \cdot \partial_j u + u_t \cdot \partial_j u_t$; then compute $\partial_j e$ and subtract.)

The considerations of Exercises 1–5 apply to the “wave map” equation

$$u_{tt} - \Delta u = \Gamma(u)(\nabla u, \nabla u), \tag{3.74}$$

where $\nabla u = \nabla_{t,x} u$ and $\Gamma(u)(\nabla u, \nabla u)$ is as in the harmonic map equation (10.25) of Chap. 14. Indeed, (3.74) is the analogue of the harmonic map equation for a map $u : M \to N$ when N is Riemannian but M is Lorentzian.

6. Suppose $n = 1$. Then $X = S^1$ (or $\mathbb{R}^1$). Show that (3.74) has a global smooth solution, for smooth Cauchy data, $u(0) = f$, $u_t(0) = g$, satisfying $f : X \to N$, $g(x) \in T_{f(x)}N$.
   (*Hint*: In this case, (3.72)–(3.73) imply $\partial_t^2 e - \partial_1^2 e = 0$, which gives a pointwise bound for $e(t, x)$.) This argument follows [Sha].

For results in higher dimensions, including global weak solutions and singularity formation, see [Sha], and references given therein.

4. Equations in the complex domain and the Cauchy–Kowalewsky theorem

Consider an mth order, nonlinear system of PDE of the form

$$\tag{4.1} \begin{gathered} \frac{\partial^m u}{\partial t^m} = A(t, x, D_x^m u, D_x^{m-1} \partial_t u, \dots, D_x \partial_t^{m-1} u), \\ u(0, x) = g_0(x), \dots, \partial_t^{m-1} u(0, x) = g_{m-1}(x). \end{gathered}$$

The Cauchy–Kowalewsky theorem is the following:

Theorem 4.1. *If A is real analytic in its arguments and g_j are real analytic, for $x \in \mathcal{O} \subset \mathbb{R}^n$, then there is a unique $u(t, x)$ that is real analytic for $x \in \mathcal{O}_1 \subset\subset \mathcal{O}$, t near 0, and satisfies (4.1).*

We established the linear case of this in Chap. 6. Here, in order to prove Theorem 4.1, we use a method of Garabedian [Gb1, Gb2], to transmutate (4.1) into a symmetric hyperbolic system for a function of (t, x, y). To begin, by a simple argument, it suffices to consider a general first-order, quasi-linear, $N \times N$ system, of the form

$$\tag{4.2} \frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j(t, x, u) \frac{\partial u}{\partial x_j} + f(t, x, u), \quad u(0, x) = g(x).$$

We assume that A_j and f are real analytic in their arguments, and we use these symbols also to denote the holomorphic extensions of these functions. Similarly, we assume g is analytic, with holomorphic extension $g(z)$. We want to solve (4.2) for u which is real analytic, that is, we want to extend u to $u(t, x, y)$, so as to be holomorphic in $z = x + iy$, so that

$$\tag{4.3} \frac{\partial u}{\partial x_j} + i \frac{\partial u}{\partial y_j} = 0.$$

Now, multiplying this by iB_j and adding to (4.2), we have

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \big[A_j + iB_j\big] \frac{\partial u}{\partial x_j} - \sum_{j=1}^{n} B_j \frac{\partial u}{\partial y_j} + f(t,x,u). \tag{4.4}$$

We arrange for this to be symmetric hyperbolic by taking

$$B_j(t,z,u) = \frac{1}{2i}\big(A_j^* - A_j\big). \tag{4.5}$$

Thus we have a local smooth solution to (4.4), given smooth initial data $u(0,x,y) = g(x,y)$. Now, if $g(x,y)$ is holomorphic for $(x,y) \in U$, we want to show that $u(t,x,y)$ is holomorphic for $(x,y) \in U_1 \subset U$ if t is close to 0. To see this, set

$$v_j = \frac{1}{2}\Big(\frac{\partial u}{\partial x_j} + i\,\frac{\partial u}{\partial y_j}\Big) = \frac{\partial u}{\partial \bar{z}_j}. \tag{4.6}$$

Then

$$\begin{aligned} \frac{\partial v_\nu}{\partial t} = & \sum_{j=1}^{n} \big[A_j + iB_j\big] \frac{\partial v_\nu}{\partial x_j} - \sum_{j=1}^{n} B_j \frac{\partial v_\nu}{\partial y_j} + \sum_{j=1}^{n} \big(\partial_{\bar{z}_\nu} A_j\big) \frac{\partial u}{\partial x_j} \\ & + \sum_{j=1}^{n} \big(i\,\partial_{\bar{z}_\nu} B_j\big) v_j + \partial_{\bar{z}_\nu} f(t,z,u). \end{aligned} \tag{4.7}$$

Since $A_j(t,z,u)$ and $f(t,z,u)$ are holomorphic in z and u,

$$\partial_{\bar{z}_\nu} A_j(t,z,u) = \sum_{\mu} \frac{\partial A_j}{\partial u_\mu} v_\nu^\mu = C_j(t,z,u) v_\nu,$$

and similarly $\partial_{\bar{z}_\nu} f(t,z,u) = F(t,z,u) v_\nu$. Thus

$$\frac{\partial v_\nu}{\partial t} = \sum_{j=1}^{n} \big[A_j + iB_j\big] \frac{\partial v_\nu}{\partial x_j} - \sum_{j=1}^{n} B_j \frac{\partial v_\nu}{\partial y_j} + E v_\nu + \sum_{j=1}^{n} G_{\nu j} v_j, \tag{4.8}$$

with

$$E = \sum_j C_j(t,z,u) + F(t,z,u), \quad G_{\nu j} = i\,\partial_{\bar{z}_\nu} B_j.$$

This is a symmetric hyperbolic, $(Nn) \times (Nn)$ system for $v = (v_\nu^\mu : 1 \le \mu \le N, 1 \le \nu \le n)$. The hypothesis that $g(x,y)$ is holomorphic for $(x,y) \in U$ means $v(0,x,y) = 0$ for $(x,y) \in U$. Thus, by finite propagation speed, $v(t,x,y) = 0$ on a neighborhood of $\{0\} \times U$.

Thus we have a solution to (4.2) which is holomorphic in $x + iy$, under the hypotheses of Theorem 4.1. We have not yet established analyticity in t; in fact, so far we have not used the analyticity of A_j and f in t. We do this now. As above, we use A_j, f, and u also to denote the holomorphic extensions to $\zeta = t + is$. Having u for $s = 0$, and desiring $\partial u/\partial t = -i\,\partial u/\partial s$, we produce $u(t, s, x, y)$ as the solution to

$$\frac{\partial u}{\partial s} = i \sum_{j=1}^{n} A_j \frac{\partial u}{\partial x_j} + if, \quad u(t, 0, x, y) = \text{ solution to (4.4)}. \tag{4.9}$$

Applying $iB_j^\#$ to (4.3) and adding to (4.9), we get

$$\frac{\partial u}{\partial s} = \sum_{j=1}^{n} \bigl[iA_j + iB_j^\#\bigr] \frac{\partial u}{\partial x_j} - \sum_{j=1}^{n} B_j^\# \frac{\partial u}{\partial y_j} + if, \tag{4.10}$$

which we arrange to be symmetric hyperbolic by taking

$$B_j^\#(t, s, x, y) = -\frac{1}{2}\bigl(A_j^* + A_j\bigr). \tag{4.11}$$

To see that the solution to (4.9) is holomorphic in $t + is$, let

$$w = \frac{1}{2}\Bigl(\frac{\partial u}{\partial t} + i\frac{\partial u}{\partial s}\Bigr) = \frac{\partial u}{\partial \overline{z_0}}. \tag{4.12}$$

By the initial condition for u at $s = 0$ given in (4.9), we have $w = 0$ for $s = 0$. Meanwhile, parallel to (4.7), w satisfies a symmetric hyperbolic system, so u is holomorphic in $t+is$. This establishes the Cauchy–Kowalewsky theorem for (4.2), and the general case (4.1) follows easily.

There are other proofs of the Cauchy–Kowalewsky theorem. Some work by estimating the terms in the power series of $u(t, x)$ about $(0, x_0)$. Such proofs are often presented near the beginning of PDE books, as they are elementary, though many students have grumbled that going through this somewhat elaborate argument at such an early stage is rather painful. The proof presented above reflects an aesthetic sensibility that prefers the use of complex function theory to power-series arguments. Another sort of proof, with a similar aesthetic, is given in [Nir]; see also [Ovs] and [Cafl].

There is an extension of the Cauchy–Kowalewsky theorem to systems (not necessarily determined), known as Cartan–Kähler theory. An account of this and many important ramifications can be found in [BCG3].

Exercises

1. Fill in the details of reducing (4.1) to (4.2).

5. Compressible fluid motion

We begin with a brief derivation of the equations of ideal compressible fluid flow on a region Ω. Suppose a fluid "particle" has position $F(t,x)$ at time t, with $F(0,x) = x$. Thus the velocity field of the fluid is

$$v(t,y) = F_t(t,x) \in T_y\Omega, \quad y = F(t,x), \tag{5.1}$$

where $F_t(t,x) = (\partial/\partial t)F(t,x)$. If $y \in \partial\Omega$, we assume that $v(t,y)$ is tangent to $\partial\Omega$. We want to write down a Lagrangian for the motion. At any time t, the kinetic energy of the fluid is

$$\begin{aligned} K(t) &= \frac{1}{2}\int_\Omega |v(t,y)|^2 \rho(t,y)\,dy \\ &= \frac{1}{2}\int_\Omega |F_t(t,x)|^2 \rho_0(x)\,dx, \end{aligned} \tag{5.2}$$

where $\rho(t,y)$ is the density of the fluid, and $\rho_0(x) = \rho(0,x)$. Thus

$$\rho_0(x) = \rho(t,y)\det D_x F(t,x), \quad y = F(t,x). \tag{5.3}$$

In the simplest models, the potential energy density is a function of fluid density alone:

$$\begin{aligned} V(t) &= \int_\Omega W\big(\rho(t,y)\big)\rho(t,y)\,dy \\ &= \int_\Omega W\big(\rho(t,F(t,x))\big)\rho_0(x)\,dx. \end{aligned} \tag{5.4}$$

Set $W(\rho) = Q(\rho^{-1})$, $\sigma_0(x) = \rho_0(x)^{-1}$. In such a case, the Lagrangian action integral is

$$L(F) = \int_I \int_\Omega \Big[\frac{1}{2}|F_t(t,x)|^2 - Q\big(\sigma_0(x)\det D_x F(t,x)\big)\Big]\rho_0(x)\,dx\,dt \tag{5.5}$$

defined on the space of maps $F : I \times \Omega \to \Omega$, where I is an arbitrary time interval $[t_0,t_1] \subset [0,\infty)$. We seek to produce a PDE describing the critical points of L.

Split $L(F)$ into $L(F) = L_K(F) - L_V(F)$, with obvious notation. Then

$$\begin{aligned} DL_K(F)w &= \iint \Big\langle F_t(t,x), \frac{d}{dt}w\big(t,F(t,x)\big)\Big\rangle \rho_0(x)\,dx\,dt \\ &= -\iint \Big\langle \frac{\partial v}{\partial t} + v\cdot\nabla_y v, \tilde{w}(t,y)\Big\rangle \rho(t,y)\,dy\,dt, \end{aligned} \tag{5.6}$$

upon an integration by parts, since $(d/dt)v\big(t, F(t,x)\big) = \partial v/\partial t + v \cdot \nabla_y v$. We have set $\tilde{w}(t,y) = w(t,x),\ y = F(t,x)$. Next,

$$DL_V(F)w = \iint Q'\big(\sigma_0(x) \det D_x F(t,x)\big) \det D_x F(t,x) \cdot \operatorname{Tr}\big(D_x F(t,x)^{-1} D_x w(t,x)\big)\, dx\, dt. \tag{5.7}$$

Now $D_x w(t,x) = D_x \tilde{w}\big(t, F(t,x)\big) = D_y \tilde{w}\big(t, F(t,x)\big) D_x F(t,x)$, so

$$\begin{aligned} \operatorname{Tr}\big(D_x F(t,x)^{-1} D_x w(t,x)\big) &= \operatorname{Tr} D_y \tilde{w}\big(t, F(t,x)\big) \\ &= \operatorname{div} \tilde{w}\big(t, F(t,x)\big). \end{aligned} \tag{5.8}$$

Hence

$$\begin{aligned} DL_V(F)w &= \iint Q'\big(\rho(t,y)^{-1}\big) \operatorname{div} \tilde{w}(t,y)\, dy\, dt \\ &= \iint Q''(\rho^{-1})\rho^{-2}\big\langle \nabla_y \rho(t,y), \tilde{w}(t,y)\big\rangle\, dy\, dt. \end{aligned} \tag{5.9}$$

Since $W(\rho) = Q(\rho^{-1})$, we have $Q''(\rho^{-1})\rho^{-2} = \rho^2 W''(\rho) - 2\rho W'(\rho) = \rho X''(\rho)$ if we set

$$X(\rho) = \rho W(\rho), \tag{5.10}$$

so we can write

$$DL_V(F)w = \iint \big\langle X''(\rho)\nabla_y \rho, \tilde{w}(t,y)\big\rangle \rho(t,y)\, dy\, dt. \tag{5.11}$$

Thus we have the Euler equations:

$$\frac{\partial v}{\partial t} + \nabla_v v + X''(\rho)\nabla \rho = 0, \tag{5.12}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0. \tag{5.13}$$

Equation (5.12) expresses the stationary condition, $DL(F)w = 0$ for all smooth vector fields w, tangent to $\partial\Omega$, while (5.13) simply expresses conservation of matter. Replacing $v \cdot \nabla_x v$ by $\nabla_v v$ as we have done above makes these equations valid when Ω is a Riemannian manifold with boundary. The boundary condition, as we have said, is

$$v(t,x) \parallel \partial\Omega, \quad x \in \partial\Omega. \tag{5.14}$$

Recognizing $\partial v/\partial t + \nabla_v v = (d/dt)v\big(t, F(t,x)\big)$ as the acceleration of a fluid particle, we rewrite (5.12) in a form reflecting Newton's law $F = ma$:

$$\rho\Big(\frac{\partial v}{\partial t} + \nabla_v v\Big) = -\nabla_x p. \tag{5.15}$$

The real-valued function p is called the *pressure* of the fluid. Comparison with (5.12) gives $p = p(\rho)$ and

$$\frac{p'(\rho)}{\rho} = X''(\rho). \tag{5.16}$$

The relation $p = p(\rho)$ is called an equation of state; the function $p(\rho)$ depends upon physical properties of the fluid.

Making use of the identity

$$\operatorname{div}(u \otimes v) = (\operatorname{div} v)u + \nabla_u v, \tag{5.17}$$

we can rewrite the system (5.12)–(5.13), with $X''(\rho)\nabla\rho$ replaced by $\nabla p/\rho$, in the form

$$\begin{aligned} (\rho v)_t + \operatorname{div}(\rho v \otimes v) + \nabla p &= 0, \\ \rho_t + \operatorname{div}(\rho v) &= 0, \end{aligned} \tag{5.18}$$

which is convenient for consideration of nonsmooth solutions.

It is natural to assume that $W(\rho)$ is an increasing function of ρ. One common model takes

$$W(\rho) = \alpha\rho^{\gamma-1}, \quad \alpha > 0, \;\; 1 < \gamma < 2. \tag{5.19}$$

In such a case, we have an equation of state of the form

$$p(\rho) = A\rho^{\gamma}, \quad A = (\gamma - 1)\alpha > 0. \tag{5.20}$$

Experiments indicate that for air, under normal conditions, this provides a good approximation to the equation of state if we take $\gamma = 1.4$. Obviously, these formulas lose validity when ρ becomes so large that air becomes as dense as a liquid, but in that situation other physical phenomena come into play, and the entire problem has to be reformulated.

We will rewrite Euler's equation, letting $\tilde{v}$ denote the 1-form corresponding to the vector field v via the Riemannian metric on Ω. Then (5.12) is equivalent to

$$\frac{\partial \tilde{v}}{\partial t} + \nabla_v \tilde{v} = -dx'(\rho), \quad X'(\rho) = \int \frac{p'(\rho)}{\rho}\, d\rho. \tag{5.21}$$

In turn, we will rewrite this, using the Lie derivative. Recall that, for any vector field Z, $\nabla_v Z = \mathcal{L}_v Z + \nabla_Z v$, by the zero-torsion condition on ∇. Using this, we deduce that

$$\langle \mathcal{L}_v \tilde{v} - \nabla_v \tilde{v}, Z\rangle = \langle \tilde{v}, \nabla_Z v\rangle = \frac{1}{2}\langle d|v|^2, Z\rangle,$$

so (5.21) is equivalent to

$$\frac{\partial \tilde{v}}{\partial t} + \mathcal{L}_v \tilde{v} = d\Big(\frac{1}{2}|v|^2 - X'(\rho)\Big). \tag{5.22}$$

A physically important object derived from the velocity field is the *vorticity*, which we define to be

$$\tilde{\xi} = d\tilde{v}, \tag{5.23}$$

for each t a 2-form on Ω. Applying the exterior derivative to (5.22) gives the

Vorticity equation

$$\frac{\partial \tilde{\xi}}{\partial t} + \mathcal{L}_v \tilde{\xi} = 0. \tag{5.24}$$

It is also useful to consider vorticity in another form. Namely, to $\tilde{\xi}$ we associate a section ξ of $\Lambda^{n-2}T$ ($n = \dim \Omega$), so that the identity

$$\tilde{\xi} \wedge \alpha = \langle \xi, \alpha\rangle \omega \tag{5.25}$$

holds, for every $(n-2)$-form α, where ω is the volume form on Ω, which we assume to be oriented. We have

$$\begin{aligned}
\mathcal{L}_v \tilde{\xi} \wedge \alpha &= \mathcal{L}_v(\tilde{\xi} \wedge \alpha) - \tilde{\xi} \wedge \mathcal{L}_v \alpha \\
&= \langle \mathcal{L}_v \xi, \alpha\rangle \omega + \langle \xi, \mathcal{L}_v \alpha\rangle \omega + (\operatorname{div} v)\langle \xi, \alpha\rangle \omega - \tilde{\xi} \wedge \mathcal{L}_v \alpha \\
&= \langle \mathcal{L}_v \xi, \alpha\rangle \omega + (\operatorname{div} v)\langle \xi, \alpha\rangle \omega,
\end{aligned}$$

so (5.24) implies

$$\frac{\partial \xi}{\partial t} + \mathcal{L}_v \xi + (\operatorname{div} v)\xi = 0. \tag{5.26}$$

This takes a neater form if we consider vorticity divided by ρ:

$$w = \frac{\xi}{\rho}. \tag{5.27}$$

Then the left side of (5.26) is equal to $\rho\big(\partial w/\partial t + \mathcal{L}_v w\big) + \big(\partial\rho/\partial t + \nabla_v \rho + \rho(\text{div } v)\big)w$, and if we use (5.13), we see that

(5.28) $$\frac{\partial w}{\partial t} + \mathcal{L}_v w = 0.$$

This vorticity equation takes special forms in two and three dimensions, respectively. When $\dim \Omega = n = 2,\ w$ is a scalar field, often denoted as

(5.29) $$w = \rho^{-1} \ \text{rot}\, v,$$

and (5.28) becomes the

2-D vorticity equation.

(5.30) $$\frac{\partial w}{\partial t} + v \cdot \text{grad}\, w = 0,$$

which is a conservation law.

If $n = 3,\ w$ is a vector field, denoted as

(5.31) $$w = \rho^{-1} \ \text{curl}\, v,$$

and (5.28) becomes the

3-D vorticity equation.

(5.32) $$\frac{\partial w}{\partial t} + [v, w] = 0,$$

or equivalently,

(5.33) $$\frac{\partial w}{\partial t} + \nabla_v w - \nabla_w v = 0.$$

The first form (5.23) of the vorticity equation implies

(5.34) $$\tilde{\xi}(0) = (F^t)^* \tilde{\xi}(t),$$

where $F^t(x) = F(t, x),\ \tilde{\xi}(t)(x) = \tilde{\xi}(t, x)$. Similarly, (5.28) yields

(5.35) $$w(t, y) = \Lambda^{n-2} DF^t(x)\, w(0, x), \quad y = F(t, x),$$

where $DF^t(x) : T_x\Omega \to T_y\Omega$ is the derivative. In case $n = 2$, this is simply $w(t, y) = w(0, x)$, the conservation law mentioned after (5.30).

One implication of (5.34) is the following. Let S be an oriented surface in Ω, with boundary C; let $S(t)$ be the image of S under F^t, and $C(t)$ the image of C; then (5.34) yields

$$\int_{S(t)} \tilde{\xi}(t) = \int_{S} \tilde{\xi}(0). \tag{5.36}$$

Since $\tilde{\xi} = d\tilde{v}$, this implies the following:

Kelvin's circulation theorem.

$$\int_{C(t)} \tilde{v}(t) = \int_{C} \tilde{v}(0). \tag{5.37}$$

We take a look at some phenomena special to the case $\dim \Omega = n = 3$, where the vorticity ξ is a vector field on Ω, for each t. Fix t_0 and consider $\xi = \xi(t_0)$. Let S be an oriented surface in Ω, transversal to ξ. A *vortex tube* $\mathcal{T}$ is defined to be the union of orbits of ξ through S, to a second transversal surface S_2. For simplicity we will assume that none of these orbits ends at a zero of the vorticity field, though more general cases can be handled by a limiting argument.

Since $d\tilde{\xi} = d^2\tilde{v} = 0$, we can use Stokes' theorem to write

$$0 = \int_{\mathcal{T}} d\tilde{\xi} = \int_{\partial\mathcal{T}} \tilde{\xi}. \tag{5.38}$$

Now $\partial\mathcal{T}$ consists of three pieces: S and S_2 (with opposite orientations) and the lateral boundary $\mathcal{L}$ the union of the orbits of ξ from ∂S to ∂S_2. Clearly, the pull-back of $\tilde{\xi}$ to $\mathcal{L}$ is 0, so (5.38) implies

$$\int_{S} \tilde{\xi} = \int_{S_2} \tilde{\xi}. \tag{5.39}$$

Applying Stokes' theorem again, for $\tilde{\xi} = d\tilde{v}$, we have

Helmholtz' theorem. *For any two curves C, C_2 enclosing a vortex tube,*

$$\int_{C} \tilde{v} = \int_{C_2} \tilde{v}. \tag{5.40}$$

This common value is called the strength of the vortex tube $\mathcal{T}$.

Also, note that if $\mathcal{T}$ is a vortex tube at $t_0 = 0$, then, for each t, $\mathcal{T}(t)$, the image of $\mathcal{T}$ under F^t, is a vortex tube, as a consequence of (5.35), with $n = 3$, since ξ and $w = \xi/\rho$ have the same integral curves. Furthermore, (5.37) implies that the strength of $\mathcal{T}(t)$ is independent of t. This conclusion is also part of Helmholtz' theorem.

If we write $\mathcal{L}_v \tilde{v}$ in terms of exterior derivatives, we obtain from (5.22) the equivalent formula

$$\frac{\partial \tilde{v}}{\partial t} + (d\tilde{v}) \rfloor v = -d\Big(\frac{1}{2}|v|^2 + X'(\rho)\Big). \tag{5.41}$$

We can use this to obtain various results known collectively as *Bernoulli's law*. First, taking the inner product of (5.41) with v, we obtain

$$\frac{1}{2}\Big(\frac{\partial}{\partial t} - \mathcal{L}_v\Big)|v|^2 = -\mathcal{L}_v X'(\rho). \tag{5.42}$$

Now, consider the special case when the flow v is irrotational (i.e., $d\tilde{v} = 0$). The vorticity equation (5.24) implies that if this holds for any t, then it holds for all t. If Ω is simply connected, we can pick $x_0 \in \Omega$ and define a velocity potential $\varphi(t, x)$ by

$$\varphi(t, x) = \int_{x_0}^{x} \tilde{v}, \tag{5.43}$$

the integral being independent of path. Thus $d\varphi = \tilde{v}$, and (5.41) implies

$$d\Big(\frac{\partial \varphi}{\partial t} + \frac{1}{2}|v|^2 + X'(\rho)\Big) = 0 \tag{5.44}$$

on Ω, for an irrotational flow on a simply connected domain Ω. In other words, in such a case,

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}|v|^2 + X'(\rho) = H(t) \tag{5.45}$$

is a function of t alone. This is Bernoulli's law for irrotational flow.

Another special type of flow is steady flow, for which $v_t = 0$ and $\rho_t = 0$. In such a case, the equation (5.42) becomes

$$\mathcal{L}_v\Big(\frac{1}{2}|v|^2 + X'(\rho)\Big) = 0, \tag{5.46}$$

that is, the function $(1/2)|v|^2 + X'(\rho)$ is constant on the integral curves of v, called *streamlines*. For steady flow, the equation (5.13) becomes

$$\operatorname{div}(\rho v) = 0, \quad \text{i.e.,} \quad d(\rho * \tilde{v}) = 0. \tag{5.47}$$

If $\dim \Omega = 2$ and Ω is simply connected, this implies that there is a function ψ on Ω, called a *stream function*, such that

$$\rho * \tilde{v} = d\psi, \quad \text{i.e.,} \quad \tilde{v} = -\frac{1}{\rho} * d\psi. \tag{5.48}$$

In particular, v is orthogonal to $\nabla\psi$, so the stream function ψ is also constant on the integral curves of v, namely, the streamlines. One is temped to deduce from (5.46) that, for some function $H : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{2}|v|^2 + X'(\rho) = H(\psi) \tag{5.49}$$

in this case, and certainly this works out in some cases.

If a flow is both steady and irrotational, then from (5.44) we get

$$d\Big(\frac{1}{2}|v|^2 + X'(\rho)\Big) = 0, \tag{5.50}$$

which is stronger than (5.46).

We next discuss conservation of energy in compressible fluid flow. The total energy

$$E(t) = K(t) + V(t) = \int\limits_{\Omega} \Big\{\frac{1}{2}|v(t,x)|^2 + W\big(\rho(t,x)\big)\Big\}\rho(t,x)\,dx \tag{5.51}$$

is constant, for smooth solutions to (5.12)–(5.13). In fact, a calculation gives

$$E'(t) = \int\limits_{\Omega} \partial_t e(t,x)\,dx = -\int\limits_{\Omega} \operatorname{div}\Phi(t,x)\,dx = 0, \tag{5.52}$$

where

$$e(t,x) = \frac{1}{2}\rho|v|^2 + X(\rho) \tag{5.53}$$

is the total energy density and

$$\Phi(t,x) = \Big(\frac{1}{2}\rho|v|^2 + X'(\rho)\rho\Big)v = (e+p)v. \tag{5.54}$$

One passes from the first integral in (5.52) to the second via

$$\partial_t e(t,x) + \operatorname{div}\Phi(t,x) = 0, \tag{5.55}$$

which is a consequence of (5.12)–(5.13), for smooth solutions.

As we will see in § 8 in the special case $n = 1$, the equation (5.55) can break down in the presence of shocks. "Entropy satisfying" solutions with shocks then have the property that $E(t)$ is a nonincreasing function of t.

Now any equation of physics in which energy is not precisely conserved must be incomplete. Dissipated energy always goes somewhere. Energy dissipated by shocks acts to heat up the fluid. Say the heat energy density of the fluid is ρh. One way to extend (5.18) is to couple a PDE of the form

$$\partial_t(\rho h) + \operatorname{div}(\rho h v) = -\partial_t e - \operatorname{div}\Phi. \tag{5.56}$$

In such a case, solutions preserve the total energy

$$\int_\Omega (e + \rho h)\, dx. \tag{5.57}$$

For smooth solutions, the left side of (5.56) is equal to

$$\rho(h_t + \nabla_v h) + h\big(\rho_t + \operatorname{div}(\rho v)\big),$$

so in that case we are equivalently adjoining the equation

$$\rho(h_t + \nabla_v h) = -e_t - \operatorname{div}\Phi. \tag{5.58}$$

The right side of (5.58) vanishes for smooth solutions, recall, so we simply have $h_t + \nabla_v h = 0$, describing the transport of heat along the fluid trajectories. (We are neglecting the diffusion of heat here.)

If we consider the total energy intensity

$$\mathcal{E} = \frac{1}{2}|v|^2 + \rho^{-1}X(\rho) + h, \tag{5.59}$$

so $\rho\mathcal{E} = e + \rho h$, we obtain

$$\begin{aligned}\partial_t(\rho\mathcal{E}) + \operatorname{div}(\rho\mathcal{E}v) + \operatorname{div}(pv)& \\ = \partial_t e + \operatorname{div}\big((e+p)v\big) + \partial_t(\rho h) + \operatorname{div}(\rho h v)&,\end{aligned}$$

whose vanishing is equivalent to (5.56). Using this, we have the augmented system

$$\begin{aligned}(\rho v)_t + \operatorname{div}(\rho v \otimes v) + \nabla p &= 0,\\ \rho_t + \operatorname{div}(\rho v) &= 0,\\ (\rho\mathcal{E})_t + \operatorname{div}(\rho\mathcal{E}v) + \operatorname{div}(pv) &= 0.\end{aligned} \tag{5.60}$$

As in (5.20), this is supplemented by an equation of state, which in this context can take a more general form than $p = p(\rho)$, namely $p = p(\rho, \mathcal{E})$. Compare with (5.62) below.

We mention another extension of the system (5.18), based on ideas from thermodynamics. Namely, a new variable, denoted as S, for "entropy," is introduced, and one adjoins $(\rho S)_t + \operatorname{div}(\rho S v) = 0$, to (5.18), so the augmented system takes the form

$$\begin{aligned}(\rho v)_t + \operatorname{div}(\rho v \otimes v) + \nabla p &= 0,\\ \rho_t + \operatorname{div}(\rho v) &= 0,\\ (\rho S)_t + \operatorname{div}(\rho S v) &= 0.\end{aligned} \tag{5.61}$$

For smooth solutions, the left side of the last equation is equal to

$$\rho(S_t + \nabla_v S) + S\big(\rho_t + \operatorname{div}(\rho v)\big),$$

so in that case we are equivalently adjoining the equation $S_t + \nabla_v S = 0$.

Adjoining the last equation in (5.61) apparently does not affect the system (5.18) itself, but, as in the case of (5.60), it opens the door for a significant change, for it is now meaningful, and in fact physically realistic, to consider more general equations of state,

$$p = p(\rho, S). \tag{5.62}$$

In particular, one often generalizes (5.20) to

$$p = A(S)\rho^\gamma. \tag{5.63}$$

Brief discussions of the thermodynamic concepts underlying (5.61) can be found in [CF] and [LL]. In [CF] there is a discussion of how the system (5.60) leads to (5.61), while [LL] discusses how (5.61) leads to (5.60).

It must be mentioned that certain aspects of the behavior of gases, related to interpenetration, are not captured in the model of a fluid as described in this section. Another model, involving the "Boltzmann equation," is used. We say no more about this, but mention the books [CIP] and [RL] for treatments and further references.

Exercises

1. Write down the equations of radially symmetric compressible fluid flow, as a system in one "space" variable.

6. Weak solutions to scalar conservation laws; the viscosity method

For real-valued $u = u(t, x)$, we will obtain global weak solutions to PDE of the form

$$\frac{\partial u}{\partial t} = \sum \partial_j F_j(u), \quad u(0) = f, \tag{6.1}$$

for $t \geq 0,\ x \in \mathbb{T}^n = M$, as limits of solutions u_ν to

$$\frac{\partial u_\nu}{\partial t} = \nu \Delta u_\nu + \sum \partial_j F_j(u_\nu), \quad u_\nu(0) = f. \tag{6.2}$$

This method of producing solutions to (6.1) is called the *viscosity method*. Recall from Proposition 1.5 of Chap. 15 that, for each $\nu > 0,\ f \in L^\infty(M)$, (6.2) has a unique global solution

$$u_\nu \in L^\infty([0,\infty) \times M) \cap C^\infty((0,\infty) \times M),$$

with

$$\|u_\nu(t)\|_{L^\infty} \leq \|f\|_{L^\infty}, \tag{6.3}$$

for each $t \geq 0$, and furthermore if $u_{j\nu}$ solve (6.2) with $u_{j\nu} = f_j$, then, for each $t > 0$,

$$\|u_{1\nu}(t) - u_{2\nu}(t)\|_{L^1} \leq \|f_1 - f_2\|_{L^1}, \tag{6.4}$$

by Proposition 1.6 in that section. We will use these facts to show that as $\nu \searrow 0$, $\{u_\nu\}$ has a limit point u solving (6.1), provided $f \in L^\infty(M) \cap BV(M)$, where, with $\mathcal{M}(M)$ denoting the space of finite Borel measures on M,

$$BV(M) = \{u \in \mathcal{D}'(M) : \nabla u \in \mathcal{M}(M)\}. \tag{6.5}$$

As shown in Chap. 13, § 1, $BV(M) \subset L^{n/(n-1)}(M)$. Of course, that $BV \subset L^\infty$ for $n = 1$ is a standard result in introductory measure theory courses. Our analysis begins with the following:

Lemma 6.1. *If $f \in BV(M)$ and u_ν solves (6.2), then*

$$\{u_\nu : \nu \in (0,1]\} \text{ is bounded in } L^\infty(\mathbb{R}^+, BV). \tag{6.6}$$

Proof. If we define $\tau_y f(x) = f(x+y)$, it is clear that

$$f \in BV \Longrightarrow \|f - \tau_y f\|_{L^1} \leq C|y|, \tag{6.7}$$

for $|y| \leq 1/2$. Now apply (6.4) with $f_1 = f,\ f_2 = \tau_y f$ to obtain, for each $t \geq 0$,

$$\|u_\nu(t) - \tau_y u_\nu(t)\|_{L^1} \leq C|y|, \tag{6.8}$$

which yields (6.6).

Now if we write $\partial_j F_j(u_\nu) = F_j'(u_\nu)\partial_j u_\nu$, and note the boundedness in the sup norm of $F_j'(u_\nu)$, we deduce that

$$\{\partial_j F_j(u_\nu) : \nu \in (0,1]\} \text{ is bounded in } L^\infty(\mathbb{R}^+, \mathcal{M}(M)). \tag{6.9}$$

Let us use the inclusion

$$\mathcal{M}(M) \subset H^{-\delta,p}(M), \quad p'\delta > n, \tag{6.10}$$

a consequence of Sobolev's imbedding theorem, which implies

$$BV(M) \subset H^{1-\delta,p}(M). \tag{6.11}$$

We think of choosing δ small and p close to 1. We deduce from (6.6) and (6.9) that $\{\nu\Delta u + \sum \partial_j F_j(u_\nu)\}$ is bounded in $L^\infty(\mathbb{R}^+, H^{-1-\delta,p}(M))$, and hence, by (6.2),

$$\{\partial_t u_\nu\} \text{ is bounded in } L^\infty(\mathbb{R}^+, H^{-1-\delta,p}). \tag{6.12}$$

Thus, for $t, t' > 0$,

$$\|u_\nu(t) - u_\nu(t')\|_{H^{-1-\delta,p}(M)} \le C|t-t'|, \tag{6.13}$$

with C independent of $\nu \in (0,1]$. We now use the following interpolation inequality, a special case of results established in Chap. 13:

$$\|v\|_{H^{\varepsilon,p}} \le C\|v\|^{1-\sigma}_{H^{1-\delta,p}} \cdot \|v\|^{\sigma}_{H^{-1-\delta,p}},$$

where $\sigma \in (0,1)$ and $\varepsilon = (1-\sigma)(1-\delta) + \sigma(-1-\delta) = 1 - 2\sigma - \delta + \sigma\delta$ is > 0 if σ is chosen small and positive. We apply this to (6.13) and the following consequence of (6.6) and (6.11):

$$\|u_\nu(t) - u_\nu(t')\|_{H^{1-\delta,p}} \le C, \tag{6.14}$$

to conclude that, for some $\sigma > 0,\ \varepsilon > 0$,

$$\{u_\nu\} \text{ is bounded in } C^\sigma([0,T], H^{\varepsilon,p}(M)), \tag{6.15}$$

for all $T < \infty$; hence, by Ascoli's theorem,

$$\{u_\nu\} \text{ is compact in } C([0,T], L^p(M)), \tag{6.16}$$

for all $T < \infty$.

From here, producing a limit point u solving (6.1) is easy. Given $T < \infty$, by (6.16) we can pass to a subsequence $\nu_k \to 0$ such that

$$u_{\nu_k} = u_k \to u \quad \text{in } C([0,T], L^p(M)); \tag{6.17}$$

by a diagonal argument we can arrange this to hold for all $T < \infty$. We can also assume $u_k(t,x) \to u(t,x)$ pointwise a.e. on $\mathbb{R}^+ \times M$. In view of the pointwise boundedness (6.3), we deduce

$$F_j(u_k) \to F_j(u) \quad \text{in } L^p([0,T] \times M), \tag{6.18}$$

as $k \to \infty$, for each T. Hence we have weak convergence:

$$\frac{\partial u_k}{\partial t} \to \frac{\partial u}{\partial t}, \quad \nu_k \Delta u_k \to 0, \quad \partial_j F_j(u_k) \to \partial_j F_j(u), \tag{6.19}$$

implying that u solves (6.1). We summarize:

Proposition 6.2. *Given $f \in L^\infty(M) \cap BV(M)$, the solutions u_ν to (6.2) have a weak limit point*

$$u \in C([0,\infty), L^p(M)) \cap L^\infty(\mathbb{R}^+ \times M) \cap L^\infty(\mathbb{R}^+, BV(M)), \tag{6.20}$$

for all $p < \infty$, solving (6.1).

As we will see below, weak solutions to (6.1) in the class (6.20) need not be unique. However, there is uniqueness for those solutions obtained by the viscosity method. A device that provides a proof of this, together with an intrinsic characterization of these viscosity solutions, is furnished by "entropy inequalities," which we now discuss.

Let $\eta : \mathbb{R} \to \mathbb{R}$ be any C^2-convex function (so $\eta'' > 0$). Note that, for $v = v(t,x)$, $\partial_t \eta(v) = \eta'(v)\, \partial_t v$ and $\partial_j^2 \eta(v) = \eta'(v)\, \partial_j^2 v + \eta''(v)(\partial_j v)^2$, so

$$\Delta \eta(v) = \eta'(v) \Delta v + \eta''(v) |\nabla_x v|^2.$$

Thus, if u_ν solves (6.2), and if we multiply each side by $\eta'(u_\nu)$, we obtain

$$\frac{\partial}{\partial t} \eta(u_\nu) = \nu \Delta \eta(u_\nu) - \nu \eta''(u_\nu) |\nabla u_\nu|^2 + \sum \partial_j q_j(u_\nu), \tag{6.21}$$

where, using $\eta'(v)\, \partial_j F_j(v) = \eta'_j(v) F'_j(v)\, \partial_j v$ and $\partial_j q_j(v) = q'_j(v)\, \partial_j v$, we require of q_j that

$$q'_j(v) = \eta'(v) F'_j(v). \tag{6.22}$$

Now, for $u_{\nu_k} \to u$ as above, we have derived weak convergence $\eta(u_{\nu_k}) \to \eta(u)$ and, by the same reasoning, $q_j(u_{\nu_k}) \to q_j(u)$, but we have no basis to say that $|\nabla u_{\nu_k}|^2 \to |\nabla u|^2$, and in fact this convergence can fail (otherwise the inequality we derive would always be an equality). Taking this into account, we abstract from (6.21) the inequality

$$\frac{\partial}{\partial t}\,\eta(u_\nu) - \sum \partial_j q_j(u_\nu) \le \nu\Delta\eta(u_\nu), \tag{6.23}$$

using convexity of η, and then, passing to the limit $u_{\nu_k} \to u$, obtain

$$\frac{\partial}{\partial t}\eta(u) - \sum \partial_j q_j(u) \le 0, \tag{6.24}$$

in the sense that we have a nonpositive *measure* on $(0,\infty) \times M$ on the left side. In other words,

$$\varphi \in C_0^\infty\big((0,\infty)\times M\big), \quad \varphi \ge 0, \tag{6.25}$$

implies

$$\iint \Big\{\eta\big(u(t,x)\big)\varphi_t - \sum q_j\big(u(t,x)\big)\,\partial_j\varphi\Big\}\,dx\,dt \ge 0. \tag{6.26}$$

By a limiting argument, we can let $\eta(u)$ tend to $|u-k|$, for any given $k \in \mathbb{R}$, and use $q_j(u) = \operatorname{sgn}(u-k)[F_j(u) - F_j(k)]$, to deduce (using the summation convention)

$$\iint \Big\{|u-k|\varphi_t - \operatorname{sgn}(u-k)\big[F_j(u) - F_j(k)\big]\,\partial_j\varphi\Big\}\,dx\,dt \ge 0, \tag{6.27}$$

for all φ satisfying (6.25). That (6.27) holds for all $k \in \mathbb{R}$ is called *Kruzhkov's entropy condition*. The following is Kruzhkov's key result:

Proposition 6.3. *If u and v belong to the space in (6.20) and both satisfy Kruzhkov's entropy condition, and if $u(0,x) = f(x),\ v(0,x) = g(x)$, then, for $t > 0$,*

$$\|u(t) - v(t)\|_{L^1} \le \|f - g\|_{L^1}. \tag{6.28}$$

Proof. Let us write the entropy condition for v in the form (using the summation convention)

$$\iint \Big\{|v-\ell|\varphi_s - \operatorname{sgn}(v-\ell)\big[F_j(v) - F_j(\ell)\big]\frac{\partial\varphi}{\partial y_j}\Big\}\,dy\,ds \ge 0, \tag{6.29}$$

for all $\ell \in \mathbb{R}$. Let $\varphi = \varphi(s,t,x,y)$ be smooth and compactly supported in $s > 0,\ t > 0$, and $\varphi \ge 0$. Now substitute $v(s,y)$ for k in (6.27), $u(t,x)$ for ℓ in (6.29), integrate both over $dx\,dy\,ds\,dt$, and sum, to get

$$\begin{aligned}\iiiint \Big\{\big|u(t,x) - v(s,y)\big|(\varphi_t + \varphi_s) - \operatorname{sgn}\big(u(t,x) - v(s,y)\big) \\ \cdot\big[F_j(u) - F_j(v)\big]\Big(\frac{\partial\varphi}{\partial x_j} + \frac{\partial\varphi}{\partial y_j}\Big)\Big\}\,dx\,dy\,ds\,dt \ge 0.\end{aligned} \tag{6.30}$$

We now consider the following functions φ:

$$\varphi(s,t,x,y) = f(t)d_h(t-s)\delta_h(x-y), \tag{6.31}$$

where $f,\ d_h,\ \delta_h \geq 0$ and, as $h \to 0,\ d_h$ and δ_h approach the delta functions on $\mathbb{R}$ and $\mathbb{T}^n = M$, respectively. With such a choice, note that $\partial\varphi/\partial x_j + \partial\varphi/\partial y_j = 0$ and $\varphi_t + \varphi_s = f'(t)d_h(t-s)\delta_h(x-y)$. Passing to the limit $h \to 0$ yields

$$\iint \bigl|u(t,x) - v(t,x)\bigr|\ f'(t)\,dx\,dt \geq 0, \tag{6.32}$$

for all nonnegative $f \in C_0^\infty\bigl((0,\infty)\bigr)$, which in turn implies

$$\frac{d}{dt}\,\|u(t) - v(t)\|_{L^1} \leq 0, \tag{6.33}$$

yielding (6.28).

We have given all the arguments necessary to establish the following:

Corollary 6.4. *Given $f \in L^\infty(M) \cap BV(M)$, the weak solutions to (6.1) belonging to the space (6.20) which are limits of solutions u_ν to (6.2) are unique. Given two such f_j, initial data for viscosity solutions u_j, we have*

$$\|u_1(t) - u_2(t)\|_{L^1} \leq \|f_1 - f_2\|_{L^1}, \tag{6.34}$$

for $t \geq 0$. Furthermore, a weak solution to (6.1) is a viscosity solution if and only if the entropy inequality (6.27) holds for all $k \in \mathbb{R}$.

As a complementary remark, we note that if u, belonging to (6.20), satisfies Kruzhkov's entropy condition, then automatically u is a weak solution to (6.1). Indeed, let v be the viscosity solution with the same initial data as u; by (6.28), $v = u$.

Note that (6.27) can be rewritten as

$$\iint |u-k|\Bigl\{\varphi_t - \sum G_j(u,k)\frac{\partial\varphi}{\partial x_j}\Bigr\}\,dx\,dt \geq 0, \tag{6.35}$$

where $G_j(u,k) = [F_j(u) - F_j(k)]/(u-k)$ is smooth in its arguments. The formula (6.30) can be similarly rewritten; also, (6.32) can be generalized to

$$\iint \bigl|u(t,x) - v(t,x)\bigr|\Bigl\{\frac{\partial\varphi}{\partial t} - \sum G_j(u,v)\frac{\partial\varphi}{\partial x_j}\Bigr\}\,dx\,dt \geq 0, \tag{6.36}$$

for a pair u, v satisfying Kruzhkov's entropy condition. Suppose their initial data are bounded in sup norm by M, which therefore bounds $u(t)$ and $v(t)$ for all $t \geq 0$; pick A so that

$$|u|, |v| \le M \Longrightarrow \sum G_j(u,v)^2 \le A^2. \tag{6.37}$$

Now pick $\varphi(t,x) = f(t)\psi(t,x)$, with f as above and ψ satisfying

$$\psi_t + A|\nabla_x \psi| \le 0, \tag{6.38}$$

so that

$$\psi_t + \sum |G_j(u,v)| \cdot |\partial_j \psi| \le 0. \tag{6.39}$$

Then (6.36) implies

$$\iint \big|u(t,x) - v(t,x)\big| f'(t)\psi(t,x)\, dx\, dt \ge 0. \tag{6.40}$$

By a limiting argument, we can let ψ be the characteristic function of a set in $[0,\infty) \times \mathbb{T}^n$ of the form

$$\{(t,x) : |x - x_0| \le B - At\}. \tag{6.41}$$

Then, refining (6.33), we deduce that

$$\int\limits_{|x-x_0| \le B-At} \big|u(t,x) - v(t,x)\big|\, dx = D(t) \searrow \quad \text{as } t \nearrow. \tag{6.42}$$

In particular, if $u(0,x) = v(0,x)$ on $\{x : |x - x_0| \le B\}$, we deduce the following result on finite propagation speed:

Proposition 6.5. *If u and v are viscosity solutions to (6.1), bounded by M, with initial data f and g which agree on a set $\{x : |x - x_0| \le B\}$, and if A is large enough that (6.37) holds, then u and v coincide on the set (6.41).*

In light of this, we have in a natural fashion unique, global entropy-satisfying weak solutions to (6.1), for $t \ge 0$, $x \in \mathbb{R}^n$, provided the initial data belong to $L^\infty(\mathbb{R}^n)$ and have bounded variation.

We next consider weak solutions to (6.1) with discontinuities of the simplest sort; namely, we suppose that $u(t,x)$ is defined for $t \ge 0$, $x \in \mathbb{R}$, and that there is a smooth curve γ, given by $x = x(t)$, such that $u(t,x)$ is smooth on either side of γ, with a simple jump across γ. If $(x,t) \in \gamma$, denote by $[u] = [u](x,t)$ the size of this jump:

$$[u] = \lim_{\varepsilon \searrow 0} u(x(t) + \varepsilon, t) - u(x(t) - \varepsilon, t). \tag{6.43}$$

If $F : \mathbb{R} \to \mathbb{R}$ is smooth, we let $[F]$ denote the jump in $F(u)$ across γ. Now, if such u solves

$$u_t + F(u)_x = 0 \tag{6.44}$$

on $(\mathbb{R}^+ \times \mathbb{R}) \setminus \gamma$, then this object on $\mathbb{R}^+ \times \mathbb{R}$ will be a measure supported on γ; u will be a weak solution everywhere provided this measure vanishes. It is a simple exercise to evaluate this measure in terms of the jumps $[u]$ and $[F]$ and the slope of γ, or equivalently the speed $s = dx/dt$, as being proportional to $s[u] - [F]$. In other words, such a u provides a weak solution to (6.44) precisely when

$$s[u] = [F] \quad \text{on } \gamma. \tag{6.45}$$

This condition is called the *jump condition*, or the Rankine–Hugoniot condition.

A special case of solutions to (6.44) off γ are functions u that are piecewise constant. Thus the jumps are constant, so s is constant, so γ is a line; we may as well call it the line $x = st$ (possibly shifting the origin on the x-axis). See Fig. 6.1. If $u = u_\ell$ on the left side of γ and $u = u_r$ on the right side of γ, the Rankine–Hugoniot condition becomes

$$s = \frac{F(u_r) - F(u_\ell)}{u_r - u_\ell}. \tag{6.46}$$

An initial-value problem with such piecewise-constant initial data is called a *Riemann problem*. Let us describe two explicit weak solutions to

$$u_t + \frac{1}{2}(u^2)_x = 0 \tag{6.47}$$

of this form, in Fig. 6.2.

Claim 6.6. *Figure 6.2A describes an entropy-satisfying solution of (6.47), while Fig. 6.2B describes an entropy-violating solution.*

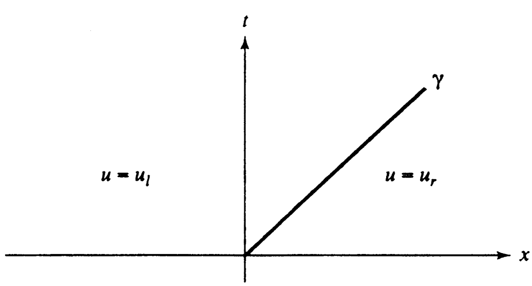

FIGURE 6.1 Shock Wave

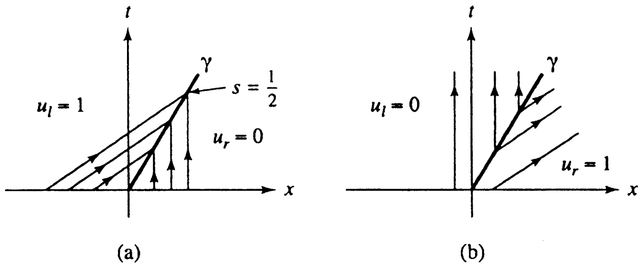

FIGURE 6.2 Entropy Satisfaction and Violation

In each figure we have drawn in integral curves of the vector fields $\partial/\partial t + F'(u)(\partial/\partial x)$ in the regions where u is smooth. Note that in Fig. 6.2A these curves run into γ, while in Fig. 6.2B these curves diverge from γ.

These assertions are consequences of the following result of Oleinik:

Proposition 6.7. *Let $u(t,x)$ be a piecewise smooth solution to (6.44) on $\mathbb{R}^+ \times \mathbb{R}$ with jump across γ, satisfying the jump condition (6.45). Then the entropy condition holds if and only if*
(i) in case $u_r < u_\ell$:
The graph of $y = F(u)$ over $[u_r, u_\ell]$ lies below the chord connecting the point $\big(u_r, F(u_r)\big)$ to $\big(u_\ell, F(u_\ell)\big)$;
(ii) in case $u_r > u_\ell$:
The graph of $y = F(u)$ over $[u_\ell, u_r]$ lies above the chord.

These two cases are illustrated in Fig. 6.3. A weak solution to (6.44) which satisfies the hypotheses of Proposition 6.7 is said to satisfy Oleinik's "condition (E)."

Proof. As a slight variation on Kruzhkov's convex functions, it suffices to consider the weakly convex functions

$$\begin{aligned} \eta(u) = 0, &\qquad \text{for } u < k, \\ u - k, &\qquad \text{for } u > k, \end{aligned}$$

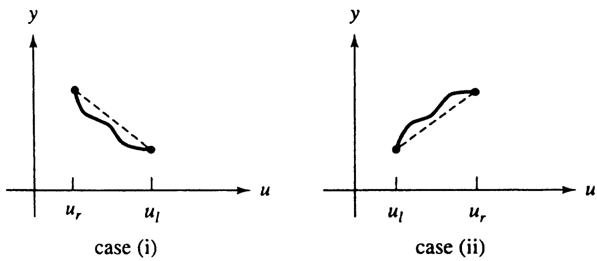

FIGURE 6.3 Oleinik's Condition (E)

plugged into the inequality $\eta_t + q_x \le 0$, with

$$q(u) = 0, \qquad \text{for } u < k,$$
$$F(u) - F(k), \qquad \text{for } u > k,$$

as k runs over $\mathbb{R}$. In fact only for k between u_r and u_ℓ is $\eta_t + q_x$ nonzero; in such a case it is a measure supported on γ, which is ≤ 0 if and only if

$$s\big[\eta(u_\ell) - \eta(u_r)\big] \le F(u_\ell) - F(u_r).$$

The jump condition (6.46) on s then implies

$$F(k) \ge \frac{u_r - k}{u_r - u_\ell} F(u_\ell) + \frac{k - u_\ell}{u_r - u_\ell} F(u_r), \tag{6.48}$$

for k between u_r and u_ℓ, which is equivalent to the content of (i)–(ii).

Note that if F is convex (i.e., $F'' > 0$), as in the example (6.47), then the content of (i) and (ii) is

$$F'(u_\ell) > s > F'(u_r) \quad (\text{for } F'' > 0), \tag{6.49}$$

a result that, for $F(u) = u^2/2$, holds in the situation of Fig. 6.2A but not in that of Fig. 6.2B.

For weak solutions to (6.1) with these simple discontinuities, if the entropy conditions are satisfied, the discontinuities are called *shock waves*. Thus the discontinuity depicted in Fig. 6.2A is a shock, but the one in Fig. 6.2B is not.

The Riemann problem for (6.47) with initial data $u_\ell = 0$, $u_r = 1$, has an entropy-satisfying solution, different from that of Fig. 6.2B, which can be obtained as a special case of the following construction. Namely, we look for a piecewise smooth solution of (6.44), with initial data $u(0,x) = u_\ell$ for $x < 0$, u_r for $x > 0$, and which is Lipschitz continuous for $t > 0$, in the form

$$u(t,x) = v(t^{-1}x). \tag{6.50}$$

The PDE (6.44) yields for v the ODE

$$v'(s)\big[F'(v(s)) - s\big] = 0. \tag{6.51}$$

We look for $v(s)$ Lipschitz on $\mathbb{R}$, satisfying alternatively $v'(s) = 0$ and $F'(v(s)) = s$ on subintervals of $\mathbb{R}$, such that $v(-\infty) = u_\ell$ and $v(+\infty) = u_r$. Let us suppose that $F(u)$ is convex ($F'' > 0$) for u between u_ℓ and u_r and that the shock condition (6.49) is violated (i.e., we suppose $u_\ell < u_r$). Since $F'(u)$ is monotone increasing on $u_\ell \le u \le u_r$, we can define an inverse map $= (F')^{-1}$,

$$G : [F'(u_\ell), F'(u_r)] \to [u_\ell, u_r].$$

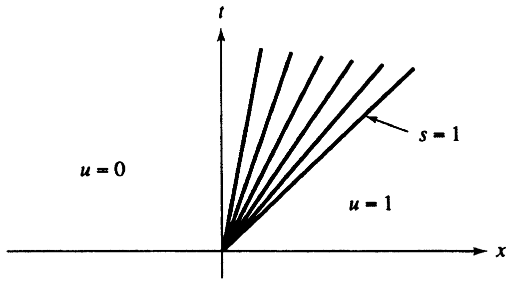

FIGURE 6.4 Rarefaction Wave

Then setting

$$v(s) = \begin{cases} u_\ell, & \text{for } s < F'(u_\ell), \\ G(s), & \text{for } F'(u_\ell) \le s \le F'(u_r), \\ u_r, & \text{for } s > F'(u_r) \end{cases} \tag{6.52}$$

completes the construction. For the PDE (6.47), with $F'(u) = u$, the solution so produced is illustrated in Fig. 6.4. There is a fan of lines through $(0, 0)$ drawn in this figure, with speeds s running from 0 to 1, and $u = s$ on the line with speed $dx/dt = s$.

Solutions to (6.44) constructed in this fashion are called *rarefaction waves*. If F is concave between u_ℓ and u_r, an analogous construction works, provided $u_\ell > u_r$.

Rarefaction waves always satisfy the entropy conditions, since if u is a weak solution to (6.44), $\eta(u)_t + q(u)_x = 0$ on any open set on which u is Lipschitz.

In case $F(u)$ is either convex or concave over all of $\mathbb{R}$, any Riemann problem for (6.44) has an entropy-satisfying solution, which is either a shock wave or a rarefaction wave. In these two respective cases we say u_ℓ is connected to u_r by a shock wave or by a rarefaction wave. If $F''(u)$ changes sign, there are other possibilities. We illustrate one here; let $u_r < u_\ell$, and say $F(u)$ is as depicted in Fig. 6.5 (with an inflection point at v_1).

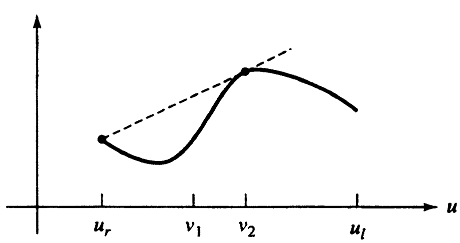

FIGURE 6.5 More Complex Nonlinearity

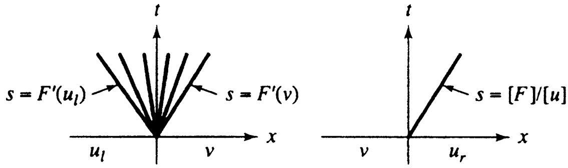

FIGURE 6.6 Rarefaction and Shock

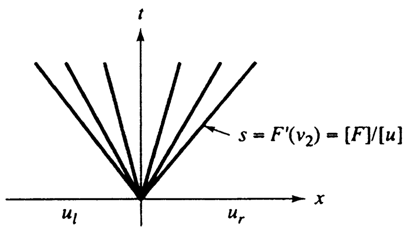

FIGURE 6.7 Rarefaction Bounded by a Shock

By the analysis above, we see that if $v_1 \leq v \leq v_2$, we can connect u_ℓ to v by a rarefaction wave, and we can connect v and u_r by a shock, as illustrated in Fig. 6.6.

These can be fitted together provided $[F(v) - F(u_r)]/(v - u_r) \geq F'(v)$. This requires $v = v_2$, so the solution is realized by a rarefaction wave bordered by a shock, as illustrated in Fig. 6.7.

We now illustrate the entropy solution to $u_t + (1/2)(u^2)_x = 0$ with initial data equal to the characteristic function of an interval, namely,

$$u(0,x) = 1, \quad \text{for } 0 \leq x \leq 1, \qquad 0 \quad \text{otherwise.} \tag{6.53}$$

For $0 \leq t \leq 2$, this solution is a straightforward amalgamation of the rarefaction wave of Fig. 6.4 and the shock wave of Fig. 6.2A. For $t > 2$, there is an interaction of the rarefaction wave and the shock wave. Let $\big(x(\sigma), t(\sigma)\big)$ denote a point on the shock front (for $t \geq 2$) where $u = \sigma$. From $[u] = \sigma,\ [F] = \sigma^2/2$, and $s = [F]/[u] = \sigma/2$, we deduce

$$\frac{x'(\sigma)}{t'(\sigma)} = \frac{\sigma}{2} = \frac{x}{2t}.$$

Hence $x'/x = t'/2t$, so $\log x = (1/2)\log t + C$, or $x = kt^{1/2}$. Since $x = 2$ at $t = 2$ on the shock front, this gives $k = \sqrt{2}$. Thus the shock front is given by

$$x = \sqrt{2t}, \quad \text{for } t \geq 2. \tag{6.54}$$

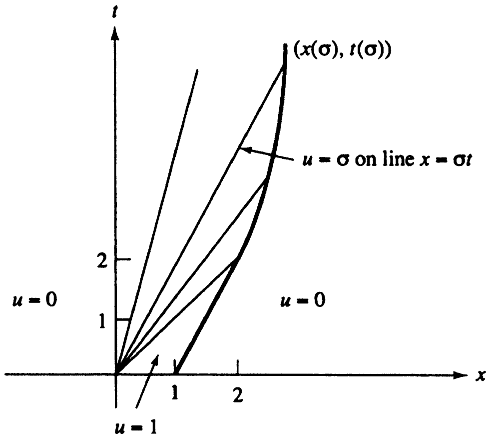

FIGURE 6.8 Curved Shock Front

This is illustrated in Fig. 6.8. Note that the interaction of these waves leads to decay:

$$\sup_x u(t,x) = \sqrt{\frac{2}{t}}, \quad \text{for } t \geq 2. \tag{6.55}$$

Exercises

Exercises 1–3 examine a difference scheme approximation to (6.1), used by [CwS] and [Kot]. Let $h = \Delta t,\ \varepsilon = \Delta x_j$, and let Λ be the n-dimensional lattice

$$\Lambda = \{x \in \mathbb{R}^n : x = \varepsilon\alpha,\ \alpha \in \mathbb{Z}^n\}.$$

We want to approximate a solution $u(t,x)$ to (6.1) at $t = hk,\ x = \varepsilon\alpha$, by $u(k,\alpha)$, defined on $\mathbb{Z}^+ \times \Lambda$, satisfying the difference scheme

$$\begin{aligned} &\frac{1}{h}\Big[u(k+1,\alpha) - \frac{1}{2n}\sum_{j=1}^{n}\Big\{u\big(k,\alpha+\delta(j)\big) + u\big(k,\alpha-\delta(j)\big)\Big\}\Big] \\ &\quad + \frac{1}{2\varepsilon}\sum_{j=1}^{n}\Big\{F_j\big(u(k,\alpha+\delta(j))\big) - F_j\big(u(k,\alpha-\delta(j))\big)\Big\} = 0, \end{aligned} \tag{6.56}$$

for $k \geq 0$, with initial condition

$$u(0,\alpha) = f(\alpha). \tag{6.57}$$

Here, $\delta(j) = (0,\dots,1,\dots,0)$, with the 1 in the jth position. We impose the "stability condition"

$$0 < h \leq \frac{\varepsilon}{An}, \quad A = \max_j \sup_{|w|\leq M} |F_j'(w)|. \tag{6.58}$$

1. Show that

$$\sup_\alpha |f(\alpha)| \le M \Longrightarrow |u(k,\alpha)| \le M. \tag{6.59}$$

(*Hint*: Write $F_j\big(u(k,\alpha+\delta(j))\big) - F_j\big(u(k,\alpha-\delta(j))\big) = \Phi_{k\alpha j}\big[u(k,\alpha+\delta(j)) - u(k,\alpha-\delta(j))\big]$. Then rewrite (6.56) as

$$\begin{aligned} u(k+1,\alpha) &= \frac{1}{2n}\sum_{j=1}^{n}\Big\{u\big(k,\alpha+\delta(j)\big) + u\big(k,\alpha-\delta(j)\big)\Big\} \\ &\quad - \frac{h}{2\varepsilon}\sum_{j=1}^{n}\Phi_{k\alpha j}\Big\{u\big(k,\alpha+\delta(j)\big) - u\big(k,\alpha-\delta(j)\big)\Big\}. \end{aligned} \tag{6.60}$$

Hence

$$u(k+1,\alpha) = \sum_{\beta\in\Lambda}\kappa_{k\alpha\beta}u(k,\beta),$$

where $\sum_\beta \kappa_{k\alpha\beta} = 1$, and, given (6.58), $\kappa_{k\alpha\beta} \ge 0$. Deduce that $|u(k+1,\alpha)| \le \sup_\beta |u(k,\beta)|$.)

2. If $v(k,\alpha)$ solves (6.56) with $v(0,\alpha) = g(\alpha)$, show that

$$\sum_{\alpha\in\Lambda}\big|u(k,\alpha) - v(k,\alpha)\big| \le \sum_{\alpha\in\Lambda}\big|f(\alpha) - g(\alpha)\big|. \tag{6.61}$$

Compare with (6.4). Deduce that

$$\sum_{j=1}^{n}\sum_{\alpha\in\Lambda}\big|u(k,\alpha) - u\big(k,\alpha+\delta(j)\big)\big| \le \sum_{j=1}^{n}\sum_{\alpha\in\Lambda}\big|f(\alpha) - f\big(\alpha+\delta(j)\big)\big|. \tag{6.62}$$

(*Hint*: With $w(k,\alpha) = u(k,\alpha) - v(k,\alpha)$, deduce from (6.56) that

$$\begin{aligned} w(k+1,\alpha) &= \frac{1}{2n}\sum_{j=1}^{n}\Big\{w\big(k,\alpha+\delta(j)\big) + w\big(k,\alpha-\delta(j)\big)\Big\} \\ &\quad -\frac{h}{2\varepsilon}\sum_{j=1}^{n}\Big\{\Psi_{k,\alpha+\delta(j)}w\big(k,\alpha+\delta(j)\big) - \Psi_{k,\alpha-\delta(j)}w\big(k,\alpha-\delta(j)\big)\Big\}, \end{aligned} \tag{6.63}$$

where $F_j\big(u(k,\alpha)\big) - F_j\big(v(k,\alpha)\big) = \Psi_{k\alpha}w\big(k,\alpha\big)$. Multiply (6.63) by $\sigma_{k\alpha} = \operatorname{sgn} w(k+1,\alpha)$ and sum over α, to get

$$\sum_\alpha |w(k+1,\alpha)| = \sum_\alpha \gamma_{k\alpha}w(k,\alpha),$$

where

$$\gamma_{k\alpha} = \frac{1}{2n}\sum_j\Big\{\Big(1-\frac{nh}{\varepsilon}\Psi_{k\alpha}\Big)\sigma_{k,\alpha-\delta(j)} + \Big(1+\frac{nh}{\varepsilon}\Psi_{k\alpha}\Big)\sigma_{k,\alpha+\delta(j)}\Big\}.$$

Using $1 \pm (nh/\varepsilon)\Psi_{k\alpha} \ge 0$, deduce that $-1 \le \gamma_{k\alpha} \le 1$.)

3. Show that

$$
(6.64)\quad
\begin{aligned}
\sum_{\alpha} h^{-1}\big|u(k+1,\alpha)-u(k,\alpha)\big| &\le A\sum_{j=1}^{n}\sum_{\alpha\in\Lambda}\varepsilon^{-1}\big|f\big(\alpha+\delta(j)\big)-f(\alpha)\big| \\
&+\frac{1}{2n}\frac{\varepsilon^2}{h}\sum_{j=1}^{n}\sum_{\alpha\in\Lambda}\varepsilon^{-2}\big|f\big(\alpha+\delta(j)\big)-2f(\alpha)+f\big(\alpha-\delta(j)\big)\big|.
\end{aligned}
$$

(*Hint*: Set $v(k,\alpha) = u(k+1,\alpha)$, and apply (6.61). Then use (6.56) to analyze $u(1,\alpha)-f(\alpha)$.)

Let us use the notation

$$
(6.65)\quad
\begin{aligned}
\Delta_t u(k,\alpha) &= \frac{1}{h}\Bigg[u(k+1,\alpha)-\frac{1}{2n}\sum_{j=1}^{n}\Big\{u\big(k,\alpha+\delta(j)\big)+u\big(k,\alpha-\delta(j)\big)\Big\}\Bigg], \\
\Delta_j v(k,\alpha) &= \frac{1}{2\varepsilon}\Big[v\big(k,\alpha+\delta(j)\big)-v\big(k,\alpha-\delta(j)\big)\Big],
\end{aligned}
$$

so (6.56) takes the form

$$
(6.66)\qquad \Delta_t u+\sum_{j=1}^{n}\Delta_j F_j(u)=0.
$$

The following is a special case of a result in [L4].

4. Let η and q_j be as in (6.22). Assume

$$0<m\le\eta''(u)\le M<\infty,$$

and strengthen (6.58) to

$$h\le\frac{\varepsilon}{An}\left(\sqrt{1+\frac{m}{M}}-1\right).$$

Show that a solution u to (6.66) also satisfies

$$
(6.67)\qquad \Delta_t\eta(u)+\sum_j\Delta_j q_j(u)\le 0.
$$

Compare with (6.24).

5. Let $u_\sigma(t,x)$ be the entropy solution to $u_t+(1/2)(u^2)_x=0$ with initial data

$$
\begin{aligned}
u_\sigma(0,x) = \sigma^{-1}, &\qquad \text{for } 0\le x\le\sigma, \\
0 &\qquad \text{otherwise.}
\end{aligned}
$$

Compare u_σ to the solution to (6.53), illustrated in Fig. 6.8.
Note that, given $0<\sigma<1$, we have $u_\sigma(t,x)=u_1(t,x)$ for large t, so there is *no backward uniqueness*.
Show that as $\sigma\to 0$, $u_\sigma\to u_0$, depicted in Fig. 6.9. Show that u_0 is an entropy solution of

$$u_t+\frac{1}{2}(u^2)_x=0,\quad u(0,x)=\delta(x).$$

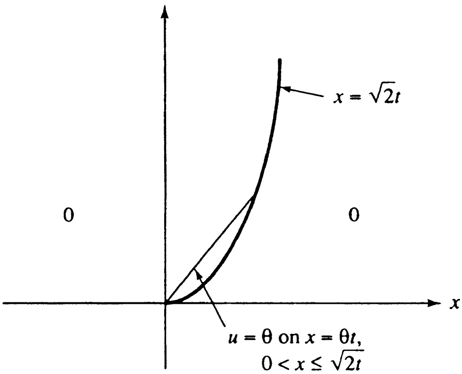

FIGURE 6.9 Solution with Initial Data $\delta(x)$

7. Systems of conservation laws in one space variable; Riemann problems

Here we consider $L \times L$ first-order systems of the form

$$u_t + F(u)_x = 0, \tag{7.1}$$

where x belongs to either $\mathbb{R}$ or $S^1 = \mathbb{R}/\mathbb{Z}$. Here, u takes values in $\mathbb{R}^L$, or perhaps in some region $\Omega \subset \mathbb{R}^L$, and $F : \Omega \to \mathbb{R}^L$ is smooth. Assume Ω is simply connected. If u is a smooth solution of (7.1), then

$$u_t + A(u)u_x = 0, \quad A(u) = D_u F(u). \tag{7.2}$$

Thus $A(u)$ is an $L \times L$ matrix. We typically make the hypothesis of strict hyperbolicity, that $A(u)$ has L real and distinct eigenvalues:

$$A(u)r_j(u) = \lambda_j(u)r_j(u), \quad \lambda_1(u) < \cdots < \lambda_L(u). \tag{7.3}$$

The vectors $r_j(u) \in \mathbb{R}^L$ are eigenvectors of $A(u)$.

The equation (7.1) is said to be a system of *conservation laws* because, if $u(t, x)$ either vanishes sufficiently rapidly as $x \to \pm\infty$ or is defined for $x \in S^1$, then

$$\int u(t, x)\, dx = C \tag{7.4}$$

is independent of t; thus the components of this vector are conserved quantities. To see this, using (7.1), we have

$$\frac{d}{dt}\int u(t, x)\, dx = -\int F(u)_x\, dx = 0,$$

by the fundamental theorem of calculus. As we will see in §8, (7.1) will sometimes give rise to other "conservation laws" for u.

We give a couple of examples of systems of conservation laws. First consider equations of isentropic compressible fluid flow. When $x \in \mathbb{R}^n$ and $n = 1$, then the system (2.11) for compressible fluid flow specializes to

$$\begin{aligned} v_t + v v_x &= -\frac{p_x}{\rho}, \\ \rho_t + v\rho_x + \rho v_x &= 0. \end{aligned} \tag{7.5}$$

We assume $p = p(\rho)$ is a given function of ρ, the most common relation being

$$p(\rho) = A\rho^\gamma, \quad A > 0, \ \ 1 < \gamma < 2, \tag{7.6}$$

as in (2.12). We can rewrite (7.5) in conservation form:

$$\begin{aligned} v_t + \bigl(\frac{1}{2}v^2 + q(\rho)\bigr)_x &= 0, \\ \rho_t + (\rho v)_x &= 0, \end{aligned} \tag{7.7}$$

where $q'(\rho) = p'(\rho)/\rho$. If $p(\rho)$ is given by (7.6), we can take

$$q(\rho) = \frac{\gamma A}{\gamma - 1}\rho^{\gamma-1}.$$

Alternatively, we can set $m = \rho v$, the momentum density, and rewrite (7.5) as

$$\begin{aligned} \rho_t + m_x &= 0, \\ m_t + \Bigl(\frac{m^2}{\rho} + p\Bigr)_x &= 0. \end{aligned} \tag{7.8}$$

In this case, we have $u = (\rho, m)$ and

$$A(u) = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + p'(\rho) & \frac{2m}{\rho} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -v^2 + p'(\rho) & 2v \end{pmatrix}, \tag{7.9}$$

which has eigenvalues and eigenvectors:

$$\lambda_\pm = v \pm \sqrt{p'(\rho)}, \qquad r_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}. \tag{7.10}$$

As a second example, consider this second-order equation, for real-valued V:

$$V_{tt} - K(V_x)_x = 0, \tag{7.11}$$

which is a special case of (3.12). As discussed in § 1 of Chap. 2, this equation arises as the stationary condition for an action integral

$$J(V) = \iint \Big[\frac{1}{2}V_t^2 - F(V_x)\Big]\, dx\, dt. \tag{7.12}$$

Here, $F(V_x)$ is the potential energy density. Thus $K(v)$ has the form

$$K(v) = F'(v).$$

If we set

$$v = V_x, \quad w = V_t, \tag{7.13}$$

we get the 2×2 system

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - K(v)_x &= 0. \end{aligned} \tag{7.14}$$

In this case, $u = (v, w)$ and

$$A(u) = \begin{pmatrix} 0 & -1 \\ -K_v & 0 \end{pmatrix}, \quad K_v = F''(v). \tag{7.15}$$

We assume $F''(v) > 0$. Then (7.14) is strictly hyperbolic; $A(u)$ has eigenvalues and eigenvectors

$$\lambda_\pm = \pm\sqrt{K_v}, \quad r_\pm = \begin{pmatrix} 1 \\ -\lambda_\pm \end{pmatrix}. \tag{7.16}$$

As in the scalar case examined in § 6, we expect classical solutions to (7.1) to break down, and we hope to extend these to weak solutions, with shocks, and so forth. Our next goal is to study the Riemann problem for (7.1),

$$\begin{aligned} u(0,x) = u_\ell, &\quad \text{for } x < 0, \\ u_r, &\quad \text{for } x > 0, \end{aligned} \tag{7.17}$$

given $u_\ell,\ u_r \in \mathbb{R}^L$, and try to obtain a solution in terms of shocks and rarefaction waves, extending the material of (6.43)–(6.52).

We first consider *rarefaction waves*, solutions to (7.1) of the form

$$u(t,x) = \varphi(t^{-1}x), \tag{7.18}$$

for $\varphi(s)$ which is Lipschitz and piecewise smooth. Now $u_t = -(x/t^2)\varphi'(x/t)$ and $u_x = (1/t)\varphi(x/t)$, so (7.2) implies

$$\big(A(\varphi(s)) - s\big)\varphi'(s) = 0. \tag{7.19}$$

Thus, on any open interval where $\varphi'(s) \neq 0$, we need, for some $j \in \{1, \dots, L\}$,

$$\lambda_j\big(\varphi(s)\big) = s, \qquad \varphi'(s) = \alpha_j(s) r_j\big(\varphi(s)\big), \tag{7.20}$$

where $r_j(u)$ is the λ_j-eigenvector of $A(u)$ and $\alpha_j(s)$ is real-valued. Differentiating the first of these identities and using the second, we have

$$\alpha_j(s)\, r_j\big(\varphi(s)\big) \cdot \nabla\lambda_j\big(\varphi(s)\big) = 1. \tag{7.21}$$

We say that (7.1) is genuinely nonlinear in the jth field if $r_j(u) \cdot \nabla\lambda_j(u)$ is nowhere zero (on the domain of definition, $\Omega \subset \mathbb{R}^L$). Granted the condition of genuine nonlinearity, one typically rescales the eigenvector $r_j(u)$, so that

$$r_j(u) \cdot \nabla\lambda_j(u) = 1. \tag{7.22}$$

Then (7.20) holds with $\alpha_j(s) = 1$.

Consequently, if (7.1) is genuinely nonlinear in the jth field and $u_\ell \in \mathbb{R}^L$ is given, then there is a smooth curve in $\mathbb{R}^L$, with one endpoint at u_ℓ, called the *j-rarefaction curve*:

$$\varphi_j^r(u_\ell; \tau), \quad 0 \le \tau \le \sigma_j, \tag{7.23}$$

for some $\sigma_j > 0$, so that

$$\varphi_j^r(u_\ell; 0) = u_\ell, \tag{7.24}$$

and, for any $\sigma \in (0, \sigma_j)$, the function u defined by

$$u(t,x) = \begin{cases} u_\ell, & \text{for } \dfrac{x}{t} < \lambda_j(u_\ell), \\ \varphi_j^r(u_\ell; \tau), & \text{for } \dfrac{x}{t} = \tau \in \big[\lambda_j(u_\ell), \lambda_j\big(\varphi_j^r(u_\ell; \sigma)\big)\big], \\ \varphi_j^r(u_\ell; \sigma) = u_r, & \text{for } \dfrac{x}{t} > \lambda_j(u_r) \end{cases} \tag{7.25}$$

is a j-rarefaction wave. See Fig. 7.1. Note that given (7.22), we have

$$\frac{d}{d\tau}\varphi_j^r(u_\ell; 0) = r_j(u_\ell). \tag{7.26}$$

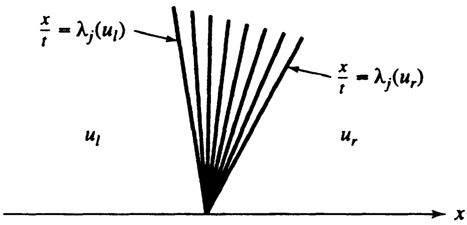

FIGURE 7.1 J-Rarefaction Wave

Next we consider weak solutions to (7.1) of the form

$$u(t,x) = \begin{cases} u_\ell, & \text{for } x < st, \\ u_r, & \text{for } x > st, \end{cases} \tag{7.27}$$

for $t > 0$, given $s \in \mathbb{R}$, the "shock speed." As in (6.45), the condition that this define a weak solution to (7.1) is the *Rankine–Hugoniot condition*:

$$s[u] = [F], \tag{7.28}$$

where $[u]$ and $[F]$ are the jumps in these quantities across the line $x = st$; in other words,

$$F(u_r) - F(u_\ell) = s(u_r - u_\ell). \tag{7.29}$$

If course, if $L > 1$, unlike in (6.46), we cannot simply divide by $u_r - u_\ell$; the identity (7.29) now implies the nontrivial relation that the vector $F(u_r) - F(u_\ell) \in \mathbb{R}^L$ be parallel to $u_r - u_\ell$. We will produce curves $\varphi_j^s(u_\ell;\tau)$, smooth on $\tau \in (-\tau_j, 0]$, for some $\tau_j > 0$, so that

$$\varphi_j^s(u_\ell; 0) = u_\ell, \tag{7.30}$$

and, for any $\tau \in (-\tau_j, 0]$, the function u defined by (7.27) is a weak solution to (7.1), with

$$u_r = \varphi_j^s(u_\ell;\tau), \quad s = s_j(\tau), \tag{7.31}$$

where $s_j(\tau)$ is also smooth on $(-\tau_j, 0]$. For notational convenience, set $\varphi(\tau) = \varphi_j^s(u_\ell;\tau)$. Thus we want

$$F\big(\varphi(\tau)\big) - F(u_\ell) = s_j(\tau)\big(\varphi(\tau) - u_\ell\big). \tag{7.32}$$

If this holds, then taking the τ-derivative yields

$$\Big(A\big(\varphi(\tau)\big) - s_j(\tau)\Big)\varphi'(\tau) = s_j'(\tau)\big(\varphi(\tau) - u_\ell\big). \tag{7.33}$$

If this holds, setting $\tau = 0$ gives

$$\big(A(u_\ell) - s_j(0)\big)\varphi'(0) = 0, \tag{7.34}$$

so

$$s_j(0) = \lambda_j(u_\ell), \tag{7.35}$$

and $\varphi'(0)$ is proportional to $r_j(u_\ell)$. Reparameterizing in τ, if $\varphi'(0) \neq 0$, we can assume

$$\varphi'(0) = r_j(u_\ell). \tag{7.36}$$

We now show that such a smooth curve $\varphi(\tau)$ exists. Guided by (7.36), we set

$$\varphi(\tau) = u_\ell + \tau r_j(u_\ell) + \tau\zeta(\tau) \tag{7.37}$$

and show that, for τ close to 0, there exists $\zeta(\tau) \in \mathbb{R}^L$ near 0, such that (7.33) holds. We will require that $\zeta(\tau) \in V_j$, the linear span of the eigenvectors of $A(u_\ell)$ other than $r_j(u_\ell)$. Then we want to solve for $\zeta \in V_j,\ \eta \in \mathbb{R}$:

$$\tau^{-1}\big[F(u_\ell + \tau r_j(u_\ell) + \tau\zeta) - F(u_\ell)\big] - \big(\lambda_j(u_\ell) + \eta\big)\big[r_j(u_\ell) + \zeta\big] = 0. \tag{7.38}$$

Denote the left side by Φ_τ, so

$$\Phi_\tau : \mathcal{O} \times \mathbb{R} \longrightarrow \mathbb{R}^L, \tag{7.39}$$

where $\mathcal{O}$ is a neighborhood of 0 in V_j. This extends smoothly to $\tau = 0$, with

$$\begin{aligned} \Phi_0(\zeta, \eta) &= A(u_\ell)\big[r_j(u_\ell) + \zeta\big] - \big(\lambda_j(u_\ell) + \eta\big)\big[r_j(u_\ell) + \zeta\big] \\ &= \Big(A(u_\ell) - \lambda_j(u_\ell) - \eta\Big)\zeta - \eta r_j(u_\ell). \end{aligned} \tag{7.40}$$

Note that $\Phi_0(0, 0) = 0$. Also,

$$D\Phi_0(0, 0)\begin{pmatrix}\zeta \\ \eta\end{pmatrix} = \big(A(u_\ell) - \lambda_j(u_\ell)\big)\zeta - \eta r_j(u_\ell), \tag{7.41}$$

which is an invertible linear map of $V_j \oplus \mathbb{R} \to \mathbb{R}^L$. The inverse function theorem implies that, at least for τ close to 0, $\Phi_\tau\big(\zeta(\tau), \eta(\tau)\big) = 0$ for a uniquely defined smooth $\big(\zeta(\tau), \eta(\tau)\big)$ satisfying $\zeta(0) = 0,\ \eta(0) = 0$.

We see that the curve $\varphi(\tau)$ is defined on a two-sided neighborhood of $\tau = 0$, but, taking a cue from §6, we will restrict this to $\tau \le 0$ to define the j-shock curve $\varphi_j^s(u_\ell; \tau)$. Comparing (7.36) with (7.26), we see that $\varphi_j^s(u_\ell; \tau)$ and the j-rarefaction curve $\varphi_j^r(u_\ell; \tau)$ fit together to form a C^1-curve, for $\tau \in (-\tau_j, \sigma_j)$; we denote this curve by $\varphi_j(u_\ell; \tau)$.

In fact, assuming genuine nonlinearity, we can arrange that $\varphi_j(u_\ell;\tau)$ be a C^2 curve, after perhaps a further reparameterization of $\varphi_j^s(u_\ell;\tau)$. To see this, we compute the second τ-derivatives at $\tau = 0$. This time, for notational simplicity, we denote the j-shock curve by $\varphi_s(\tau)$ and the j-rarefaction curve by $\varphi_r(\tau)$.

Recall that, given (7.22), the second equation in (7.20) becomes

$$\varphi_r'(\tau) = r_j\big(\varphi_r(\tau)\big). \tag{7.42}$$

Differentiation of this plus use of (7.26) yields

$$\varphi_r''(0) = \nabla_{r_j(u_\ell)} r_j(u_\ell). \tag{7.43}$$

Next, we take the τ-derivative of (7.33). Set $\mathcal{A}(\tau) = A\big(\varphi_s(\tau)\big)$. We get

$$\begin{aligned}\Big(\mathcal{A}(\tau) - s_j(\tau)\Big)\varphi_s''(\tau) + \Big(\mathcal{A}'(\tau) - s_j'(\tau)\Big)\varphi_s'(\tau) \\ = s_j''(\tau)\big(\varphi_s(\tau) - u_\ell\big) + s_j'(\tau)\varphi_s'(\tau).\end{aligned} \tag{7.44}$$

Thus, since $s_j(0) = \lambda_j(u_\ell)$ and $\varphi_s'(0) = r_j(u_\ell)$, we have

$$\Big(\mathcal{A}(0) - \lambda_j(u_\ell)\Big)\varphi_s''(0) = \Big(s_j'(0) - \mathcal{A}'(0)\Big)r_j(u_\ell). \tag{7.45}$$

Now $\mathcal{A}(\tau)r_j\big(\varphi_s(\tau)\big) = \lambda_j\big(\varphi_s(\tau)\big)r_j\big(\varphi_s(\tau)\big)$. Let us write this identity as $\big(\mathcal{A}(\tau) - \lambda_j(\tau)\big)r_j(\tau) = 0$ and differentiate, to obtain

$$\Big(\mathcal{A}(0) - \lambda_j(u_\ell)\Big)r_j'(0) = \Big(\lambda_j'(0) - \mathcal{A}'(0)\Big)r_j(u_\ell). \tag{7.46}$$

Subtracting from (7.45), we get

$$\Big(A(u_\ell) - \lambda_j(u_\ell)\Big)\big(\varphi_s''(0) - r_j'(0)\big) = \big(2s_j'(0) - \lambda_j'(0)\big)r_j(u_\ell). \tag{7.47}$$

Now the left side of (7.47) belongs to V_j, which is complementary to the span of $r_j(u_\ell)$, so both sides of (7.47) must vanish. This implies

$$s_j'(0) = \frac{1}{2}\lambda_j'(0) = \frac{1}{2}\varphi_s'(0)\cdot\nabla\lambda_j(u_\ell) = \frac{1}{2}, \tag{7.48}$$

and, since $\varphi_s''(0) - r_j'(0)$ belongs to the null space of $A(u_\ell) - \lambda_j(u_\ell)$,

$$\varphi_s''(0) = r_j'(0) + \beta r_j(u_\ell), \tag{7.49}$$

for some $\beta \in \mathbb{R}$.

Note that $r_j'(0)$ coincides with the quantity in (7.43). We claim that we can reparameterize $\varphi_s(\tau)$ so that $\beta = 0$ in (7.49), by taking

$$\widetilde{\varphi}_s(\tau) = \varphi(\tau + \alpha\tau^2), \tag{7.50}$$

for appropriate α. Indeed, we have $\widetilde{\varphi}_s(0) = \varphi_s(0)$, $\widetilde{\varphi}_s'(0) = \varphi_s'(0)$, and

$$\widetilde{\varphi}_s''(0) = \varphi_s''(0) + 2\alpha\varphi_s'(0) = \varphi_s''(0) + 2\alpha r_j(u_\ell). \tag{7.51}$$

Thus, taking $\alpha = -\beta/2$ in (7.50) accomplishes this goal. Replacing $\varphi_s(\tau)$ by (7.50), we arrange that the curve $\varphi_j(u_\ell;\tau)$ is C^2 in τ.

Note that if $u_r = \varphi_j^s(u_\ell;\tau)$, for some $\tau \in (-\tau_j, 0]$, the identity (7.48) together with (7.35) implies that the shock speed $s = s_j(\tau)$ of the weak solution (7.17) satisfies $\lambda_j(u_r) < s < \lambda_j(u_\ell)$, at least if τ is close enough to 0. In view of the ordering of the eigenvalues of $A(u)$, we have the inequalities

$$\begin{aligned} \lambda_{j-1}(u_\ell) < s < \lambda_j(u_\ell), \\ \lambda_j(u_r) < s < \lambda_{j+1}(u_r), \end{aligned} \tag{7.52}$$

for τ sufficiently close to 0. These are called the *Lax j-shock conditions*. The corresponding weak solutions are called *shock waves*.

The function $\varphi_j(u_\ell;\tau)$ is in fact C^2 in $(u_\ell;\tau)$. We can define a C^2-map

$$\Psi(u_\ell;\tau_1,\dots,\tau_L) = \varphi_L\big(\varphi_{L-1}\big(\cdots(\varphi_1(u_\ell);\tau_1)\cdots\big);\tau_{L-1}\big);\tau_L\big). \tag{7.53}$$

Since $(d/d\tau)\varphi_j(u_\ell;0) = r_j(u_\ell)$ and the eigenvectors $r_j(u_\ell)$ form a basis of $\mathbb{R}^L$, we can use the inverse function theorem to conclude the following:

Proposition 7.1. *Assume the $L \times L$ system (7.1) is strictly hyperbolic and genuinely nonlinear in each field. Given $u_\ell \in \Omega$, there is a neighborhood $\mathcal{O}$ of u_ℓ such that if $u_r \in \mathcal{O}$, then there is a weak solution to (7.1) with initial data*

$$\begin{aligned} u(0,x) = u_\ell, &\quad \text{for } x < 0, \\ u_r, &\quad \text{for } x > 0, \end{aligned} \tag{7.54}$$

consisting of a set of rarefaction waves and/or shock waves satisfying the Lax conditions (7.52).

See Fig. 7.2 for an illustration, with $L = 4$.

We consider how Proposition 7.1 applies to some examples. First consider the system (7.14), arising from the second-order equation (7.11). Here, with $r_\pm$ and $\lambda_\pm$ given by (7.16), we have

$$r_\pm \cdot \nabla\lambda_\pm = \pm\frac{1}{2}K_v^{-1/2}K_{vv}. \tag{7.55}$$

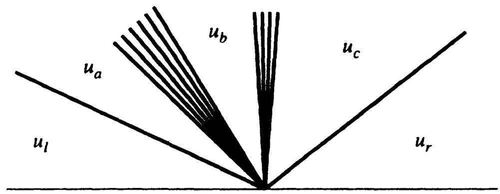

FIGURE 7.2 Shocks and Rarefactions

The strict hyperbolicity assumption is $K_v \neq 0$ on Ω. Given this, the hypothesis of genuine nonlinearity is that K_{vv} is nowhere vanishing. To achieve (7.22), we would rescale $r_\pm$, changing it to

$$r_\pm = -2\frac{\sqrt{K_v}}{K_{vv}}\begin{pmatrix} \mp 1 \\ \sqrt{K_v} \end{pmatrix}. \tag{7.56}$$

In this case, given $u_\ell = (v_\ell, w_\ell) \in \Omega \subset \mathbb{R}^2$, the rarefaction curves emanating from u_ℓ are the forward orbits of the vector fields r_+ and r_-, starting at u_ℓ. The jump condition (7.29) takes the form

$$\begin{aligned} w_\ell - w_r &= s(v_r - v_\ell), \\ K(v_\ell) - K(v_r) &= s(w_r - w_\ell), \end{aligned} \tag{7.57}$$

in this case. This requires $K(v_\ell) - K(v_r) = s^2(v_\ell - v_r)$, so

$$\frac{w_\ell - w_r}{v_\ell - v_r} = \pm\sqrt{\frac{K(v_\ell) - K(v_r)}{v_\ell - v_r}}. \tag{7.58}$$

This defines a pair of curves through u_ℓ; half of each such curve makes up a shock curve.

One occurrence of (7.11) is to describe longitudinal waves in a string, with $V(t,x)$ denoting the position of a point of the string, constrained to move along the x-axis. Physically, a real string would greatly resist being compressed to a degree that $V_x = v \to 0$. Thus a reasonable potential energy function $F(v)$ has the property that $F(v) \to +\infty$ as $v \searrow 0$; recall $K(v) = F'(v)$. A situation yielding a strictly hyperbolic, genuinely nonlinear PDE is depicted in Fig. 7.3, in which F is convex, K is monotone increasing and concave, K_v is positive and monotone decreasing, and K_{vv} is negative. Here, $\Omega = \{(v,w) : v > 0\}$.

We illustrate the rarefaction and shock curves through a point $u_\ell \in \Omega$, in Fig. 7.4, for such a case.

A specific example is

$$F(v) = \frac{1+v^2}{v}, \quad K(v) = 1 - \frac{1}{v^2}, \quad K'(v) = \frac{2}{v^3}, \quad K''(v) = -\frac{6}{v^4}. \tag{7.59}$$

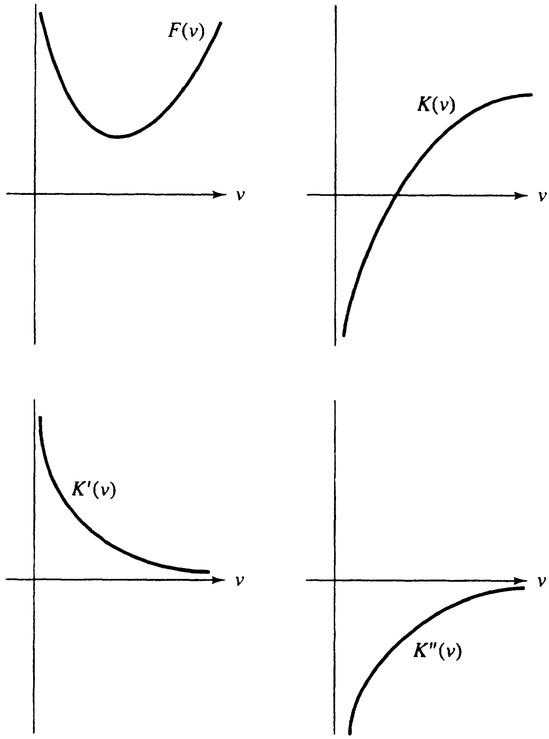

FIGURE 7.3 Typical String Potential

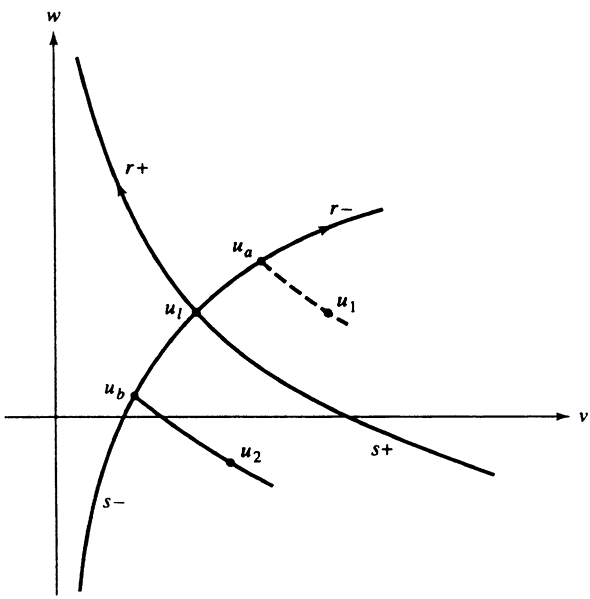

FIGURE 7.4 Rarefaction and Shock Curves Through u_ℓ

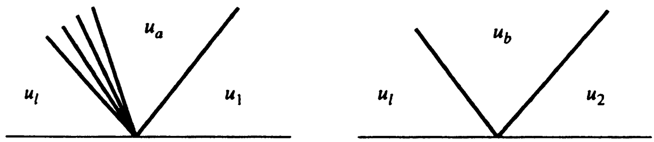

FIGURE 7.5 Solutions to Riemann Problems

In Fig. 7.5 we illustrate the solution of the Riemann problem (7.17), with $u_r = u_1$ and $u_r = u_2$, respectively, where u_1 and u_2 are as pictured in Fig. 7.4.

However, it must be noted that such an example as (7.59) is exceptional. A real elastic substance would tend to have a potential energy function $F(v)$ that increases much more rapidly for large (or even moderate) v. A specific example is

$$\begin{aligned} F(v) &= \frac{1}{v} + \frac{1}{1-v} + v, \quad K(v) = -\frac{1}{v^2} + \frac{1}{(1-v)^2} + 1, \\ K'(v) &= \frac{2}{v^3} + \frac{2}{(1-v)^3}, \quad K''(v) = -6\Big(\frac{1}{v^4} - \frac{1}{(1-v)^4}\Big), \end{aligned} \tag{7.60}$$

on $\Omega = \{(v, w) : 0 < v < 1\}$. In this case, the system (7.14) is genuinely nonlinear except on the line $\{v = 1/2\}$.

Another situation giving rise to (7.11) is a model of a string, vibrating in $\mathbb{R}^2$, but (magically) constrained to have only transverse vibrations, so a point whose coordinate on the string is x is at the point $\big(x, V(t, x)\big) \in \mathbb{R}^2$ at time t. In such a case, $\Omega = \mathbb{R}^2$ and $F(v)$ has the form $F(v) = f(v^2)$, so

$$K(v) = 2f'(v^2)v.$$

Thus $K(v)$ is a smooth *odd* function on $\mathbb{R}$. Hence K_{vv} is also odd and must vanish at $v = 0$. Thus genuine nonlinearity must fail at $v = 0$. We will return to this a little later; see (7.85)–(7.91).

We next investigate how Proposition 7.1 applies to the equations of isentropic compressible fluid flow, in the form (7.8), which can be cast in the form

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - K(v, w)_x &= 0, \end{aligned} \tag{7.61}$$

a generalization of the form (7.14), if we set

$$v = \rho, \quad w = -m, \quad K(v, w) = \frac{w^2}{v} + p(v). \tag{7.62}$$

(This v is not the v in (7.5).) For smooth solutions, (7.61) takes the form (7.2) with

$$A(u) = -\begin{pmatrix} 0 & 1 \\ K_v & K_w \end{pmatrix}. \tag{7.63}$$

This has eigenvalues and eigenvectors

$$\lambda_\pm = -\frac{1}{2}K_w \pm \frac{1}{2}\sqrt{K_w^2 + 4K_v}, \qquad r_\pm = \begin{pmatrix} 1 \\ -\lambda_\pm \end{pmatrix}. \tag{7.64}$$

With $K(v, w)$ given by (7.62), we have

$$\lambda_\pm = \frac{m}{\rho} \pm \sqrt{p'(\rho)}, \tag{7.65}$$

which is equivalent to (7.10). In this case,

$$r_\pm \cdot \nabla\lambda_\pm = \pm\frac{1}{\sqrt{p'(\rho)}}\Big(p''(\rho) + \frac{p'(\rho)}{\rho}\Big) = \pm\sqrt{A\gamma}\;\gamma\;\rho^{(\gamma-3)/2}, \tag{7.66}$$

the last identity holding when $p(\rho) = A\rho^\gamma$, as in (7.6). Thus the system (7.8) is genuinely nonlinear in the region $\Omega = \{(\rho, m) : \rho > 0\}$.

A number of important cases of Riemann problems are not covered by Proposition 7.1. We will take a look at some of them here, though our treatment will not be nearly exhaustive.

First, we consider a condition that is directly opposite to the hypothesis of genuine nonlinearity. We say the jth field is linearly degenerate provided

$$r_j(u) \cdot \nabla\lambda_j(u) = 0 \quad \text{on } \Omega. \tag{7.67}$$

In such a case, the integral curve of $R_j = r_j \cdot \nabla$ through u_ℓ, which we denote now by $\varphi_j^c(u_\ell; \tau)$ instead of (7.23), does not produce a set of data u_r for which there is a rarefaction wave solution to (7.17), of the form (7.18)–(7.20), but we do have the following.

Lemma 7.2. *Under the linear degeneracy hypothesis (7.67), if we set*

$$s = \lambda_j(u_\ell), \tag{7.68}$$

and let $u_r = \varphi_j^c(u_\ell; \tau)$ for any τ (for which the flow is defined), then

$$\begin{aligned} u(t, x) = u_\ell, &\quad \textit{for } x < st, \\ u_r, &\quad \textit{for } x > st \end{aligned} \tag{7.69}$$

defines for $t \geq 0$ a weak solution to the Riemann problem (7.17); that is, the Rankine–Hugoniot condition (7.29) is satisfied. Furthermore,

$$\lambda_j(u_r) = s. \tag{7.70}$$

Proof. Setting $\varphi(\tau) = \varphi_j^c(u_\ell;\tau)$, we have

$$\varphi'(\tau) = r_j\big(\varphi(\tau)\big), \quad \varphi'(0) = u_\ell. \tag{7.71}$$

By the definition of r_j, this implies

$$\big[A\big(\varphi(\tau)\big) - \lambda_j\big(\varphi(\tau)\big)\big]\varphi'(\tau) = 0. \tag{7.72}$$

Now the Rankine–Hugoniot condition (7.29) for $u_r = \varphi(\tau)$, with $s = \lambda_j(u_\ell)$, holds for all τ if and only if

$$\frac{d}{d\tau}\Big[F\big(\varphi(\tau)\big) - \lambda_j(u_\ell)\varphi(\tau)\Big] = 0, \tag{7.73}$$

or equivalently,

$$\big[A\big(\varphi(\tau)\big) - \lambda_j(u_\ell)\big]\varphi'(\tau) = 0. \tag{7.74}$$

On the other hand,

$$\frac{d}{d\tau}\lambda_j\big(\varphi(\tau)\big) = \varphi'(\tau)\cdot\nabla\lambda_j\big(\varphi(\tau)\big) = r_j\big(\varphi(\tau)\big)\cdot\nabla\lambda_j\big(\varphi(\tau)\big) = 0,$$

so

$$\lambda_j\big(\varphi(\tau)\big) = \lambda_j(u_\ell) = s, \quad \forall\ \tau. \tag{7.75}$$

This implies (7.70) and also shows that the left sides of (7.72) and (7.74) are equal, so the lemma is proved.

When the jth field is linearly degenerate, the weak solution to the Riemann problem defined by (7.69) is called a *contact discontinuity*. The term "contact" refers to the identity (7.70), that is, to

$$\lambda_j(u_\ell) = s = \lambda_j(u_r), \tag{7.76}$$

which contrasts with the shock condition (7.52). Note that in defining $\varphi_j^c(u_\ell;\tau)$, we do not restrict τ to be ≤ 0, as for a j-shock curve, nor do we restrict τ to be ≥ 0, as for a j-rarefaction curve. Rather, τ runs over an interval containing 0 in its interior.

There is a straightforward extension of Proposition 7.1:

Proposition 7.3. *Assume that the $L\times L$ system (7.1) is strictly hyperbolic and that each field is either genuinely nonlinear or linearly degenerate. Given $u_\ell \in \Omega$, there is a neighborhood $\mathcal{O}$ of u_ℓ such that if $u_r \in \mathcal{O}$, then there is a weak solution to (7.1) with initial data*

$$u(0,x) = u_\ell, \quad \textit{for } x < 0, \qquad u_r, \quad \textit{for } x > 0, \tag{7.77}$$

consisting of a set of rarefaction waves, shock waves satisfying the Lax condition (7.52), and contact discontinuities.

An important example is the following system:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2)_x + p(\rho, S)_x &= 0, \\ (\rho S)_t + (\rho S v)_x &= 0, \end{aligned} \tag{7.78}$$

for a one-dimensional compressible fluid that is not isentropic; here $S(t,x)$ is the "entropy," and the equation of state $p = p(\rho)$ is generalized to $p = p(\rho, S)$. Compare with (5.61)–(5.62). Using $m = \rho v$ as before, we can write this system as

$$\begin{aligned} \rho_t + m_x &= 0, \\ m_t + \Bigl(\frac{m^2}{\rho} + p(\rho, S)\Bigr)_x &= 0, \\ (\rho S)_t + (mS)_x &= 0. \end{aligned} \tag{7.79}$$

Note that, for smooth solutions, we can replace the last equation by

$$S_t + \frac{m}{\rho} S_x = 0. \tag{7.80}$$

In this case, we have $u = (\rho, m, S)$ and

$$\begin{aligned} A(u) &= \begin{pmatrix} 0 & 1 & 0 \\ -\frac{m^2}{\rho^2} + \frac{\partial p}{\partial \rho} & \frac{2m}{\rho} & \frac{\partial p}{\partial S} \\ 0 & 0 & \frac{m}{\rho} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -v^2 + \frac{\partial p}{\partial \rho} & 2v & \frac{\partial p}{\partial S} \\ 0 & 0 & v \end{pmatrix}. \end{aligned} \tag{7.81}$$

Note that $A(u)$ leaves invariant the two-dimensional space $\{(a, b, 0)\}$, so as in (7.10) we have eigenvalues and eigenvectors

$$\lambda_\pm = v \pm \sqrt{\frac{\partial p}{\partial \rho}}, \quad r_\pm = (1, \lambda_\pm, 0)^t. \tag{7.82}$$

Also, by inspection $A(u)^t$ has eigenvector $(0,0,1)^t$, with eigenvalue v, which must also be an eigenvalue of $A(u)$; we have

$$\lambda_0 = v = \frac{m}{\rho}, \quad r_0 = \Big(1, v, -\frac{p_\rho}{p_s}\Big)^t = \Big(1, \frac{m}{\rho}, -\frac{p_\rho}{p_s}\Big)^t. \tag{7.83}$$

Thus

$$r_0 \cdot \nabla\lambda_0 = -\frac{m}{\rho^2} + \frac{m}{\rho} \cdot \frac{1}{\rho} = 0. \tag{7.84}$$

Of course $r_\pm \cdot \nabla\lambda_\pm$ are still given by (7.66). Thus we have one linearly degenerate field and two genuinely nonlinear fields for the 3×3 system (7.79).

In § 10 we will see that the study of a string vibrating in a plane gives rise to a 4×4 system that, in some cases, has two linearly degenerate fields and two genuinely nonlinear fields, though for such a system there are also more complicated possibilities.

We now return to the 2×2 system (7.14), i.e.,

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - K(v)_x &= 0, \end{aligned} \tag{7.85}$$

in cases such as those mentioned after (7.60), that is,

$$K(v) = 2f'(v^2)v, \tag{7.86}$$

where $\Omega = \mathbb{R}^2$ and f' is smooth, with $f'(0) > 0$. Thus, as computed before, we have

$$\lambda_\pm = \pm\sqrt{K_v}, \quad r_\pm = (1, -\lambda_\pm)^t, \quad r_\pm \cdot \nabla\lambda_\pm = \pm\frac{1}{2}K_v^{-1/2}K_{vv}, \tag{7.87}$$

or, with the $\pm$ subscript replaced by j; $\pm 1 = (-1)^j$, $j = 1, 2$,

$$R_j\lambda_j = (-1)^j \frac{1}{2} K_{vv} K_v^{-1/2}. \tag{7.88}$$

The genuine nonlinearity condition fails on the line $v = 0$. We will assume that f' is behaved so that $K_v > 0$ on $\mathbb{R}$, $K_{vv} > 0$ on $(0,\infty)$, $K_{vv} < 0$ on $(-\infty, 0)$, and $K_{vvv}(0) > 0$. Set

$$\Omega^j_\pm = \{(v,w) : \pm R_j\lambda_j > 0\}, \tag{7.89}$$

so in the case we are considering, the regions $\Omega^j_\pm$ are pictured in Fig. 7.6.

In Fig. 7.7 we depict the various shock and rarefaction curves emanating from u_ℓ on the left and those emanating from u_r on the right. The rarefaction curves, which are integral cuves of R_j, terminate upon hitting the vertical axis $\{v = 0\}$.

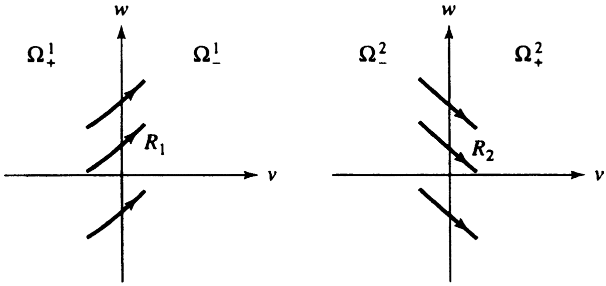

FIGURE 7.6 The Regions $\Omega^j_\pm$

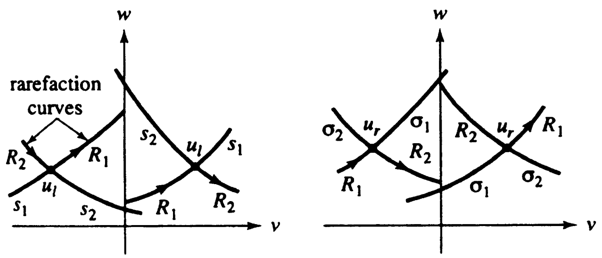

FIGURE 7.7 Rarefaction and Shock Curves Through u_ℓ and u_r

On the other hand, the shock curves continue to produce solutions to the Riemann problem even after they cross this axis, though the Lax shock conditions might break down eventually. Note that the rarefaction curves from u_ℓ are flow-outs of R_j in Ω^j_+ and flow-outs of $-R_j$ in Ω^j_-.

We look at the question of how to solve the Riemann problem when u_ℓ cannot be connected to curves that avoid the vertical axis $v = 0$.

In Fig. 7.8 we indicate in one case how to extend the curve $\varphi_1(u_\ell; \tau)$ for positive τ, beyond the point where this curve (which initially, for $\tau > 0$, is an integral curve of R_1) intersects the vertical axis.

To decide precisely which u_r lie on this continued curve, it is easiest to work backward from u_r, along the shock curve σ_1, continued across the vertical axis into the region $\{v < 0\}$. Let u^a denote the first point along σ_1 at which the Lax shock condition fails. Thus the solution to the Riemann problem with initial data u^a for $x < 0$, u_r for $x > 0$, has a one-sided contact discontinuity, in the sense that the speed s satisfies

$$\lambda_1(u^a) = s < \lambda_1(u_r). \tag{7.90}$$

Then the flow-out from u^a under $-R_1$ gives rise to u_ℓ that are connected to u_r by a solution such as that indicated in Fig. 7.9.

Thus the solution consists of a rarefaction wave connecting u_ℓ to u^a, followed by a jump discontinuity that is a one-sided contact and one-sided shock, as stated in (7.90).

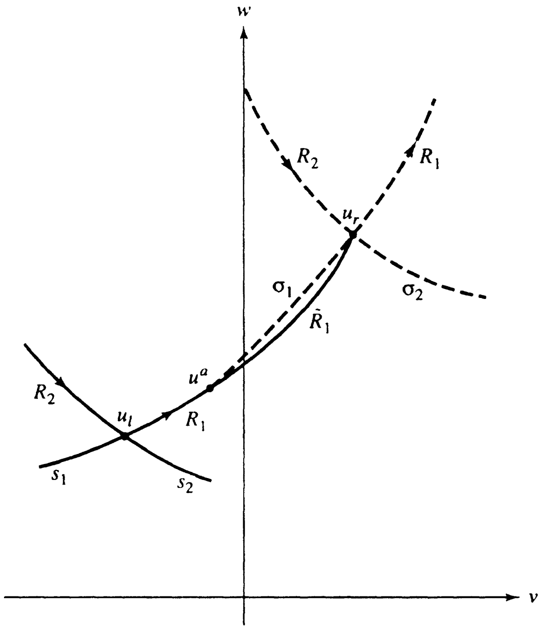

FIGURE 7.8 Connecting u_ℓ to u_r

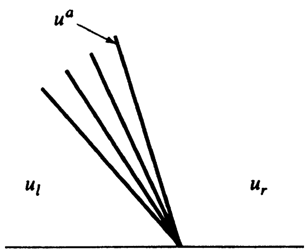

FIGURE 7.9 One-Sided Contact Discontinuity

In Fig. 7.10, we take the case illustrated by Fig. 7.8 and relabel the old u_r as u_m, taking a new u_r, connected to u_m by $S_2 \cup \widetilde{S}_2$, consisting of the shock curve out of u_m, continued beyond the vertical axis until the Lax shock condition fails, at u^b, and then followed by the flow-out from u^b under $-R_2$.

The resulting solution to the Riemann problem is depicted in Fig. 7.11. First we have the 1-rarefaction connecting u_ℓ to u^a, followed by the jump discontinuity connecting u^a to u_m, as in Fig. 7.9. Then we have the jump discontinuity connecting u_m to u^b, satisfying the shock/contact condition

$$\lambda_2(u_m) < s = \lambda_2(u^b). \tag{7.91}$$

Finally, u^b is connected to u_r by a 2-rarefaction.

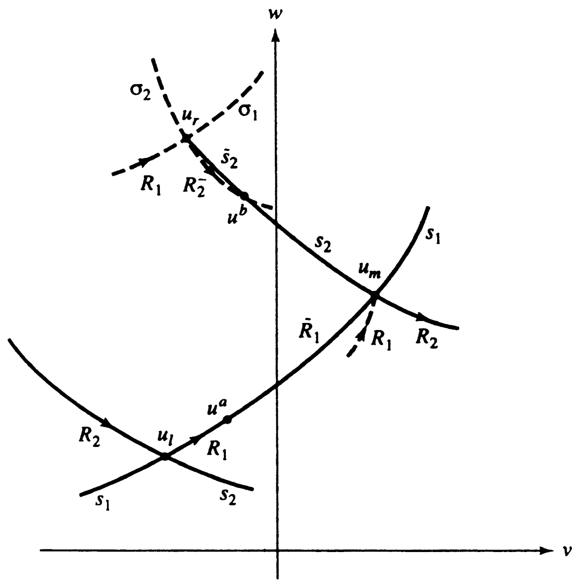

FIGURE 7.10 Connecting u_ℓ to u_r

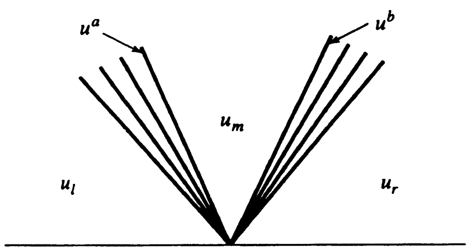

FIGURE 7.11 Two One-Sided Contacts

Figures 7.9 and 7.11 should remind one of Fig. 6.7, depicting the solution to a Riemann problem for a scalar conservation law, satisfying Oleinik's condition (E). In fact, it can be verified that the discontinuities produced by the construction above satisfy the following admissibility condition. Say a weak solution to (7.85) is equal to (v_ℓ, w_ℓ) for $x < st$ and to (v_r, w_r) for $x > st$, $t \geq 0$. Then the admissibility condition is that, for all v between v_ℓ and v_r, either

$$\frac{K(v) - K(v_\ell)}{v - v_\ell} \leq \frac{K(v_r) - K(v_\ell)}{v_r - v_\ell} \qquad (\text{if } s \leq 0), \tag{7.92}$$

or

$$\frac{K(v) - K(v_\ell)}{v - v_\ell} \geq \frac{K(v_r) - K(v_\ell)}{v_r - v_\ell} \qquad (\text{if } s \geq 0). \tag{7.93}$$

Compare this to the formulation (6.48) of condition (E).

In [Liu1] there is a treatment of a class of 2×2 systems, containing the case just described, in which an extension of condition (E) is derived. See also [Wen]. This study is extended to $n \times n$ systems in [Liu2].

Further interesting phenomena for the Riemann problem arise when there is breakdown of strict hyperbolicity. Material on this can be found in [KK2, SS2], and in the collection of articles in [KK3]. We will not go into such results here, though some mention will be made in § 10.

In addition to solving the Riemann problem when u_ℓ and u_r are close, one also wants solutions, when possible, when u_ℓ and u_r are far apart. There are a number of results along these lines, which can be found in [DD,KK1,Liu2,SJ]. We restrict our discussion of this to a single example.

We give an example, from [LS], of a strictly hyperbolic, genuinely nonlinear system for which the Riemann problem is solvable for arbitrary $u_\ell, u_r \in \Omega$, but some of the solutions do not fit into the framework of Proposition 7.1. Namely, consider the 2×2 system (7.5)–(7.6) describing compressible fluid flow, for $u = (v, \rho)$, with $\Omega = \{(v, \rho) : \rho > 0\}$. As seen in (7.10), if we switch to (m, ρ)-coordinates, with $m = \rho v$, then there are eigenvalues $\lambda_\pm = m/\rho \pm \sqrt{p'(\rho)}$ and eigenvectors $R_\pm = \partial/\partial\rho + \lambda_\pm \partial/\partial m$. Thus integral curves of $R_\pm$ satisfy $\dot\rho = 1,\ \dot m = m/\rho \pm \sqrt{p'(\rho)}$; hence

$$\dot v = \frac{\dot m \rho - m \dot\rho}{\rho^2} = \pm \frac{\sqrt{p'(\rho)}}{\rho},$$

that is, integral curves of $R_\pm$ through (v_ℓ, ρ_ℓ) are given by

$$v - v_\ell = \pm \int_{\rho_\ell}^{\rho} \frac{\sqrt{p'(s)}}{s}\, ds = \pm\sqrt{A\gamma} \int_{\rho_\ell}^{\rho} s^{(\gamma-3)/2}\, ds. \tag{7.94}$$

If $\gamma \in (1, 2)$, as assumed in (7.6), then these rarefaction curves intersect the axis $\rho = 0$. Note that if we normalize $R_\pm$ so that $R_\pm \lambda_\pm = 1$, then

$$R_\pm = \left(A\gamma^3 \rho^{\gamma-3}\right)^{-1/2} \left[\frac{\sqrt{p'(\rho)}}{\rho} \frac{\partial}{\partial v} \pm \frac{\partial}{\partial\rho} \right]. \tag{7.95}$$

Furthermore, specializing (7.58), we see that the shock curves from u_ℓ are given by

$$v - v_\ell = -\left[\frac{\rho - \rho_\ell}{\rho\rho_\ell} \big(p(\rho) - p(\rho_\ell)\big) \right]^{1/2}, \qquad \text{for } \pm(\rho_\ell - \rho) > 0. \tag{7.96}$$

Note that these shock curves never reach the axis $\rho = 0$. See Fig. 7.12 for a picture of the shock and rarefaction curves emanating from u_ℓ.

Now, as in Fig. 7.13, pick $u_0 = (v_0, \rho_0) \in \Omega$ and consider the "triangular" region $\mathcal{T}$, with apex at u_0, bounded by the integral curves of R_- (forward) and

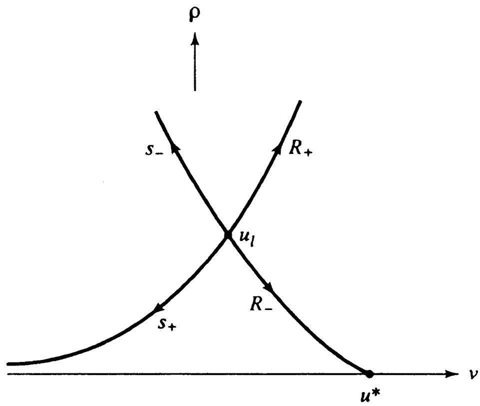

FIGURE 7.12 Shock and Rarefaction Curves Through u_ℓ

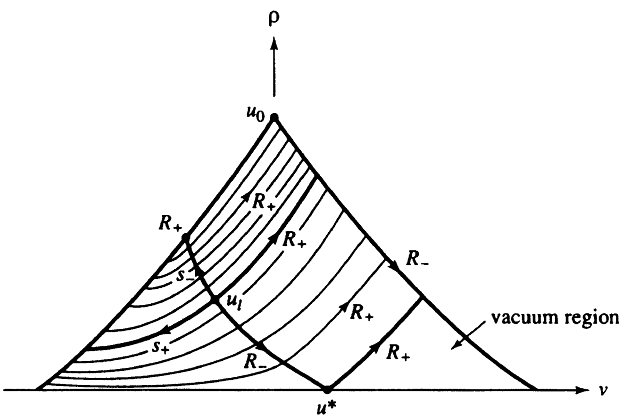

FIGURE 7.13 Vacuum Region

of R_+ (backward) through u_0, and by the axis $\rho = 0$. This is a bounded region. Given any $u_\ell, u_r \in \mathcal{T}$, we will produce a solution to the Riemann problem, whose intermediate state also belongs to $\mathcal{T}$ (or at least to $\overline{\mathcal{T}}$).

In fact, as seen in Fig. 7.13, if $u_\ell \in \mathcal{T}$, the rarefaction and shock waves described before suffice to do this for u_r in all of $\mathcal{T}$ except for a smaller triangular region in the lower right corner of $\mathcal{T}$, which we call the "vacuum region." This is bounded by part of $\partial\mathcal{T}$, plus part of the integral curve of R_+ emanating from u^*, where u^* is the point of intersection of the R_--integral curve through u_ℓ with $\{\rho = 0\}$.

What we do if u_r belongs to this vacuum region is indicated in Figs. 7.14 and 7.15. Namely, u_ℓ is connected to the vacuum by a rarefaction wave, whose speed on the left is $\lambda_-(u_\ell) = v_\ell - \sqrt{p'(\rho_\ell)}$ and whose speed on the right is $\lambda_-(u^*) = \lambda_-(v^*, 0) = v^*$ (since $p'(0) = 0$ when (7.6) holds). Next, if $u^a = (v^a, 0)$ is the point on the axis $\{\rho = 0\}$ from which issues the R_+-integral curve through u_r,

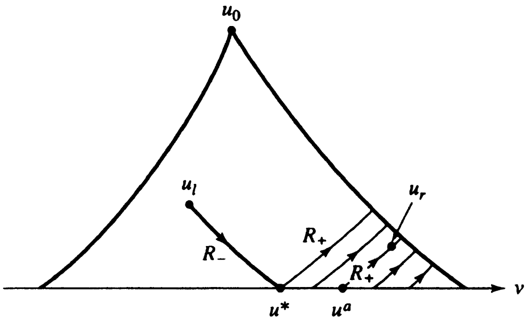

FIGURE 7.14 Connecting u_ℓ to u_r Through the Vacuum

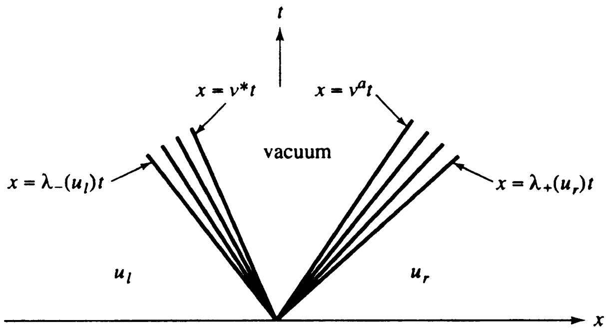

FIGURE 7.15 Associated Solution to the Riemann Problem

then the vacuum is connected to u_r by a rarefaction wave whose speed on the left is $\lambda_+(u^a) = v^a > v^*$ (if $u^a \neq u^*$) and whose speed on the right is $\lambda_+(u_r)$. In the special case that $u^a = u^*$, the vacuum state disappears, except for a single ray, along which the two rarefaction waves fit together.

This concludes our discussion in this section of examples of the Riemann problem. In § 10 there is further discussion for equations of vibrating strings.

Continuing a theme from § 6, we next explore the relation between the shock condition (7.52) and the possibility that the solution u is a limit as $\varepsilon \searrow 0$ of solutions to

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_x^2 u_\varepsilon. \tag{7.97}$$

Here, we will look for solutions to (7.97) of the form

$$u_\varepsilon(t,x) = v\big(\varepsilon^{-1}(x - ct)\big). \tag{7.98}$$

This satisfies (7.97) if and only if

$$\frac{d}{d\tau}\big[F(v) - cv(\tau)\big] = v''(\tau), \tag{7.99}$$

or equivalently, if and only if there exist $b \in \mathbb{R}^L$ such that

$$v'(\tau) = F(v) - cv - b = \Phi_{cb}(v). \tag{7.100}$$

In other words, $v(\tau)$ should be an integral curve of the vector field Φ_{cb}. The requirement that the limit $u(t, x)$ satisfy the Riemann problem (7.17) is equivalent to

$$v(-\infty) = u_\ell, \qquad v(+\infty) = u_r. \tag{7.101}$$

Consequently, u_ℓ and u_r should be critical points of the vector field Φ_{cb}, connected by a "heteroclinic orbit." If this happens, we say u_ℓ is connected to u_r via a "viscous profile."

For u_ℓ and u_r to be critical points of Φ_{cb}, we need

$$F(u_\ell) - cu_\ell = b = F(u_r) - cu_r, \tag{7.102}$$

hence

$$F(u_r) - F(u_\ell) = c(u_r - u_\ell). \tag{7.103}$$

This is precisely the Rankine–Hugoniot condition (7.29), with $s = c$. Now, consider the behavior of the vector field Φ_{cb} near each of these critical points. The linearization near $u_0 = u_\ell$ or u_r is given by

$$V(u_0 + v) = \big(A(u_0) - s\big)v. \tag{7.104}$$

Now, if (7.52) holds (i.e., $\lambda_j(u_r) < s < \lambda_j(u_\ell)$), and if u_r and u_ℓ are sufficiently close, then $A(u_\ell) - s$ has $L - (j - 1)$ positive eigenvalues and $j - 1$ negative eigenvalues, while $A(u_r) - s$ has $L - j$ positive eigenvalues and j negative eigenvalues.

The qualitative theory of ODE guarantees the existence of a heteroclinic orbit from u_ℓ to u_r (if they are sufficiently close). We will not give the proof here, but confine our discussion to a presentation of Fig. 7.16, illustrating the 2×2 case in which u_ℓ is connected to u_r by a 1-shock. The ODE theory involved here has been developed quite far, in order also to investigate cases where u_ℓ and u_r are not close but can still be shown to be connected by a viscous profile. The book [Smo] gives a detailed discussion of this.

We mention a variant of the viscosity method described above, which was used in [DD]. Namely, we look at a family of solutions to

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon t\, \partial_x^2 u_\varepsilon \tag{7.105}$$

of the form

$$u_\varepsilon(t, x) = v_\varepsilon(t^{-1}x), \tag{7.106}$$

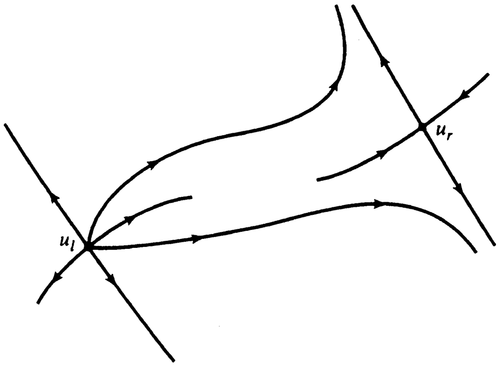

FIGURE 7.16 Forcing a Heteroclinic Orbit

where $v_\varepsilon(\tau)$ solves

$$\varepsilon v_\varepsilon''(\tau) = \big[A(v_\varepsilon) - \tau\big] v_\varepsilon'(\tau), \tag{7.107}$$

and

$$v_\varepsilon(-\infty) = u_\ell, \quad v_\varepsilon(+\infty) = u_r. \tag{7.108}$$

Setting $w_\varepsilon(\tau) = v_\varepsilon'(\tau)$, we get a $(2L)\times(2L)$ first-order system for $V_\varepsilon = (v_\varepsilon, w_\varepsilon)$:

$$V_\varepsilon'(\tau) = \Psi\big(\tau, V_\varepsilon(\tau)\big) = \big(w_\varepsilon(\tau), \varepsilon^{-1}[A(v_\varepsilon) - \tau] w_\varepsilon(\tau)\big), \tag{7.109}$$

with

$$V_\varepsilon(-\infty) = (u_\ell, 0), \quad V_\varepsilon(+\infty) = (u_r, 0). \tag{7.110}$$

The paper [DD] considered such solutions when (7.1) is a 2×2 system, satisfying

$$\frac{\partial F_1}{\partial u_2} < 0, \quad \frac{\partial F_2}{\partial u_1} < 0, \quad \forall\, u \in \mathbb{R}^2, \tag{7.111}$$

a condition that guarantees strict hyperbolicity. In particular, it is shown in [DD] that this viscosity method leads to a solution to the Riemann problem for all data (u_ℓ, u_r) whenever (7.1) is a symmetric hyperbolic 2×2 system, satisfying (7.111).

We mention another "viscosity method" that has been applied to 2×2 systems of the form (7.14). Namely, for $\varepsilon > 0$, consider

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - K(v)_x &= \varepsilon v_{xt}. \end{aligned} \tag{7.112}$$

This comes via $v = u_x,\ w = u_t$, from the equation

$$u_{tt} - K(u_x)_x = \varepsilon u_{xxt}, \tag{7.113}$$

which arises in the study of viscoelastic bars; see [Sh1] and [Sl]. We look for a traveling wave solution $U = (v, w)$ of the form $U\big((x - st)/\varepsilon\big)$, satisfying $U(-\infty) = (v_\ell, w_\ell),\ U(+\infty) = (v_r, w_r)$. Thus we require

$$\begin{aligned} -s(v - v_\ell) - (w - w_\ell) &= 0, \\ -s(w - w_\ell) - \big[K(v) - K(v_\ell)\big] &= -sv'(\tau), \end{aligned} \tag{7.114}$$

hence

$$sv'(\tau) = K(v) - K(v_\ell) - s^2(v - v_\ell); \quad v(-\infty) = v_\ell,\ v(+\infty) = v_r. \tag{7.115}$$

For this to be possible, one requires that $\psi(v) = K(v) - K(v_\ell) - s^2(v - v_\ell)$ vanish at $v = v_r$ as well as $v = v_\ell$; this together with the first part of (7.114) constitutes precisely the Rankine–Hugoniot condition, that $\widetilde{U} = (v_\ell, w_\ell)$ for $x < st,\ (v_r, w_r)$ for $x > st,\ t \geq 0$, be a weak solution to (7.14). In addition, in order to solve (7.115), one requires that v_ℓ be a source for the vector field $(\operatorname{sgn} s)\psi(v)\partial/\partial v$ on $\mathbb{R}$, that v_r be a sink, and that there be no other zeros of $\psi(v)$ for v between v_ℓ and v_r. Thus we require

$$\frac{K(v) - K(v_\ell)}{v - v_\ell} - s^2 > 0,$$

for v between v_ℓ and v_r $<$ if $s > 0$, and the reverse inequality if $s < 0$. Note that this implies the admissibility condition (7.92)–(7.93), given that $K(v_r) - K(v_\ell) = s^2(v_r - v_\ell)$. See the exercises after § 8 for more on this viscosity method.

There is a method for approximating a solution to (9.1) with general initial data, via solving a sequence of Riemann problems, called the *Glimm scheme*, after [Gl1], where it is used as a tool to establish the existence of global solutions for certain classes of initial-value problems. The method is the following: Divide the x-axis into intervals J_ν of length ℓ. In each interval J_ν, pick a point x_ν, at random, evaluate $u(0, x_\nu) = a_\nu$, and now consider the piecewise-constant initial data so obtained. Assuming, for example, that (8.1) is strictly hyperbolic and genuinely nonlinear, and $|u(0, x)| \leq C$, one can obtain for small h a weak solution $v(t, x)$ to (8.1) on $(t, x) \in [0, h] \times \mathbb{R}$, consisting locally of solutions to Riemann problems; see Fig. 7.17. Now, pick a new sequence y_ν of random points in J_ν, evaluate $v(h, y_\nu) = b_\nu$, and repeat this construction to define $v(t, x)$ for $(t, x) \in [h, 2h] \times \mathbb{R}$. Continue. In [Gl1] there are results giving conditions under which one has $v = v_{\ell,h}$ well defined for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, and convergent to a weak solution as $\ell \to 0,\ h = c_0\ell$. Further results can be found in [GL, DiP1, Liu5]; see also the treatment in [Smo]. In § 9 we will describe a different method, due to [DiP4], to establish global existence for a class of systems of conservation laws.

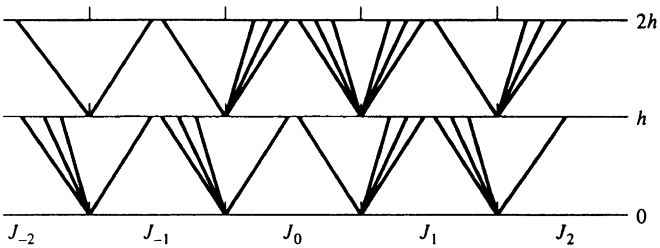

FIGURE 7.17 Setup for Glimm's Scheme

Exercises

In Exercises 1–3, we consider some shock interaction problems for a system of the form (7.1). Assume (7.1) is a 2×2 sysyem, strictly hyperbolic and genuinely nonlinear. Assume u_ℓ and u_r are sufficiently close together.

1. Suppose that, for $t < t_0$, u takes three constant values, u_ℓ, u_m, u_r, in regions separated by shocks of the *opposite* family, with shock speeds s_+, s_-. Assume the faster shock is to the left. Thus these shocks must intersect; say they do so at $t = t_0$ (see Fig. 7.18). Show that the solution to the Riemann problem at $t = t_0$, with data u_ℓ, u_r, consists of two shocks, s_-, s_+, as depicted in Fig. 7.18. In particular, there are no rarefaction waves.
2. Suppose that, for $t < t_0$, u takes on three constant values u_ℓ, u_m, u_r, in regions separated by shocks of the *same* family, say s_+, and assume that the left shock has higher speed than the right shock. Thus these two shocks must intersect; say they do so at $t = t_0$ (see Fig. 7.19). Show that the solution to the Riemann problem at $t = t_0$, with data u_ℓ, u_r, consists of a shock of the same family as those that interacted, together (perhaps) with either a shock wave or a rarefaction wave of the other family. (*Hint*: Study Fig. 7.4.)
   If only the second possibility can occur when two shocks of the same family collide, the 2×2 system is said to satisfy the "shock interaction condition." This condition was introduced by Glimm and Lax; see [GL].
3. Show that the shock interaction condition holds, at least for sufficiently weak shocks, provided that $\Omega = \mathbb{R}^2$ and, for each $u_\ell \in \Omega$, the curves $\varphi_1(u_\ell; \tau)$ and $\varphi_2(u_\ell; \tau)$ are

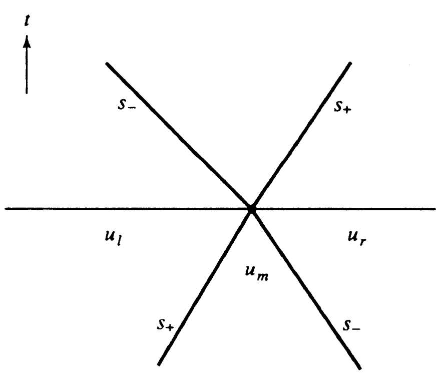

FIGURE 7.18 Situation for Exercise 1

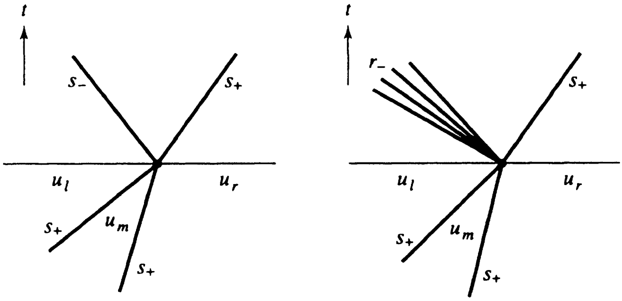

FIGURE 7.19 Situation for Exercise 2

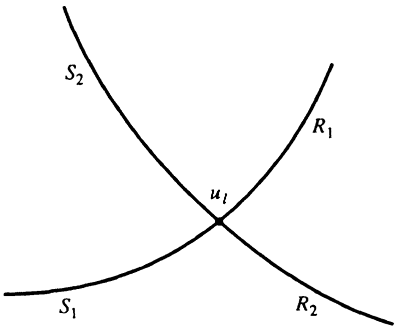

FIGURE 7.20 Situation for Exercise 3

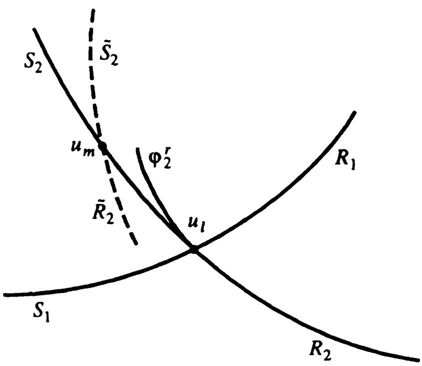

FIGURE 7.21 Possible Attack on Exercise 3

both strongly convex, as in Fig. 7.20. Here, $\varphi_j(u_\ell;\tau)$ is obtained by piecing together the rarefaction curve $\varphi_j^r(u_\ell;\tau)$ and the shock curve $\varphi_j^s(u_\ell;\tau)$. (*Hint*: Show that if, for example, u_m lies on the 2-shock curve from u_ℓ, as in Fig. 7.21, then the 2-wave curve $\varphi_2(u_m;\cdot) = \widetilde{S}_2 \cup \widetilde{R}_2$ is as pictured in that figure, as is the continuation of $\varphi_2^r(u_\ell;\tau)$ for $\tau < 0$. To do this, you will need to look at $\partial_\tau^3\varphi_j(u_\ell;\pm 0)$. See [SJ].)

4. Strengthen Proposition 7.1 as follows. Under the hypotheses of that proposition:

   Claim. *Given $u_0 \in \Omega$, there is a neighborhood $\mathcal{O}$ of u_0 such that if $u_\ell, u_r \in \mathcal{O}$, then there is a weak solution to (7.1) with initial data $u(0,x) = u_\ell$ for $x < 0$, u_r for $x > 0$.*

   What is the difference? Similarly strengthen Proposition 7.3.
5. Consider shock wave solutions to the system produced in Exercise 1 of § 5, namely, spherically symmetric shocks in compressible fluids.
6. Show that a solution to the system (7.1) is given by

$$u(t,x) = v\big(\varphi(t,x)\big), \tag{7.116}$$

   where φ is real-valued, satisfying the scalar conservation law

$$\varphi_t + \lambda_j\big(v(\varphi)\big)\varphi_x = 0, \tag{7.117}$$

   for some j, and $v'(s)$ is parallel to $r_j\big(v(s)\big)$, with λ_j, r_j as in (7.3).
   Such a solution is called a *simple wave*. Rarefaction waves are a special case, called centered simple waves.
   Considering (7.117), study the breakdown of simple waves.

8. Entropy-flux pairs and Riemann invariants

As in § 7, we work with an $L \times L$ system of conservation laws in one space variable:

$$u_t + F(u)_x = 0, \tag{8.1}$$

where u takes values in $\Omega \subset \mathbb{R}^L$ and $F : \Omega \to \mathbb{R}^L$ is smooth. Thus smooth solutions also satisfy

$$u_t + A(u)u_x = 0, \quad A(u) = D_uF(u). \tag{8.2}$$

As noted in § 7, if $u(t,x)$ vanishes sufficiently rapidly as $x \to \pm\infty$, then

$$\int u(t,x)\,dx \in \mathbb{R}^L \tag{8.3}$$

is independent of t; so each component of (8.3) is a conserved quantity.

An *entropy-flux pair* is a pair of functions

$$\eta,\ q : \Omega \longrightarrow \mathbb{R} \tag{8.4}$$

with the property that the equation (8.1) implies

$$\eta(u)_t + q(u)_x = 0 \tag{8.5}$$

as long as u is smooth. If there is such a pair, again given appropriate behavior as $x \to \pm\infty$, we have

$$\frac{d}{dt}\int \eta\big(u(t,x)\big)\,dx = 0, \tag{8.6}$$

so

$$\int \eta\big(u(t,x)\big)\,dx = I_\eta(u) \tag{8.7}$$

is independent of t, hence is another conserved quantity, provided $u(t,x)$ is smooth. As we'll see below, the situation is different for nonsmooth, weak solutions to (8.1).

To produce a more operational characterization of entropy-flux pairs, apply the chain rule to the left side of (8.5), to get $u_t \cdot \nabla\eta(u) + u_x \cdot \nabla q(u)$, and substitute $u_t = -A(u)u_x$ from (8.2), to get

$$\eta(u)_t + q(u)_x = u_x \cdot \big(-A(u)^t\nabla\eta(u) + \nabla q(u)\big). \tag{8.8}$$

Thus the condition for (η, q) to be an entropy-flux pair for (8.1) is that

$$A(u)^t\nabla\eta(u) = \nabla q(u). \tag{8.9}$$

Note that (8.9) consists of L equations in two unknowns. Thus it is overdetermined if $L \ge 3$. For $L \ge 3$, some special structure is usually required to produce nontrivial entropy-flux pairs. For example, if $A(u)$ is *symmetric*, so $\partial F_j/\partial u_k = \partial F_k/\partial u_j$, and if $\Omega \subset \mathbb{R}^L$ is simply connected, we can set $F_\ell(u) = \partial g/\partial u_\ell$. In such a case,

$$\eta(u) = \frac{1}{2}\sum u_j^2, \quad q(u) = \sum u_j F_j(u) - g(u), \tag{8.10}$$

is seen to define an entropy-flux pair. Note that in this case η is a strictly *convex* function of u.

If $L = 2$, then (8.9) is a system of two equations in two unknowns. We can convert it to a single equation for η as follows (assuming Ω is simply connected). The condition that $A_{kj}(u)\,\partial\eta/\partial u_k$ be a gradient field is that

$$\frac{\partial}{\partial u_\ell}\Big(A_{kj}(u)\frac{\partial\eta}{\partial u_k}\Big) = \frac{\partial}{\partial u_j}\Big(A_{k\ell}(u)\frac{\partial\eta}{\partial u_k}\Big), \tag{8.11}$$

for all j, ℓ. We use the summation convention and hence sum over k in (8.11). We need verify (8.11) only for $j < \ell$, hence for $j = 1,\ \ell = 2$, if $L = 2$. Carrying out the differentiation, we can write (8.11) as

$$\left[\delta^m{}_\ell A_{kj}(u) - \delta^m{}_j A_{k\ell}(u)\right]\frac{\partial^2\eta}{\partial u_m \partial u_k} = 0, \quad \forall\, j < \ell. \tag{8.12}$$

In case $L = 2$, this becomes the single equation

$$B_{11}(u)\frac{\partial^2\eta}{\partial u_1^2} + 2B_{12}(u)\frac{\partial^2\eta}{\partial u_2\partial u_1} + B_{22}(u)\frac{\partial^2\eta}{\partial u_2^2} = 0, \tag{8.13}$$

with

$$\begin{aligned} B_{11}(u) &= -A_{12}(u) = -\frac{\partial F_1}{\partial u_2}, \\ B_{22}(u) &= A_{21}(u) = \frac{\partial F_2}{\partial u_1}, \\ 2B_{12}(u) &= A_{11}(u) - A_{22}(u) = \frac{\partial F_1}{\partial u_1} - \frac{\partial F_2}{\partial u_2}. \end{aligned} \tag{8.14}$$

Lemma 8.1. *If (8.2) is a 2×2 system, then (8.13) is a linear hyperbolic equation for η if and only if (8.2) is strictly hyperbolic.*

Proof. The equation (8.13) is hyperbolic if and only if the matrix $B(u) = \big(B_{jk}(u)\big)$ has negative determinant. We have

$$\begin{aligned} \det B(u) &= -A_{12}A_{21} - \frac{1}{4}(A_{11} - A_{22})^2 \\ &= -\frac{1}{4}(A_{11}^2 - 2A_{11}A_{22} + A_{22}^2 + 4A_{12}A_{21}). \end{aligned}$$

Meanwhile,

$$\det\big(A(u) - \lambda\big) = \lambda^2 - (A_{11} + A_{22})\lambda + A_{11}A_{22} - A_{12}A_{21},$$

so $A(u)$ has two real and distinct eigenvalues if and only if

$$(A_{11} + A_{22})^2 - 4(A_{11}A_{22} - A_{12}A_{21}) > 0.$$

This last quantity is seen to be equal to $-4 \det B(u)$, so the lemma is proved.

We will be particularly interested in producing entropy-flux pairs (η, q) such that η is convex. The reason for doing so is explained by the following result, which extends (6.21)–(6.24).

Proposition 8.2. *Consider solutions u_ε of*

$$\partial_t u_\varepsilon + A(u_\varepsilon)\partial_x u_\varepsilon = \varepsilon\partial_x^2 u_\varepsilon, \quad \varepsilon > 0. \tag{8.15}$$

Suppose that, as $\varepsilon \searrow 0$, u_ε *converges boundedly a.e. to* u*, a weak solution of*

$$\partial_t u + \partial_x F(u) = 0. \tag{8.16}$$

If (η, q) *is an entropy-flux pair and* η *is convex, then*

$$\eta(u)_t + q(u)_x \le 0, \tag{8.17}$$

in the sense that this is a nonpositive measure.

Here, if F, η, and q are defined on an open set $\Omega \subset \mathbb{R}^L$, we assume $u_\varepsilon(t, x) \in K \subset\subset \Omega$.

Proof. Take the dot product of (8.15) with $\nabla\eta(u_\varepsilon)$ to get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \nabla\eta(u_\varepsilon) \cdot \partial_x^2 u_\varepsilon. \tag{8.18}$$

Use the identity

$$\eta(v)_{xx} = \nabla\eta(v) \cdot v_{xx} + \sum_{j,k} \eta_{jk}(v)(\partial_x v_j)(\partial_x v_k), \quad \eta_{jk}(v) = \frac{\partial^2 \eta}{\partial v_k \partial v_j}, \tag{8.19}$$

to get

$$\begin{aligned} \eta(u_\varepsilon)_t + q(u_\varepsilon)_x &= \varepsilon \eta(u_\varepsilon)_{xx} - \varepsilon \sum \eta_{jk}(u_\varepsilon)(\partial_x u_{j\varepsilon})(\partial_x u_{k\varepsilon}) \\ &\le \varepsilon \eta(u_\varepsilon)_{xx}, \end{aligned} \tag{8.20}$$

by convexity of η. Now passing to the limit $\varepsilon \to 0$ gives

$$\eta(u_\varepsilon) \to \eta(u), \qquad q(u_\varepsilon) \to q(u), \tag{8.21}$$

boundedly and a.e., hence weak* in L^∞, while the right side of (8.20) tends to 0 in the distributional topology. This yields (8.17).

The inequality (8.17) is called an *entropy condition.*

Suppose u is a weak solution to (8.1) which is smooth on a region $\mathcal{O} \subset \mathbb{R}^2$ except for a simple jump across a curve $\gamma \subset \mathcal{O}$. If (η, q) is an entropy-flux pair, then $\eta(u)_t + q(u)_x = 0$ on $\mathcal{O} \setminus \gamma$. Suppose (8.17) holds for u. Then the negative measure $\eta(u)_t + q(u)_x = -\mu$ is supported on γ; in fact, for continuous φ with compact support in $\mathcal{O}$,

$$-\int \varphi \, d\mu = \int \big(s[\eta] - [q]\big)\varphi \, d\sigma, \tag{8.22}$$

where $[\eta]$ and $[q]$ are the jumps of η and q across γ, in the direction of increasing t; $s = dx/dt$ on γ is the shock speed; and $d\sigma$ is the arclength along γ. Consequently, such an entropy-satisfying weak solution of (8.1) has the property

$$s[\eta] - [q] \le 0 \quad \text{on} \ \ \gamma. \tag{8.23}$$

We remarked in § 7 that if u_ℓ and u_r are close and the Riemann problem (7.17) has a solution consisting of a j-shock, satisfying the Lax shock condition (7.52), then u_ℓ and u_r are connected by a viscous profile; we sketched a proof for 2×2 systems. It follows from Proposition 8.2 that such solutions satisfy the entropy condition (8.17), for all convex entropies.

We give some explicit examples of entropy-flux pairs. First consider the system (7.14), namely,

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - K(v)_x &= 0, \end{aligned} \tag{8.24}$$

for which $\lambda_\pm$ and $r_\pm$ are given by (7.16). In this case, one can use

$$\eta(v, w) = \frac{1}{2} w^2 + \int_{v_0}^{v} K(s)\, ds, \quad q(v, w) = -wK(v). \tag{8.25}$$

Note that η is strongly convex as long as $K'(v) > 0$.

For the equation (7.5) of isentropic compressible fluid flow, we can set

$$\eta(v, \rho) = \frac{1}{2} v^2 \rho + X(\rho), \quad X'(\rho) = \int_0^\rho \frac{p'(s)}{s}\, ds, \tag{8.26}$$

which is the total energy, with flux

$$q(v, \rho) = \Bigl(\frac{1}{2} v^2 \rho + X'(\rho)\rho\Bigr) v. \tag{8.27}$$

In the (ρ, m)-coordinates used to express the PDE in conservation form (7.8), we have

$$\eta(\rho, m) = \frac{m^2}{2\rho} + X(\rho). \tag{8.28}$$

In this case

$$D^2\eta = \begin{pmatrix} \frac{p'(\rho)}{\rho} + \frac{m^2}{\rho^3} & -\frac{m}{\rho^2} \\ -\frac{m}{\rho^2} & \frac{1}{\rho} \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} p'(\rho) + v^2 & -v \\ -v & 1 \end{pmatrix}, \tag{8.29}$$

so $\eta(\rho, m)$ is strongly convex as long as $p'(\rho) > 0$.

We aim to present a construction of P. Lax of a large family of entropy-flux pairs, for 2×2 systems. In order to do this, and also for further analysis in § 9, it is useful to introduce the concept of a Riemann invariant. If $A(u) = D_u F(u)$ has eigenvalues and eigenvectors $\lambda_j(u),\ r_j(u)$, as in (7.3), we say a smooth function $\xi : \Omega \to \mathbb{R}$ is a k-Riemann invariant provided $r_k \cdot \nabla\xi = 0$.

In the case of a system of the form (7.14), where $r_\pm = (1, -\lambda_\pm)^t,\ \lambda_\pm = \lambda_\pm(v)$, we see that Riemann invariants are constant on integral curves of $\partial/\partial v - \lambda_\pm(v)\,\partial/\partial w$, that is, curves satisfying $dv/dw = -1/\lambda_\pm(v)$, so we can take

$$\xi_\pm(v, w) = w + \int_{v_0}^{v} \lambda_\pm(s)\, ds = w \pm \int_{v_0}^{v} \sqrt{K'(s)}\, ds. \tag{8.30}$$

Also, any functions of these are Riemann invariants.

In the case of the system (7.8) for compressible fluids (in (ρ, m) coordinates), where we have $r_\pm = (1, \lambda_\pm)^t,\ \lambda_\pm = m/\rho \pm \sqrt{p'(\rho)}$, the Riemann invariants are constant on integral curves of $\partial/\partial\rho + \lambda_\pm \partial/\partial m$ (i.e., curves satisfying $dm/d\rho = m/\rho \pm \sqrt{p'(\rho)}$). If we switch to (ρ, v)-coordinates, with $v = m/\rho$, then $dm/d\rho = \rho\, dv/d\rho + v$, so these level curves satisfy $\rho\, dv/d\rho = \pm\sqrt{p'(\rho)}$. Hence we can take

$$\xi_\pm(\rho, v) = v \mp \int_{\rho_0}^{\rho} \frac{\sqrt{p'(s)}}{s}\, ds = v \mp \frac{2\sqrt{A\gamma}}{\gamma - 1}\, \rho^{(\gamma-1)/2}, \tag{8.31}$$

the latter identity holding when $p(\rho) = A\rho^\gamma$, with $\gamma > 1$, and we take $\rho_0 = 0$.

The following is a useful characterization of Riemann invariants.

Proposition 8.3. *Suppose that (8.1) is a strictly hyperbolic 2×2 system and that Ω has a coordinate system (ξ_1, ξ_2), such that ξ_k is a k-Riemann invariant. Then, for $k = 1, 2$,*

$$A(u)^t \nabla\xi_k(u) = \lambda_j(u) \nabla\xi_k(u), \quad j \neq k. \tag{8.32}$$

Conversely, for $j = 1, 2$,

$$A(u)^t \nabla\xi(u) = \lambda_j(u)\nabla\xi(u) \Longrightarrow \xi \text{ is a } k\text{-Riemann invariant},\ k \neq j. \tag{8.33}$$

Proof. Since $\{\nabla\xi_1(u), \nabla\xi_2(u)\}$ is a basis of $\mathbb{R}^2$ for each $u \in \Omega$, we see that

$$r_k(u) \cdot \zeta(u) = 0 \Longrightarrow \zeta(u) = \alpha(u)\nabla\xi_k(u), \tag{8.34}$$

for some scalar $\alpha(u)$. Meanwhile,

$$A(u)^t \zeta(u) = \lambda_j(u)\zeta(u) \Longleftrightarrow r_k \cdot A(u)^t \zeta = \lambda_j r_k \cdot \zeta. \tag{8.35}$$

Since also $r_k \cdot A(u)^t \zeta = \zeta \cdot A(u) r_k = \lambda_k r_k \cdot \zeta$ and $\lambda_j \neq \lambda_k$, we see that

$$\begin{aligned} A(u)^t \zeta(u) = \lambda_j(u) &\Longrightarrow r_k(u) \cdot \zeta(u) = 0 \\ &\Longrightarrow \zeta(u) = \alpha(u) \nabla \xi_k(u), \end{aligned} \tag{8.36}$$

the last implication by (8.34). However, since $A(u)^t$ does have a nonzero λ_j-eigenspace, this yields (8.32). It also establishes the converse, (8.33).

Proposition 8.3 has the following consequence:

Proposition 8.4. *Suppose that (8.1) is a strictly hyperbolic* 2×2 *system and that* Ω *has a coordinate system* ξ_k, $k = 1, 2$, *consisting of* k*-Riemann invariants. If* u *is a Lipschitz solution of (8.1), then*

$$\begin{aligned} \partial_t \xi_1(u) + \lambda_2(u) \partial_x \xi_1(u) = 0, \\ \partial_t \xi_2(u) + \lambda_1(u) \partial_x \xi_2(u) = 0. \end{aligned} \tag{8.37}$$

Proof. For $j \neq k$, we have

$$\begin{aligned} \partial_t \xi_j(u) + \lambda_k(u) \partial_x \xi_j(u) &= \partial_t u \cdot \nabla \xi_j(u) + \lambda_k(u) \partial_x u \cdot \nabla \xi_j(u) \\ &= \partial_t u \cdot \nabla \xi_j(u) + \partial_x u \cdot A(u)^t \nabla \xi_j(u) \\ &= \bigl(\partial_t u + A(u) \partial_x u\bigr) \cdot \nabla \xi_j(u), \end{aligned} \tag{8.38}$$

the second identity by (8.32). This proves (8.37).

Following [L4], we now present a geometrical-optics-type construction of solutions to (8.9), for certain 2×2 systems, which yields convex entropy functions in favorable circumstances. We look for solutions of the form

$$\begin{aligned} \eta &= e^{k\varphi}\bigl(\eta_0 + k^{-1}\eta_1 + \cdots + k^{-N}\eta_N + \widetilde{\eta}_N\bigr), \\ q &= e^{k\varphi}\bigl(q_0 + k^{-1}q_1 + \cdots + k^{-N}q_N + \widetilde{q}_N\bigr), \end{aligned} \tag{8.39}$$

where $\varphi = \varphi(u)$, $\eta_j = \eta_j(u)$, $q_j = q_j(u)$, k is a parameter that will be taken large, and we will have $\widetilde{\eta}_N, \widetilde{q}_N = O(k^{-N})$. In fact, plugging this ansatz into (8.9) and equating like powers of k, we obtain

$$q_0 \nabla \varphi = \eta_0 A(u)^t \nabla \varphi, \tag{8.40}$$

and, for $0 \leq j \leq N-1$,

$$n q_{j+1} \nabla \varphi + \nabla q_j = \eta_{j+1} A(u)^t \nabla \varphi + A(u)^t \nabla \eta_j. \tag{8.41}$$

If $\eta_0 \neq 0$, (8.40) says

$$A(u)^t \nabla \varphi = \frac{q_0}{\eta_0} \nabla \varphi, \tag{8.42}$$

so q_0/η_0 is an eigenvalue of $A(u)^t$ and $\nabla\varphi$ an associated eigenvector. By Proposition 8.3, the equation (8.40) holds provided we take

$$q_0 = \lambda_\ell \eta_0, \quad \varphi = \xi_k, \quad k \neq \ell, \tag{8.43}$$

where λ_ℓ is one of the two eigenvalues of $A(u)$ and ξ_k is a k-Riemann invariant. We have solved the eikonal equation for φ. For definiteness, let us take $k = 1$, $\ell = 2$.

Rather than tackle (8.41) directly, let us note that (8.9) is equivalent to

$$R_\nu q = \lambda_\nu R_\nu \eta, \quad R_\nu = r_\nu \cdot \nabla, \quad \nu = 1, 2. \tag{8.44}$$

Thus we can rewrite (8.40) and (8.41) as

$$q_0 R_\nu \varphi = \eta_0 \lambda_\nu R_\nu \varphi, \quad \nu = 1, 2, \tag{8.45}$$

and, for $0 \le j \le N-1$,

$$q_{j+1} R_\nu \varphi + R_\nu q_j = \eta_{j+1} \lambda_\nu R_\nu \varphi + \lambda_\nu R_\nu \eta_j, \quad \nu = 1, 2. \tag{8.46}$$

Clearly, (8.43) yields φ, q_0, η_0 satisfying (8.45). We have

$$R_1 \varphi = 0, \quad R_2 \varphi = R_2 \xi_1. \tag{8.47}$$

Thus (8.46) takes the form

$$R_1 q_j = \lambda_1 R_1 \eta_j, \quad (q_{j+1} - \lambda_2 \eta_{j+1})(R_2 \xi_k) = \lambda_2 R_2 \eta_j - R_2 q_j. \tag{8.48}$$

For $j = 0$, using $q_0 = \lambda_2 \eta_0$, we obtain the transport equation

$$R_1 \eta_0 + \frac{R_1 \lambda_2}{\lambda_2 - \lambda_1} \eta_0 = 0, \tag{8.49}$$

which is an ODE along each integral curve of R_1. This specifies η_0, given initial data on a curve transverse to R_1, and then q_0 is specified by (8.43). We can arrange that $\eta_0 > 0$. Note that this specification of η_0, q_0 is independent of the choice of 1-Riemann invariant $\varphi = \xi_1$.

Similarly the higher transport equations (i.e., (8.48) for $j \ge 1$), give η_j and q_j, for $j \ge 1$. Compare the geometrical optics construction in § 6 of Chap. 6. Once the transport equations have been solved to high order, one is left with a nonhomogeneous, linear hyperbolic system to solve, to obtain exact solutions (η, q) to (8.9).

It is also useful to write the transport equation (8.46) using the Riemann invariants (ξ_1, ξ_2) as coordinates on Ω, if that can be done. We obtain for $\eta = \eta(\xi_1, \xi_2)$ and $q = q(\xi_1, \xi_2)$ the system

$$\frac{\partial q}{\partial \xi_1} = \lambda_2 \frac{\partial \eta}{\partial \xi_1}, \quad \frac{\partial q}{\partial \xi_2} = \lambda_1 \frac{\partial \eta}{\partial \xi_2}, \tag{8.50}$$

equivalent to (8.9) and to (8.44), and, if $\varphi = \xi_1$, then (8.46) becomes

$$\frac{\partial q_j}{\partial \xi_2} = \lambda_1 \frac{\partial \eta_j}{\partial \xi_2}, \quad q_{j+1} - \lambda_2 \eta_{j+1} = \lambda_2 \frac{\partial \eta_j}{\partial \xi_1} - \frac{\partial q_j}{\partial \xi_1}. \tag{8.51}$$

The equation (8.49) takes the form

$$\frac{\partial \eta_0}{\partial \xi_2} + \frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2}{\partial \xi_2} \eta_0 = 0. \tag{8.52}$$

As stated in (8.43), we have $q_0 = \lambda_2 \eta_0$. We also record one implication of (8.51) for η_1, q_1:

$$q_1 - \lambda_2 \eta_1 = -\frac{\partial \lambda_2}{\partial \xi_1} \eta_0. \tag{8.53}$$

In particular, if $\eta_0 > 0$, then $q_1 - \lambda_2 \eta_1$ has the opposite sign to $\partial \lambda_2 / \partial \xi_1$ (if this is nonvanishing, which is the case if (8.1) is genuinely nonlinear).

Since it is of interest to have convex entropies η, we make note of the following result, whose proof involves a straightforward calculation:

Proposition 8.5. *If $\eta(k)$ is given by (8.39), with $\eta_0 > 0$ on Ω, then, for k sufficiently large and positive, $\eta(k)$ is strongly convex on any given $\Omega_0 \subset\subset \Omega$, provided $\nabla \varphi \neq 0$ on Ω and, at any point $u_0 \in \Omega$, if $V = a_1\, \partial/\partial u_1 + a_2\, \partial/\partial u_2$, is a unit vector orthogonal to $\nabla \varphi(u_0)$, then*

$$V^2 \varphi(u_0) > 0. \tag{8.54}$$

If φ satisfies the hypotheses of Proposition 8.5, we say φ is (strongly) quasi-convex. Clearly, (8.54) implies that a tangent line to $\{\varphi = c\}$ at u_0 lies in $\{\varphi > c\}$ on a punctured neighborhood of u_0. Equivalently, φ is quasi-convex on $\Omega \subset \mathbb{R}^2$ if and only if the curvature vector of each level curve $\{\varphi = c\}$ at any point $u_0 \in \Omega$ is antiparallel to the vector $\nabla \varphi(u_0)$. Note that if Ω is convex and φ is quasi-convex on Ω, then each region $\{\varphi \leq c\}$ is convex.

Thus a favorable situation for exploiting the construction (8.39) to obtain a strongly convex entropy is one where Ω has a coordinate system (ξ_1, ξ_2) consisting of quasi-convex Riemann invariants. Note that if this is the case, we can form $\zeta_j = e^{\lambda \xi_j}$, for some large constant λ, and obtain a coordinate system consisting of strongly convex Riemann invariants.

Consider the Riemann invariants $\xi_\pm$ of (8.30), for the system (7.14), containing models of elasticity. We see that ξ_+ and $-\xi_-$ are quasi-convex, where $K''(v) > 0$, and that $-\xi_+$ and ξ_- are quasi-convex, where $K''(v) < 0$, granted that $K'(v)$ is nowhere vanishing. As for the Riemann invariants (8.31) for the system (7.7) of compressible fluid flow, with variables (v, ρ), given $1 < \gamma < 3$ we have ξ_+ and $-\xi_-$ quasi-convex on $\{(v, \rho) : \rho > 0\}$.

We end this section with the remark that the proof of Proposition 8.4 provides just a taste of the use of geometrical optics in nonlinear PDE, extending such developments of geometrical optics for linear PDE as discussed in Chap. 6. For further results on nonlinear geometrical optics, one can consult [JMR] and [Kic], and references given therein. In particular, [Kic] describes how constructions of nonlinear geometrical optics lead to such "soliton equations" as the Korteweg–deVries equation, the sine-Gordon equation, and the "nonlinear Schrödinger equation." Studies of propagation of weak singularities of solutions to nonlinear equations, initiated in [Bon] and [RR], have also been pursued in a number of papers. Expositions of some of these results are given in [Bea, H, Tay].

Exercises

1. Assume (η, q) is an entropy-flux pair for (8.1), and fix $u_0 \in \Omega$. Show that

$$\begin{aligned} \widetilde{\eta}(u) &= \eta(u) - \eta(u_0) - (u - u_0) \cdot \nabla\eta(u_0), \\ \widetilde{q}(u) &= q(u) - q(u_0) - \big(F(u) - F(u_0)\big) \cdot \nabla\eta(u_0) \end{aligned} \tag{8.55}$$

is also an entropy-flux pair. Note that if η is strictly convex, then $\widetilde{\eta}(u) \geq 0$, and it vanishes if and only if $u = u_0$.

Exercises 2–4 involve an $L \times L$ system of conservation laws in n space variables:

$$u_t + \sum_{j=1}^{n} \partial_j F_j(u) = 0, \tag{8.56}$$

where $F_j : \Omega \to \mathbb{R}^L$, Ω open in $\mathbb{R}^L$. An entropy-flux pair is a pair of functions

$$\eta : \Omega \to \mathbb{R}, \quad q : \Omega \to \mathbb{R}^n,$$

satisfying

$$A_j(u)^t \nabla\eta(u) = \nabla q_j(u), \quad 1 \leq j \leq n, \tag{8.57}$$

where $A_j(u) = D_u F_j(u), q(u) = \big(q_1(u), \dots, q_n(u)\big)$. This material is from [FL2].

2. Show that if (8.57) holds, then any smooth solution to (8.56) also satisfies

$$\eta(u)_t + \sum_j \partial_j q_j(u) = 0.$$

3. Show that if each $A_j(u)$ is a symmetric $L \times L$ matrix, then an entropy-flux pair is given by

$$\eta(u) = \frac{1}{2}|u|^2, \quad q_j(u) = \sum_\ell u_\ell F_{j\ell}(u) - g_j(u),$$

where $F_j(u) = \big(F_{j1}(u), \dots, F_{jL}(u)\big), \; F_{j\ell}(u) = \partial g_j / \partial u_\ell$.

4. Show that if there is an entropy-flux pair (η, q) such that η is strongly convex, then the positive-definite, $L \times L$ matrix $(\partial^2\eta/\partial u_j\,\partial u_k)$ is a symmetrizer for (8.56).

In Exercises 5–7, let $u_\varepsilon = (v_\varepsilon, w_\varepsilon)$ be smooth solutions to

$$\begin{aligned} \partial_t v_\varepsilon - \partial_x w_\varepsilon &= 0, \\ \partial_t w_\varepsilon - \partial_x K(v_\varepsilon) &= \varepsilon \partial_x \partial_t v_\varepsilon, \end{aligned} \tag{8.58}$$

for $t \geq 0$. Assume $\varepsilon > 0$.

5. If either $x \in S^1$ or the functions in (8.58) decrease fast enough as $|x| \to \infty$, show that

$$E(t) = \int \Big[\frac{1}{2} w(t,x)^2 + K\big(v(t,x)\big)\Big]\,dx$$

satisfies

$$\frac{dE}{dt} = -\varepsilon \int w_x(t,x)^2\,dx.$$

6. If (η, q) is an entropy-flux pair for

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - K(v)_x &= 0, \end{aligned} \tag{8.59}$$

show that

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon \frac{\partial \eta}{\partial w}(u_\varepsilon)\,\partial_x^2 w_\varepsilon.$$

If

$$\frac{\partial \eta}{\partial w}(u_\varepsilon) = \zeta'(w_\varepsilon), \tag{8.60}$$

and ζ is convex (which holds for (η, q) given by (8.25)), deduce that

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) \leq \varepsilon \partial_x^2 \zeta(w_\varepsilon).$$

7. Now suppose that, as $\varepsilon \searrow 0$, u_ε converges boundedly to $u = (v, w)$, a weak solution to (8.59). If (η, q) is an entropy-flux pair and (8.60) holds, with ζ convex, deduce that

$$\eta(u)_t + q(u)_x \leq 0, \tag{8.61}$$

in the same sense as (8.17). Taking (η, q) as in (8.25), deduce that if u has a jump across γ, as in (8.23), then

$$\begin{aligned} K(v_\ell)(v_r - v_\ell) &\leq \int_{v_\ell}^{v_r} K(\sigma)\,d\sigma - \frac{1}{2}(w_r - w_\ell)^2, \\ K(v_r)(v_r - v_\ell) &\leq \int_{v_\ell}^{v_r} K(\sigma)\,d\sigma + \frac{1}{2}(w_r - w_\ell)^2. \end{aligned}$$

9. Global weak solutions of some 2×2 systems

Here we establish existence, for all $t \geq 0$, of entropy-satisfying weak solutions to a class of 2×2 systems of conservation laws in one space variable:

$$u_t + F(u)_x = 0, \quad u(0,x) = f(x). \tag{9.1}$$

We will take $x \in S^1 = \mathbb{R}/\mathbb{Z}$; modifications for $x \in \mathbb{R}$ are not difficult. We assume f takes values in a certain convex open set $\Omega \subset \mathbb{R}^2$ and $F : \Omega \to \mathbb{R}^2$ is smooth. As before, we set $A(u) = DF(u)$, a 2×2 matrix-valued function of u. We assume strict hyperbolicity; namely, $A(u)$ has real, distinct eigenvalues $\lambda_1(u) < \lambda_2(u)$, with associated eigenvectors $r_1(u),\ r_2(u)$. We will assume that Ω has a global coordinate system (ξ_1, ξ_2), where $\xi_j \in C^\infty(\Omega)$ is a j-Riemann invariant. In fact, we assume that ξ maps Ω diffeomorphically onto a region

$$\mathcal{R} = \{\xi : A_1 < \xi_1 < B_1, A_2 < \xi_2 < B_2\},$$

where $-\infty \leq A_j < B_j \leq +\infty$. The assumptions stated in this paragraph will be called the "standard hypotheses" on (9.1).

We will obtain a solution to (9.1) as a limit of solutions to

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_x^2 u_\varepsilon, \quad u_\varepsilon(0,x) = f(x). \tag{9.2}$$

Methods of Chap. 15, § 1 (particularly Proposition 1.3 there), yield, for any $\varepsilon > 0$, a solution $u_\varepsilon(t)$, defined for $0 \leq t < T(\varepsilon)$, given any $f \in L^\infty(S^1)$, taking values in a compact subset of Ω. The solution is C^∞ on $(0, T(\varepsilon)) \times S^1$ and continues as long as we have

$$u_\varepsilon(t,x) \in K, \tag{9.3}$$

for some compact $K \subset \Omega$. For now we make the hypothesis that (9.3) holds, for all $t \geq 0$. We also have the identity

$$\|u_\varepsilon(t)\|_{L^2}^2 + \varepsilon \int_0^t \|\partial_x u_\varepsilon(s)\|_{L^2}^2 \, ds = \|f\|_{L^2}^2. \tag{9.4}$$

To study the behavior of the solutions u_ε to (9.2) as $\varepsilon \to 0$, we use the theory of Young measures, developed in § 11 of Chap. 13. By Proposition 11.3 of Chap. 13, there exists a sequence $u_j = u_{\varepsilon_j}$, with $\varepsilon_j \to 0$, and an element $(u, \lambda) \in Y^\infty(\mathbb{R}^+ \times S^1)$ such that

$$u_j \to (u, \lambda) \quad \text{in } Y^\infty(\mathbb{R}^+ \times S^1). \tag{9.5}$$

By Proposition 11.1 and Corollary 11.2 of Chap. 13,

$$F(u_j) \to \overline{F} \ \text{ weak}^* \text{ in } \ L^\infty(\mathbb{R}^+ \times S^1), \tag{9.6}$$

where

$$\overline{F}(x) = \int_{\mathbb{R}^2} F(y)\, d\lambda_{t,x}(y), \quad \text{a.e. } (t,x) \in \mathbb{R}^+ \times S^1. \tag{9.7}$$

Since $\varepsilon\partial_x^2 u_j \to 0$ in $\mathcal{D}'(\mathbb{R}^+ \times S^1)$, this implies

$$\partial_t u + \partial_x \overline{F} = 0. \tag{9.8}$$

To conclude that u is a weak solution to (9.1), we need to show that $\overline{F} = F(u)$, which will follow if we can show that the convergence $u_j \to (u,\lambda)$ in $Y^\infty(\mathbb{R}^+ \times S^1)$ is *sharp* (i.e., $\lambda = \gamma_u$), or equivalently, that $\lambda_{t,x}$ is a point mass on $\mathbb{R}^2$, for almost every $(t,x) \in \mathbb{R}^+ \times S^1$.

Following [DiP4] we use entropy-flux pairs as a tool for examining λ, in a chain of reasoning parallel to, but somewhat more elaborate than, that used to treat the scalar case in § 11 of Chap. 13.

For any smooth entropy-flux pair (η, q), we have

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon\partial_x^2 \eta(u_\varepsilon) - \varepsilon\partial_x u_\varepsilon \cdot \eta''(u_\varepsilon)\, \partial_x u_\varepsilon, \tag{9.9}$$

where $\eta''(u_\varepsilon)$ is the 2×2 Hessian matrix of second-order partial derivatives of η. We have the identity

$$\varepsilon \int_0^T \int \partial_x u_\varepsilon \cdot \eta''(u_\varepsilon)\, \partial_x u_\varepsilon \, dx\, dt = \int \eta\big(f(x)\big)\, dx - \int \eta\big(u_\varepsilon(T,x)\big)\, dx. \tag{9.10}$$

We rewrite (9.9) as

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon\partial_x^2 \eta(u_\varepsilon) - R_\varepsilon, \tag{9.11}$$

with

$$R_\varepsilon \ \text{ bounded in } \ L^1(\mathbb{R}^+ \times S^1). \tag{9.12}$$

If η is convex, this follows directly from (9.10), since then the left side of (9.10) is the integral of a positive quantity. But even if η is not assumed to be convex, we can appeal to (9.4) to say $\sqrt{\varepsilon}\partial_x u_\varepsilon$ is bounded in $L^2(\mathbb{R}^+ \times S^1)$, and this plus (9.3) implies (9.12).

Since $\partial_x \eta(u_\varepsilon) = \eta'(u_\varepsilon)\, \partial_x u_\varepsilon$, we also deduce from (9.3)–(9.4) that the quantity $\sqrt{\varepsilon}\partial_x \eta(u_\varepsilon)$ is bounded in $L^2(\mathbb{R}^+ \times S^1)$. Hence

$$\varepsilon\partial_x^2 \eta(u_\varepsilon) \to 0 \ \text{ in } \ H^{-1}(\mathbb{R}^+ \times S^1), \ \text{ as } \ \varepsilon \to 0. \tag{9.13}$$

Now we can apply Lemma 12.6 of Chap. 13 (Murat's lemma) to deduce from (9.11)–(9.13) that

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) \ \text{ is precompact in } \ H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times S^1). \tag{9.14}$$

Now, let (η_1, q_1) and (η_2, q_2) be any two entropy-flux pairs, and consider the vector-valued functions

$$v_j = \big(\eta_1(u_j), q_1(u_j)\big), \quad w_j = \big(q_2(u_j), -\eta_2(u_j)\big), \tag{9.15}$$

where u_j is as in (9.5). By (9.14), we have

$$\operatorname{div} v_j, \quad \operatorname{rot} w_j \quad \text{precompact in } \ H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times S^1). \tag{9.16}$$

Also the L^∞ bound on u_j implies that v_j and w_j are bounded in $L^\infty(\mathbb{R}^+ \times S^1)$, and a fortiori in $L^2_{\mathrm{loc}}(\mathbb{R}^+ \times S^1)$. Therefore, we can apply the div-curl lemma, either in the form developed in the exercises after § 8 of Chap. 5 or in the form developed in the exercises after § 6 of Chap. 13. We have

$$v_j \cdot w_j \to v \cdot w \ \text{ in } \ \mathcal{D}'(\mathbb{R}^+ \times S^1), \quad v = (\overline{\eta}_1, \overline{q}_1), \ w = (\overline{q}_2, -\overline{\eta}_2). \tag{9.17}$$

In view of the L^∞-bounds, we hence have

$$\begin{gathered} \eta_1(u_j) q_2(u_j) - \eta_2(u_j) q_1(u_j) \longrightarrow \overline{\eta}_1 \overline{q}_2 - \overline{\eta}_2 \overline{q}_1 \\ \text{weak}^* \ \text{in} \ L^\infty(\mathbb{R}^+ \times S^1). \end{gathered} \tag{9.18}$$

Recall that we want to show that any measure $\nu = \lambda_{t,x}$, arising in the disintegration of the measure λ in (9.5), is supported at a point. We are assuming that there are global coordinates (ξ_1, ξ_2) on Ω consisting of Riemann invariants. Let

$$R = \{\xi : a_j^- \le \xi_j \le a_j^+\} \tag{9.19}$$

be a minimal rectangle (in ξ-coordinates) containing the support of ν. The following provides the key technical result:

Lemma 9.1. *If $a_1^- < a_1^+$, then each closed vertical side of R must contain a point where $\partial \lambda_2 / \partial \xi_1 = 0$.*

Proof. We have from (9.18) that

$$\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle, \tag{9.20}$$

for all entropy-flux pairs (η_j, q_j). Let $\big(\eta(k), q(k)\big)$ be a family of entropy-flux pairs of the form (8.39), with $k \in \mathbb{R}$, $|k|$ large, so that $\eta(k) > 0$. Thus, for $|k|$ large, we can define a probability measure μ_k by

$$\langle \mu_k, f \rangle = \frac{\langle \nu, \eta(k) f \rangle}{\langle \nu, \eta(k) \rangle}. \tag{9.21}$$

We can take a subsequence $k_n \to +\infty$ such that

$$\mu_{k_n} \to \mu^+, \ \mu_{-k_n} \to \mu^-, \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega). \tag{9.22}$$

In view of the exponential factor $e^{k\xi_1}$ in $\eta(k)$, it is clear that

$$\operatorname{supp} \mu^{\pm} \subset R \cap \{\xi_1 = a_1^{\pm}\}. \tag{9.23}$$

Now set $\lambda_2^{\pm} = \langle \mu^{\pm}, \lambda_2 \rangle$. We claim that

$$\langle \nu, q - \lambda_2^{\pm} \eta \rangle = \langle \mu^{\pm}, q - \lambda_2 \eta \rangle, \tag{9.24}$$

for every entropy-flux pair (η, q). To establish this, use (9.20) with $(\eta_1, q_1) = (\eta, q)$ and $(\eta_2, q_2) = \big(\eta(k), q(k)\big)$. We get

$$\frac{\langle \nu, \eta q(k) - \eta(k) q \rangle}{\langle \nu, \eta(k) \rangle} = \langle \nu, \eta \rangle \frac{\langle \nu, q(k) \rangle}{\langle \nu, \eta(k) \rangle} - \langle \nu, q \rangle. \tag{9.25}$$

Since, by (8.43), $q_0 = \lambda_2 \eta_0$ in the expansion (8.39), we have

$$\frac{\langle \nu, q(k) \rangle}{\langle \nu, \eta(k) \rangle} = \frac{\langle \nu, \lambda_2 \eta(k) \rangle}{\langle \nu, \eta(k) \rangle} + O(k^{-1}) = \langle \mu_k, \lambda_2 \rangle + O(k^{-1}). \tag{9.26}$$

Now, letting $k = \pm k_n$ and passing to the limit yield $\langle \mu^{\pm}, \lambda_2 \rangle = \lambda_2^{\pm}$ for (9.26). Similarly,

$$\frac{\langle \nu, \eta q(k) \rangle}{\langle \nu, \eta(k) \rangle} \to \langle \mu^{\pm}, \lambda_2 \eta \rangle, \tag{9.27}$$

so (9.25) yields (9.24) in the limit.

Now, use (9.20) with $(\eta_1, q_1) = \big(\eta(k), q(k)\big)$, $(\eta_2, q_2) = \big(\eta(-k), q(-k)\big)$. Thus

$$\frac{\langle \nu, \eta(k) q(-k) - \eta(-k) q(k) \rangle}{\langle \nu, \eta(k) \rangle \langle \nu, \eta(-k) \rangle} = \frac{\langle \nu, q(-k) \rangle}{\langle \nu, \eta(-k) \rangle} - \frac{\langle \nu, q(k) \rangle}{\langle \nu, \eta(k) \rangle}. \tag{9.28}$$

The right side converges to $\lambda_2^- - \lambda_2^+$ as $k = k_n \to +\infty$. Meanwhile, note that $\eta(k) q(-k) - \eta(-k) q(k) = O(k^{-1})$. Also $\langle \nu, \eta(k) \rangle \langle \nu, \eta(-k) \rangle \to +\infty$, faster than $e^{k(a_1^+ - a_1^- - \varepsilon)}$, by the definition of R, if $a_1^- < a_1^+$. Thus the left side of (9.28) tends to zero. We deduce that

$$\lambda_2^+ = \lambda_2^-. \tag{9.29}$$

The identities (9.24) and (9.29) imply that

$$\langle \mu^+, q - \lambda_2 \eta \rangle = \langle \mu^-, q - \lambda_2 \eta \rangle, \tag{9.30}$$

for every entropy-flux pair (η, q). Now with $(\eta, q) = \big(\eta(k), q(k)\big)$, we have

$$\langle \mu^\pm, q - \lambda_2 \eta \rangle = e^{k a_1^\pm} \big\langle \mu^\pm, (q_1 - \lambda_2 \eta_1) k^{-1} + O(k^{-2}) \big\rangle, \tag{9.31}$$

where $\eta_1 k^{-1}$ and $q_1 k^{-1}$ are the second terms in the expansion (8.39). If $a_1^- < a_1^+$, the identity of these two expressions forces $\langle \mu^\pm, q_1 - \lambda_2 \eta_1 \rangle = 0$. By (8.53), this implies

$$\Big\langle \mu^\pm, \eta_0 \frac{\partial \lambda_2}{\partial \xi_1} \Big\rangle = 0. \tag{9.32}$$

Since $\mu^\pm$ are probability measures and $\eta_0 > 0$, this forces $\partial \lambda_2 / \partial \xi_1$ to change sign on supp $\mu^\pm$, proving the lemma.

Corollary 9.2. *If (9.1) is genuinely nonlinear, so $\partial \lambda_1 / \partial \xi_2$ and $\partial \lambda_2 / \partial \xi_1$ are both nowhere vanishing, then ν is supported at a point.*

We therefore have the following result:

Theorem 9.3. *Assume that (9.1) satisfies the standard hypotheses and that solutions u_ε to (9.2) satisfy (9.3). If (9.1) is genuinely nonlinear, then there is a sequence $u_{\varepsilon_j} \to u$, converging boundedly and pointwise a.e., such that u solves (9.1). Also, u satisfies the entropy inequality $\partial_t \eta(u) + \partial_x q(u) \le 0$, for every entropy-flux pair (η, q) such that η is convex (on a neighborhood of K).*

Certain cases of (9.1) that satisfy the standard hypotheses but for which genuine nonlinearity fails, not everywhere on Ω, but just on a curve, are amenable to treatment via the following extension of Lemma 9.1:

Lemma 9.4. *If both characteristic fields of (9.1) are genuinely nonlinear outside a curve $\xi_2 = \psi(\xi_1)$, with ψ strictly monotone, then ν is supported at a point.*

Proof. By Lemma 9.1, each closed side of the rectangle R must intersect this curve, so it must go through a pair of opposite vertices of R; call them P and Q. By (9.32), we see that μ^+ and μ^- must be supported at these points. Thus (9.24) and (9.29) imply that

$$q(Q) - \lambda_2(Q)\eta(Q) = q(P) - \lambda_2(P)\eta(P). \tag{9.33}$$

We have the same sort of identity with λ_2 replaced by λ_1, so

$$\big[\lambda_2(Q) - \lambda_1(Q)\big]\eta(Q) = \big[\lambda_2(P) - \lambda_1(P)\big]\eta(P), \tag{9.34}$$

for every entropy η. In particular, we can take $\eta(u) = u_j - Q_j,\ j = 1,2$, to deduce from (9.34) that $P = Q$, since the strict hyperbolicity hypothesis implies $\lambda_2(P) - \lambda_1(P) \neq 0$. This implies R is a point, so the lemma is proved.

For an example of when this applies, consider the system (7.14), namely,

$$\begin{aligned} v_t - w_x &= 0, \\ w_t - K(v)_x &= 0, \end{aligned} \tag{9.35}$$

which, by (7.16), is strictly hyperbolic provided $K'(v) \neq 0$. By (7.55),

$$r_\pm \cdot \nabla\lambda_\pm = \pm\frac{1}{2}K_v^{-1/2}K_{vv}, \tag{9.36}$$

so we have genuine nonlinearity provided $K''(v) \neq 0$. However, in cases modeling the transverse vibrations of a string, by (7.12), we might have, for example,

$$K(v) = v + av^3, \tag{9.37}$$

for some positive constant a. Then $K'(v) = 1 + 3av^2 > 0$, but $K''(v) = 6av$ vanishes, at $v = 0$. In this case, Riemann invariants are given by (8.30), that is,

$$\xi_\pm = w \pm \int_0^v \sqrt{K'(s)}\, ds, \tag{9.38}$$

so genuine nonlinearity fails on the line $\xi_+ = \xi_-$ (i.e., $\xi_2 = \xi_1$). Thus Lemma 9.4 applies in this case.

To make use of Theorem 9.3 and the analogous consequence of Lemma 9.4, we need to verify (9.3). The following result of [CCS] is sometimes useful for this:

Proposition 9.5. *Let $\overline{\mathcal{O}} \subset \Omega \subset \mathbb{R}^2$ be a compact, convex region whose boundary consists of a finite number of level curves γ_j of Riemann invariants, ξ_j, such that $\nabla\xi_j$ points away from $\mathcal{O}$ on γ_j; more precisely,*

$$(u - y) \cdot \nabla\xi_j(u) > 0, \ \text{ for } \ u \in \gamma_j,\ y \in \mathcal{O}. \tag{9.39}$$

If $f \in L^\infty(S^1)$ and $f(x) \in K \subset\subset \mathcal{O}$ for all $x \in S^1$, then, for any $\varepsilon > 0$, the solution to

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon\partial_x^2 u_\varepsilon, \quad u_\varepsilon(0,x) = f(x) \tag{9.40}$$

exists on $[0,\infty) \times S^1$, and $u_\varepsilon(t,x) \in \overline{\mathcal{O}}$.

Proof. We remark that it suffices to prove the result under the further hypothesis that $f \in C^\infty(S^1)$. First, for any $\delta > 0$, consider

$$\partial_t u_{\varepsilon\delta} + \partial_x F(u_{\varepsilon\delta}) = \varepsilon \partial_x^2 u_{\varepsilon\delta} - \delta \nabla \rho(u_{\varepsilon\delta}), \quad u_{\varepsilon\delta}(0, x) = f(x), \tag{9.41}$$

where we pick $y \in \mathcal{O}$ and take $\rho(u) = |u - y|^2$. This has a unique local solution. If we show that $u_{\varepsilon\delta}(t, x) \in \mathcal{O}$, for all (t, x), then it has a solution on $[0, \infty) \times S^1$.

If it is not true that $u_{\varepsilon\delta}(t, x) \in \mathcal{O}$ for all (t, x), there is a first $t_0 > 0$ such that, for some $x_0 \in S^1$, $u(t_0, x_0) \in \partial\mathcal{O}$. Say $u(t_0, x_0)$ lies on the level curve γ_j. Take the dot product of (9.41) with $\nabla \xi_j(u_{\varepsilon\delta})$, to get (via (8.32))

$$\begin{aligned} &\partial_t \xi_j(u_{\varepsilon\delta}) + \lambda_k(u_{\varepsilon\delta})\, \partial_x \xi_j(u_{\varepsilon\delta}) \\ &= \varepsilon \partial_x^2 \xi_j(u_{\varepsilon\delta}) - \varepsilon(\partial_x u_{\varepsilon\delta}) \cdot \xi_j''(u_{\varepsilon\delta})\, \partial_x u_{\varepsilon\delta} - \delta \nabla \xi_j(u_{\varepsilon\delta}) \cdot \nabla \rho(u_{\varepsilon\delta}). \end{aligned} \tag{9.42}$$

Our geometrical hypothesis on $\mathcal{O}$ implies

$$(\partial_x u_{\varepsilon\delta}) \cdot \xi_j''(u_{\varepsilon\delta}) \partial_x u_{\varepsilon\delta} \geq 0 \quad \text{and} \quad \nabla \xi_j(u_{\varepsilon\delta}) \cdot \nabla \rho(u_{\varepsilon\delta}) > 0, \tag{9.43}$$

at (t_0, x_0). Meanwhile, the characterization of (t_0, x_0) implies

$$\partial_x \xi_j(u_{\varepsilon\delta}) = 0, \quad \text{and} \quad \partial_x^2 \xi_j(u_{\varepsilon\delta}) \leq 0, \tag{9.44}$$

at (t_0, x_0). Plugging (9.43)–(9.44) into (9.42) yields $\partial_t \xi_j(u_{\varepsilon\delta}) < 0$ at (t_0, x_0), an impossibility. Thus $u_{\varepsilon\delta} \in \mathcal{O}$ for all $(t, x) \in [0, \infty) \times S^1$.

Now, if (9.40) has a solution on $[0, T) \times S^1$, analysis of the nonhomogeneous linear parabolic equation satisfied by $w_{\varepsilon\delta} = u_\varepsilon - u_{\varepsilon\delta}$ shows that $u_{\varepsilon\delta} \to u_\varepsilon$ on $[0, T) \times S^1$, as $\delta \to 0$, so it follows that $u_\varepsilon(t, x) \in \overline{\mathcal{O}}$, and hence that (9.40) has a global solution, as asserted.

As an example of a case to which Proposition 9.5 applies, consider the system (9.35), with $K(v)$ given by (9.37), modeling transverse vibrations of a string. There are arbitrarily large, invariant regions $\mathcal{O}$ in $\Omega = \mathbb{R}^2$ of the form depicted in Fig. 9.1. Here, $\partial\mathcal{O} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, as depicted, and we take

$$\begin{aligned} &\xi_\pm = w \pm \int_0^v \sqrt{K'(s)}\, ds, \\ &\xi_1 = \xi_+ \text{ on } \gamma_1, \qquad \xi_3 = -\xi_+ \text{ on } \gamma_3, \\ &\xi_2 = \xi_- \text{ on } \gamma_2, \qquad \xi_4 = -\xi_- \text{ on } \gamma_4. \end{aligned} \tag{9.45}$$

Another example, with $\Omega = \{(v, w) : 0 < v < 1\}$, is depicted in Fig. 9.2. This applies also to the system (9.35), but with $K(v)$ given by (7.60). It models longitudinal waves in a string. In this case, there are invariant regions of the form $\mathcal{O}$ containing arbitrary compact subsets of Ω.

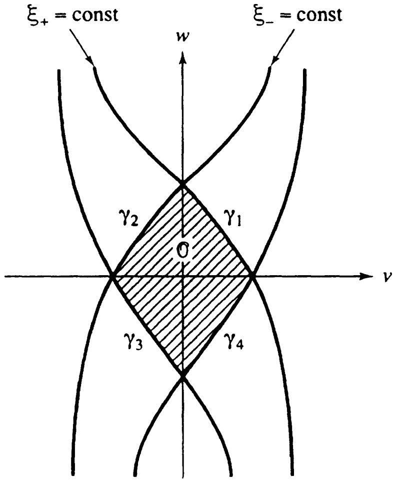

FIGURE 9.1 Setting for Proposition 9.5

Since we have seen that Lemma 9.4 applies in these cases, it follows that the conclusion of Theorem 9.3, that is, the existence of global entropy-satisfying solutions, holds, given initial data with range in any compact subset of Ω.

Exercises

1. As one specific way to end the proof of Proposition 9.5, show that $w_{\varepsilon\delta\sigma} = u_{\varepsilon\delta} - u_{\varepsilon\sigma}$ satisfies

$$\partial_t w_{\varepsilon\delta\sigma} + \partial_x (G_{\varepsilon\delta\sigma} w_{\varepsilon\delta\sigma}) = \varepsilon\partial_x^2 w_{\varepsilon\delta\sigma} - \delta\nabla\rho(u_{\varepsilon\delta}) + \sigma\nabla\rho(u_{\varepsilon\sigma}), \quad w_{\varepsilon\delta\sigma}(0, x) = 0,$$

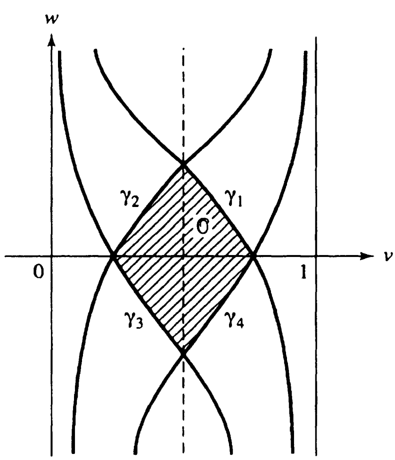

FIGURE 9.2 Another Setting for Proposition 9.5

where $G_{\varepsilon\delta\sigma} = \int_0^1 DF\big(su_{\varepsilon\delta} + (1-s)u_{\varepsilon\sigma}\big)\, ds$. Deduce that there exists $K < \infty$ such that, for $\varepsilon, \delta, \sigma \in (0, 1]$,

$$\frac{d}{dt}\|w_{\varepsilon\delta\sigma}(t)\|^2_{L^2} \le \frac{K}{\varepsilon}\|w_{\varepsilon\delta\sigma}(t)\|^2_{L^2} + K(\delta + \sigma),$$

granted that $u_{\varepsilon\delta}(t, x), u_{\varepsilon\sigma}(t, x) \in \mathcal{O}$, for all (t, x). Use Gronwall's inequality to estimate $\|w_{\varepsilon\delta\sigma}(t)\|^2_{L^2}$, showing that, for fixed $\varepsilon \in (0, 1]$, it tends to zero as $\delta, \sigma \to 0$, locally uniformly in $t \in [0, \infty)$. Use this to show that $u_{\varepsilon\delta} \to u_\varepsilon$, as $\delta \to 0$.

10. Vibrating strings revisited

As we have mentioned, the equation for a string vibrating in $\mathbb{R}^k$ was derived in § 1 of Chap. 2, from an action integral of the form

$$J(u) = \iint\limits_{I\times\Omega} \Big[\frac{m}{2}\big|u_t(t, x)\big|^2 - F\big(u_x(t, x)\big)\Big]\, dx\, dt, \tag{10.1}$$

where $x \in \Omega = [0, L]$, $t \in I = (t_0, t_1)$. Assume that the mass density m is a positive constant. The stationary condition is

$$mu_{tt} - F'(u_x)_x = 0, \tag{10.2}$$

which is a second-order, $k \times k$ system. If we set

$$v = u_x, \quad w = u_t, \tag{10.3}$$

we get a first-order, $(2k) \times (2k)$ system:

$$\begin{aligned} v_t - w_x &= 0,\\ w_t - K(v)_x &= 0, \end{aligned} \tag{10.4}$$

where

$$K(v) = \frac{1}{m} F'(v). \tag{10.5}$$

Let us assume that $F(u_x)$ is a function of $|u_x|^2$ alone:

$$F(u_x) = f(|u_x|^2). \tag{10.6}$$

Then $K(v)$ has the form

$$K(v) = \frac{2}{m} f'(|v|^2)v. \tag{10.7}$$

We can write (10.4) in quasi-linear form as

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I \\ DK(v) & 0 \end{pmatrix}\begin{pmatrix} v_x \\ w_x \end{pmatrix}, \tag{10.8}$$

where, for $b \in \mathbb{R}^k$,

$$DK(v)b = \frac{2}{m} f'(|v|^2)b + \frac{4}{m} f''(|v|^2)(v \cdot b)v, \tag{10.9}$$

that is,

$$DK(v) = \frac{2}{m} f'(|v|^2)I + \frac{4}{m} f''(|v|^2)|v|^2 P_v, \tag{10.10}$$

P_v being the orthogonal projection of $\mathbb{R}^k$ onto the line spanned by v (if $v \neq 0$).

Writing (10.8) as

$$U_t - A(U)U_x = 0, \tag{10.11}$$

for $U = (v, w)^t$, we see that the eigenvalues of $A(U)$ are given by

$$\text{Spec } A(u) = \{\pm\sqrt{\lambda_j} : \lambda_j \in \text{ Spec } DK(v)\}. \tag{10.12}$$

Now, if $k = 1$, $DK(v)$ is scalar:

$$DK(v) = \frac{1}{m} F''(v). \tag{10.13}$$

As long as $F''(v) > 0$, the system (10.8) is strictly hyperbolic, with characteristic speeds

$$\lambda_\pm = \pm\sqrt{\frac{1}{m} F''(v)}.$$

In this case, the system (10.4) describes longitudinal vibrations of a string. The Riemann problem for this system was considered in (7.55)–(7.60), and DiPerna's global existence theorem was applied in the discussion of Fig. 9.2.

On the other hand, if $k > 1$, then

$$\text{Spec } DK(v) = \left\{\frac{2}{m} f'(|v|^2), \frac{2}{m} f'(|v|^2) + \frac{4}{m} f''(|v|^2)|v|^2\right\}, \tag{10.14}$$

where the first listed eigenvalue has multiplicity $k - 1$ and the last one has multiplicity 1. We can rewrite these eigenvalues, using the notion of "tension" $T(r)$, defined so that

$$F'(v) = T(|v|)\frac{v}{|v|}, \tag{10.15}$$

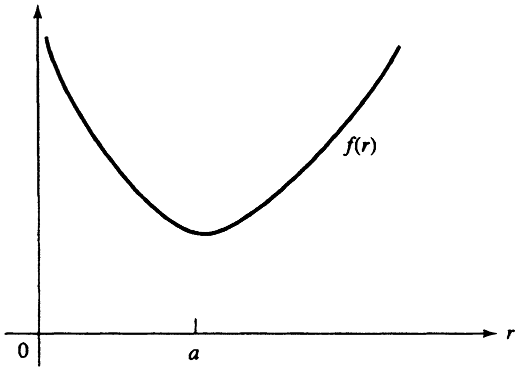

FIGURE 10.1 Graph of f

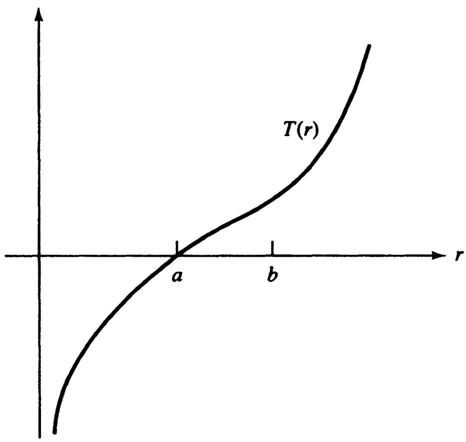

FIGURE 10.2 Graph of T

that is, so $2f'(|v|^2) = T(|v|)/|v|$. A calculation gives

$$2f'(|v|^2) + 4f''(|v|^2)|v|^2 = T'(|v|),$$

so

$$\text{Spec } DK(v) = \left\{ \frac{1}{m} \frac{T\big(|v|\big)}{|v|}, \ \frac{1}{m} T'\big(|v|\big) \right\}. \tag{10.16}$$

The basic expected behavior of the function $f(r)$ is discussed in § 7, around the formula (7.60). The function $f(r)$ should be expected to behave as in Fig. 10.1. For r larger than a certain a, $f(r)$ should increase. On the other hand, also $f(r)$ should get very large as $r \searrow 0$, since the material of the string would resist compression. This means for the tension $T(r)$ that $T(r) > 0$ for $r > a$ and $T(r) < 0$ for $0 < r < a$. On the other hand, we expect $T(r)$ to increase whenever r increases, so $T'(r) > 0$ for all r. Such behavior of the tension is depicted in Fig. 10.2.

We conclude that when $|v| > a$, then

$$\text{Spec } A(U) = \left\{ \pm\sqrt{\frac{1}{m}\frac{T(|v|)}{|v|}}, \pm\sqrt{\frac{1}{m}T'\big(|v|\big)} \right\} \tag{10.17}$$

consists of real numbers. If $k = 2$, we have four eigenvalues. These are all distinct as long as $f''(|v|^2) \neq 0$, that is, as long as the function $f(r)$ is strongly convex. If this convexity fails somewhere on $|v| > a$, then the system (10.4) will not be strictly hyperbolic, but it will be symmetrizable hyperbolic, as long as $|v| > a$.

On the other hand, when $|v| < a$, then Spec $A(U)$, which is still given by (10.17), has two purely imaginary elements, as well as two real elements (the former being eigenvalues with multiplicity $k - 1$). Thus (10.4) is not hyperbolic in the region $|v| < a$.

Let us concentrate for now on the region $|v| > a$, where (10.4) is hyperbolic, and examine whether it is genuinely nonlinear. We consider the case $k = 2$. Let us denote the two eigenvalues of $DK(v)$ given by (10.16) by $\lambda_j(v)$:

$$\lambda_1(v) = \frac{1}{m}\frac{T(|v|)}{|v|}, \quad \lambda_2(v) = \frac{1}{m}T'\big(|v|\big). \tag{10.18}$$

Thus

$$\begin{aligned} f''(|v|^2) > 0 &\Longrightarrow \lambda_2(v) > \lambda_1(v), \\ f''(|v|^2) < 0 &\Longrightarrow \lambda_2(v) < \lambda_1(v). \end{aligned} \tag{10.19}$$

From (10.10) we see that we can take as eigenvectors of $DK(v)$

$$r_2(v) = v, \quad r_1(v) = Jv, \tag{10.20}$$

where $J : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation by 90°. It follows that eigenvectors of $A(U)$ corresponding to the eigenvalues

$$\mu_{j\pm} = \pm\sqrt{\lambda_j(v)} \tag{10.21}$$

are given by

$$\mathbf{r}_{j\pm} = \begin{pmatrix} r_j(v) \\ \mu_{j\pm} r_j(v) \end{pmatrix}. \tag{10.22}$$

Thus

$$\mathbf{r}_{j\pm} \cdot \nabla \mu_{j\pm} = \pm r_j(v) \cdot \nabla \sqrt{\lambda_j(v)} = \pm \frac{1}{2\sqrt{\lambda_j(v)}} r_j(v) \cdot \nabla \lambda_j(v). \tag{10.23}$$

Now, by (10.20), if we use polar coordinates (r, θ) on $\mathbb{R}^2$, with $r^2 = v_1^2 + v_2^2$, then

$$R_2 = r \frac{\partial}{\partial r}, \quad R_1 = \frac{\partial}{\partial \theta}, \tag{10.24}$$

so

$$r_1 \cdot \nabla \lambda_1(v) = 0, \quad r_2 \cdot \nabla \lambda_2(v) = \frac{1}{m} T''(|v|)|v|. \tag{10.25}$$

Thus we see that within the hyperbolic region $|v| > a$, (10.4) has two linearly degenerate fields and two fields that are genuinely nonlinear as long as $T''(|v|) \neq 0$.

We now describe an interesting complication that arises in the treatment of the system (10.4) when $k \geq 2$. Namely, even if initial data lie entirely in the hyperbolic region, the solution might not stay in the hyperbolic region. Consider the following simple example, with $k = 2$. We let $v(0, x), w(0, x)$ be initial data for a purely longitudinal wave, so that the motion of the string is confined to the x_1-axis. Thus, we take

$$v(0, x) = \begin{pmatrix} v_1(0, x) \\ 0 \end{pmatrix}, \quad w(0, x) = \begin{pmatrix} w_1(0, x) \\ 0 \end{pmatrix}. \tag{10.26}$$

For all $t \geq 0$, the solution will have the form

$$v(t, x) = \begin{pmatrix} v_1(t, x) \\ 0 \end{pmatrix}, \quad w(t, x) = \begin{pmatrix} w_1(t, x) \\ 0 \end{pmatrix}, \tag{10.27}$$

where the pair (v_1, w_1) satisfies the $k = 1$ case of (10.4).

Suppose $K(v)$ is given by (10.7) and $T(|v|) = 2f'\big(|v|^2\big)|v|$ has the behavior indicated in Fig. 10.2; also, let us assume f is real analytic on $(0, \infty)$. Introduce another variable η, and let $\big(v_1(t, x, \eta), w_1(t, x, \eta)\big)$ solve the $k = 1$ case of (10.4), with initial data

$$\begin{aligned} v_1(0, x, \eta) &= a + \eta, \\ w_1(0, x, \eta) &= -b \ \sin x, \end{aligned} \tag{10.28}$$

where b is some positive constant. By the Cauchy–Kowalewsky theorem, there is a $T > 0$ such that there is a unique, real-analytic solution defined for $|t| < T$, for all $x \in \mathbb{R}$ (periodic of period 2π), and for all $|\eta| \leq a/2$. Note that

$$\partial_t v_1(0, 0, \eta) = -b. \tag{10.29}$$

It follows easily from the implicit function theorem that, for all $\eta > 0$ sufficiently small, there exists $t(\eta) \in (0, T)$ such that

$$v_1\big(t(\eta), 0, \eta\big) < a. \tag{10.30}$$

This is a well-behaved solution to the longitudinal wave problem, but the solution so produced to the $k = 2$ case of (10.4), having the form (10.27), clearly has the property that

$$\big(v(t(\eta), 0, \eta), w(t(\eta), 0, \eta)\big)^t = \big(v_1(t(\eta), 0, \eta), 0; w_1(t(\eta), 0, \eta), 0\big)^t \tag{10.31}$$

does not belong to the hyperbolic region, despite the fact that the initial data do.

Note that, for the system under consideration, solutions to the Riemann problem for (v_1, w_1) have the behavior discussed in § 7, illustrated by Fig. 7.4 there. For example, the situation illustrated in Fig. 10.3 can arise. Here, we have Riemann data

$$U_{1\ell} = (v_{1\ell}, w_{1\ell})^t, \quad U_{1r} = (v_{1r}, w_{1r})^t, \quad v_{1\ell} > a, \ \ v_{1r} > a, \tag{10.32}$$

but the intermediate state U_{1m} has the form

$$U_{1m} = (v_{1m}, w_{1m})^t, \quad v_{1m} < a. \tag{10.33}$$

This is also a well-behaved solution to the longitudinal wave problem, but the solution so produced to the $k = 2$ case of (10.4) is the following. We have the Riemann problem

$$U_\ell = (v_{1\ell}, 0; w_{1\ell}, 0)^t, \quad U_r(v_{1r}, 0; w_{1r}, 0)^t, \tag{10.34}$$

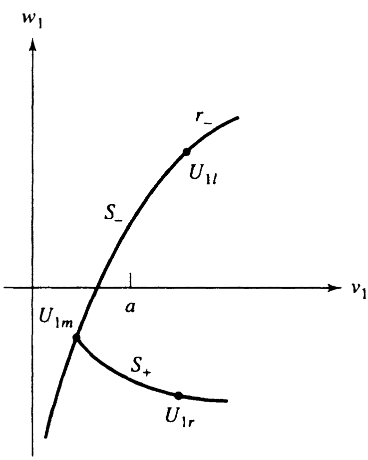

FIGURE 10.3 Connecting $U_{1\ell}$ to U_{1r}

and a weak solution to (10.4), involving two jumps, with intermediate state

$$U_m = (v_{1m}, 0; w_{1m}, 0)^t. \tag{10.35}$$

Clearly, U_m does not belong to the hyperbolic region for the $k = 2$ case of (10.4), even though U_ℓ and U_r do belong to the hyperbolic region.

M. Shearer, [Sh1, Sh2] (see also [CRS]), has proposed a method for solving Riemann problems for a class of systems, including the system for vibrating strings considered here, which leaves the hyperbolic region invariant. The solution produced by this method to the Riemann problem described in the preceding paragraph has an intermediate state U_m different from the one described above. The situation is depicted in Fig. 10.4. Here $U_{1m} = (v_{1m}, w_{1m})$ with $v_{1m} < 0$ (in fact, $v_{1m} < -a$). The physical interpretation is that the string develops a kink and doubles back on itself. For motion strictly confined to one dimension, this would not be allowable, as it involves interpenetration of different string segments. If the string has two or more dimensions in which to move, one thinks of these segments as lying side by side; however, one does not ask which segment lies on which side; for example, for $k = 2$, one does not ask whether the configuration is as in Fig. 10.5A or as in Fig. 10.5B.

Of course, there is only one real-analytic solution with the initial data (10.26)–(10.28); there is no real-analytic modification that stays in the hyperbolic region.

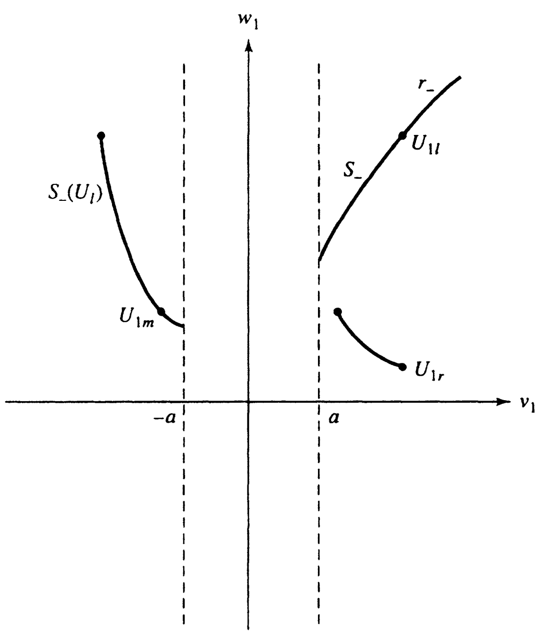

FIGURE 10.4 More Complex Connection

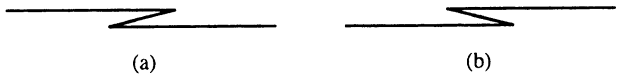

FIGURE 10.5 Crinkled Strings

In [PS] there is a study of the behavior of approximations of such solutions via Glimm's scheme. It is found that typically these approximations do not converge, even weakly, to the smooth solution.

Further material on systems that change type can be found in [KS]. Also, papers in [KK3] deal with systems for which strict hyperbolicity can fail, as can happen in the hyperbolic region for (10.4) if $f''(r) = 0$ for some $r > a^2$.

Exercises

1. Work out the equations for radially symmetric vibrations of a two-dimensional membrane in $\mathbb{R}^3$. Perform an analysis parallel to that done in this section for the vibrating string system.

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17

Euler and Navier–Stokes Equations for Incompressible Fluids

Introduction

This chapter deals with equations describing motion of an incompressible fluid moving in a fixed compact space M, which it fills completely. We consider two types of fluid motion, with or without viscosity, and two types of compact space, a compact smooth Riemannian manifold with or without boundary. The two types of fluid motion are modeled by the Euler equation

$$\frac{\partial u}{\partial t} + \nabla_u u = -\text{grad } p, \qquad \text{div} u = 0, \tag{0.1}$$

for the velocity field u, in the absence of viscosity, and the Navier–Stokes equation

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \mathcal{L} u - \text{grad } p, \qquad \text{div } u = 0, \tag{0.2}$$

in the presence of viscosity. In (0.2), ν is a positive constant and $\mathcal{L}$ is the second-order differential operator

$$\mathcal{L} u = \text{div Def } u, \tag{0.3}$$

which on flat Euclidean space is equal to Δu, when $\text{div } u = 0$. If there is a boundary, the Euler equation has boundary condition $n \cdot u = 0$, that is, u is tangent to the boundary, while for the Navier–Stokes equation one poses the no-slip boundary condition $u = 0$ on ∂M.

In § 1 we derive (0.1) in several forms; we also derive the vorticity equation for the object that is curl u when $\dim M = 3$. We discuss some of the classical physical interpretations of these equations, such as Kelvin's circulation theorem and Helmholtz' theorem on vortex tubes, and include in the exercises other topics, such as steady flows and Bernoulli's law. These phenomena can be compared with analogues for compressible flow, discussed in § 5 of Chap. 16.

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Sections 2–5 discuss the existence, uniqueness and regularity of solutions to (0.1) and (0.2), on regions with or without boundary. We have devoted separate sections to treatments first without boundary and then with boundary, for these equations, at a cost of a small amount of redundancy. By and large, different analytical problems are emphasized in the separate sections, and their division seems reasonable from a pedagogical point of view.

The treatments in §§ 2–5 are intended to parallel to a good degree the treatment of nonlinear parabolic and hyperbolic equations in Chaps. 15 and 16. Among the significant differences, there is the role of the vorticity equation, which leads to global solutions when $\dim M = 2$. For $\dim M \geq 3$, the question of whether smooth solutions exist for all $t \geq 0$ is still open, with a few exceptions, such as small initial data for (0.2). These problems, as well as variants, such as free boundary problems for fluid flow, remain exciting and perplexing.

In § 6 we tackle the question of how solutions to the Navier–Stokes equations on a bounded region behave when the viscosity tends to zero. We stick to two special cases, in which this difficult question turns out to be somewhat tractable. The first is the class of 2D flows on a disk that are circularly symmetric. The second is a class of 3D circular pipe flows, whose detailed description can be found in § 6. These cases yield convergence of the velocity fields to the fields solving associated Euler equations, though not in a particularly strong norm, due to boundary layer effects. Section 7 investigates how such velocity convergence yields information on the convergence of the flows generated by such time-varying vector fields.

In Appendix A we discuss boundary regularity for the Stokes operator, needed for the analysis in § 5.

1. Euler's equations for ideal incompressible fluid flow

An incompressible fluid flow on a region Ω defines a one-parameter family of volume-preserving diffeomorphisms

$$F(t,\cdot) : \Omega \longrightarrow \Omega, \tag{1.1}$$

where Ω is a Riemannian manifold with boundary; if $\partial\Omega$ is nonempty, we suppose it is preserved under the flow. The flow can be described in terms of its velocity field

$$u(t,y) = F_t(t,x) \in T_y\Omega, \quad y = F(t,x), \tag{1.2}$$

where $F_t(t,x) = (\partial/\partial t)F(t,x)$. If $y \in \partial\Omega$, we assume $u(t,y)$ is tangent to $\partial\Omega$. We want to derive Euler's equation, a nonlinear PDE for u describing the dynamics of fluid flow. We will assume the fluid has uniform density.

If we suppose there are no external forces acting on the fluid, the dynamics are determined by the constraint condition, that $F(t,\cdot)$ preserve volume, or equivalently, that $\operatorname{div} u(t,\cdot) = 0$ for all t. The Lagrangian involves the kinetic energy alone, so we seek to find critical points of

$$L(F) = \frac{1}{2} \int_I \int_\Omega \big\langle F_t(t,x), F_t(t,x)\big\rangle \, dV \, dt, \tag{1.3}$$

on the space of maps $F : I \times \Omega \to \Omega$ (where $I = [t_0, t_1]$), with the volume-preserving property.

For simplicity, we first treat the case where Ω is a domain in $\mathbb{R}^n$. A variation of F is of the form $F(s,t,x)$, with $\partial F/\partial s = v(t, F(t,x))$, at $s = 0$, where div $v = 0$, v is tangent to $\partial\Omega$, and $v = 0$ for $t = t_0$ and $t = t_1$. We have

$$\begin{aligned} DL(F)v &= \iint \Big\langle F_t(t,x), \frac{d}{dt} v\big(t, F(t,x)\big)\Big\rangle \, dV \, dt \\ &= \iint \Big\langle u\big(t, F(t,x)\big), \frac{d}{dt} v\big(t, F(t,x)\big)\Big\rangle \, dV \, dt \\ &= -\iint \Big\langle \frac{\partial u}{\partial t} + u \cdot \nabla_x u, v\Big\rangle \, dV \, dt. \end{aligned} \tag{1.4}$$

The stationary condition is that this last integral vanish for all such v, and hence, for each t,

$$\int_\Omega \Big\langle \frac{\partial u}{\partial t} + u \cdot \nabla_x u, v\Big\rangle \, dV = 0, \tag{1.5}$$

for all vector fields v on Ω (tangent to $\partial\Omega$), satisfying div $v = 0$.

To restate this as a differential equation, let

$$V_\sigma = \big\{ v \in C^\infty(\overline{\Omega}, T\Omega) : \operatorname{div} v = 0, \ v \text{ tangent to } \partial\Omega \big\}, \tag{1.6}$$

and let P denote the orthogonal projection of $L^2(\Omega, T\Omega)$ onto the closure of the space V_σ. The operator P is often called the *Leray projection*. The stationary condition becomes

$$\frac{\partial u}{\partial t} + P(u \cdot \nabla_x u) = 0, \tag{1.7}$$

in addition to the conditions

$$\operatorname{div} u = 0 \quad \text{on } \Omega \tag{1.8}$$

and

$$u \text{ tangent to } \partial\Omega. \tag{1.9}$$

For a general Riemannian manifold Ω, one has a similar calculation, with $u \cdot \nabla_x u$ in (1.5) generalized simply to $\nabla_u u$, where ∇ is the Riemannian connection on Ω. Thus (1.7) generalizes to

Euler's equation, first form.

$$\frac{\partial u}{\partial t} + P(\nabla_u u) = 0. \tag{1.10}$$

Suppose now $\overline{\Omega}$ is compact. According to the Hodge decomposition, the orthogonal complement in $L^2(\Omega, T)$ of the range of P is equal to the space

$$\{\text{grad } p : p \in H^1(\Omega)\}.$$

This fact is derived in the problem set following § 9 in Chap. 5, entitled "Exercises on spaces of gradient and divergence-free vector fields"; see (9.79)–(9.80). Thus we can rewrite (1.10) as

Euler's equation, second form.

$$\frac{\partial u}{\partial t} + \nabla_u u = -grad\ p. \tag{1.11}$$

Here, p is a scalar function, determined uniquely up to an additive constant (assuming Ω is connected). The function p is identified as "pressure."

It is useful to derive some other forms of Euler's equation. In particular, let $\tilde{u}$ denote the 1-form corresponding to the vector field u via the Riemannian metric on Ω. Then (1.11) is equivalent to

$$\frac{\partial \tilde{u}}{\partial t} + \nabla_u \tilde{u} = -dp. \tag{1.12}$$

We will rewrite this using the Lie derivative. Recall that, for any vector field X,

$$\nabla_u X = \mathcal{L}_u X + \nabla_X u,$$

by the zero-torsion condition on ∇. Using this, we deduce that

$$\langle \mathcal{L}_u \tilde{u} - \nabla_u \tilde{u}, X \rangle = \langle \tilde{u}, \nabla_X u \rangle. \tag{1.13}$$

In case $\langle \tilde{u}, v \rangle = \langle u, v \rangle$ (the Riemannian inner product), we have

$$\langle \tilde{u}, \nabla_X u \rangle = \frac{1}{2} \langle d\, |u|^2, X \rangle, \tag{1.14}$$

using the notation $|u|^2 = \langle u, u \rangle$, so (1.12) is equivalent to

Euler's equation, third form.

$$\frac{\partial \tilde{u}}{\partial t} + \mathcal{L}_u \tilde{u} = d\Big(\frac{1}{2}|u|^2 - p\Big). \tag{1.15}$$

Writing the Lie derivative in terms of exterior derivatives, we obtain

$$\frac{\partial \tilde{u}}{\partial t} + (d\tilde{u}) \rfloor u = -d\Big(\frac{1}{2}|u|^2 + p\Big). \tag{1.16}$$

Note also that the condition $\operatorname{div} u = 0$ can be rewritten as

$$\delta \tilde{u} = 0. \tag{1.17}$$

In the study of Euler's equation, a major role is played by the *vorticity*, which we proceed to define. In its first form, the vorticity will be taken to be

$$\tilde{w} = d\tilde{u}, \tag{1.18}$$

for each t a 2-form on Ω. The Euler equation leads to a PDE for vorticity; indeed, applying the exterior derivative to (1.15) gives immediately the

Vorticity equation, first form.

$$\frac{\partial \tilde{w}}{\partial t} + \mathcal{L}_u \tilde{w} = 0, \tag{1.19}$$

or equivalently, from (1.16),

$$\frac{\partial \tilde{w}}{\partial t} + d\big(\tilde{w} \rfloor u\big) = 0. \tag{1.20}$$

It is convenient to express this in terms of the covariant derivative. In analogy to (1.13), for any 2-form β and vector fields X and Y, we have

$$\begin{aligned}(\nabla_u \beta - \mathcal{L}_u \beta)(X, Y) &= \beta(\nabla_X u, Y) + \beta(X, \nabla_Y u) \\ &= (\beta \# \nabla u)(X, Y),\end{aligned} \tag{1.21}$$

where the last identity defines $\beta \# \nabla u$. Thus we can rewrite (1.19) as

Vorticity equation, second form.

$$\frac{\partial \tilde{w}}{\partial t} + \nabla_u \tilde{w} - \tilde{w} \# \nabla u = 0. \tag{1.22}$$

It is also useful to consider vorticity in another form. Namely, to $\tilde{w}$ we associate a section w of $\Lambda^{n-2} T$ ($n = \dim \Omega$), so that the identity

$$\tilde{w} \wedge \alpha = \langle w, \alpha \rangle \omega \tag{1.23}$$

holds, for every $(n-2)$-form α, where ω is the volume form on Ω. (We assume Ω is oriented.) The correspondence $\tilde{w} \leftrightarrow w$ given by (1.23) depends only on the volume element ω. Hence

$$\operatorname{div} u = 0 \Longrightarrow \mathcal{L}_u \tilde{w} = \widetilde{\mathcal{L}_u w}, \tag{1.24}$$

so (1.19) yields the

Vorticity equation, third form.

$$\frac{\partial w}{\partial t} + \mathcal{L}_u w = 0. \tag{1.25}$$

This vorticity equation takes special forms in two and three dimensions, respectively. When $\dim \Omega = n = 2$, w is a scalar field, often denoted as

$$w = \operatorname{rot} u, \tag{1.26}$$

and (1.25) becomes the

2-D vorticity equation.

$$\frac{\partial w}{\partial t} + u \cdot \operatorname{grad} w = 0. \tag{1.27}$$

This is a conservation law. As we will see, this has special implications for two-dimensional incompressible fluid flow. If $n = 3$, w is a vector field, denoted as

$$w = \operatorname{curl} u, \tag{1.28}$$

and (1.25) becomes the

3-D vorticity equation.

$$\frac{\partial w}{\partial t} + [u, w] = 0, \tag{1.29}$$

or equivalently,

$$\frac{\partial w}{\partial t} + \nabla_u w - \nabla_w u = 0. \tag{1.30}$$

Note that (1.28) is a generalization of the notion of the curl of a vector field on flat $\mathbb{R}^3$. Compare with material in the second exercise set following § 8 in Chap. 5.

The first form of the vorticity equation, (1.19), implies

$$\tilde{w}(0) = \left(F^t\right)^* \tilde{w}(t), \tag{1.31}$$

where $F^t(x) = F(t,x)$, $\tilde{w}(t)(x) = \tilde{w}(t,x)$. Similarly, the third form, (1.25), yields

$$w(t,y) = \Lambda^{n-2} DF^t(x)\, w(0,x), \quad y = F(t,x), \tag{1.32}$$

where $DF^t(x) : T_x\Omega \to T_y\Omega$ is the derivative. In case $n = 2$, this last identity is simply $w(t,y) = w(0,x)$, the conservation law mentioned after (1.27).

One implication of (1.31) is the following. Let S be an oriented surface in Ω, with boundary C; let $S(t)$ be the image of S under F^t, and $C(t)$ the image of C; then (1.31) yields

$$\int_{S(t)} \tilde{w}(t) = \int_S \tilde{w}(0). \tag{1.33}$$

Since $\tilde{w} = d\tilde{u}$, this implies the following:

Kelvin's circulation theorem.

$$\int_{C(t)} \tilde{u}(t) = \int_C \tilde{u}(0). \tag{1.34}$$

We take a look at some phenomena special to the case $\dim \Omega = n = 3$, where the vorticity w is a vector field on Ω, for each t. Fix t_0, and consider $w = w(t_0)$. Let S be an oriented surface in Ω, transversal to w. A vortex tube $\mathcal{T}$ is defined to be the union of orbits of w through S, to a second transversal surface S_2 (see Fig. 1.1). For simplicity we will assume that none of these orbits ends at a zero of the vorticity field, though more general cases can be handled by a limiting argument.

Since $d\tilde{w} = d^2\tilde{u} = 0$, we can use Stokes' theorem to write

$$0 = \int_{\mathcal{T}} d\tilde{w} = \int_{\partial\mathcal{T}} \tilde{w}. \tag{1.35}$$

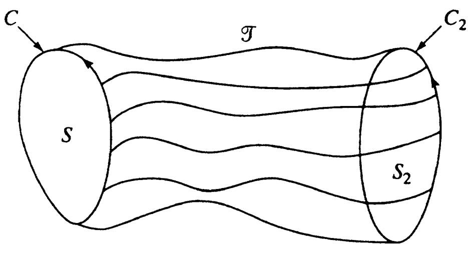

FIGURE 1.1 Vortex Tube

Now $\partial\mathcal{T}$ consists of three pieces, S and S_2 (with opposite orientations) and the lateral boundary $\mathcal{L}$, the union of the orbits of w from ∂S to ∂S_2. Clearly, the pull-back of $\tilde{w}$ to $\mathcal{L}$ is 0, so (1.35) implies

$$\int_S \tilde{w} = \int_{S_2} \tilde{w}. \tag{1.36}$$

Applying Stokes' theorem again, for $\tilde{w} = d\tilde{u}$, we have

Helmholtz' theorem. *For any two curves C, C_2 enclosing a vortex tube,*

$$\int_C \tilde{u} = \int_{C_2} \tilde{u}. \tag{1.37}$$

This common value is called the strength of the vortex tube $\mathcal{T}$.

Also note that if $\mathcal{T}$ is a vortex tube at $t_0 = 0$, then, for each t, $\mathcal{T}(t)$, the image of $\mathcal{T}$ under F^t, is a vortex tube, as a consequence of (1.32) (with $n = 3$), and furthermore (1.34) implies that the strength of $\mathcal{T}(t)$ is independent of t. This conclusion is also part of Helmholtz' theorem.

To close this section, we note that the Euler equation for an ideal incompressible fluid flow with an external force f is

$$\frac{\partial u}{\partial t} + \nabla_u u = -\text{grad } p + f,$$

in place of (1.11). If f is conservative, of the form $f = -\text{grad } \varphi$, then (1.12) is replaced by

$$\frac{\partial \tilde{u}}{\partial t} + \nabla_u \tilde{u} = -d(p + \varphi).$$

Thus the vorticity $\tilde{w} = d\tilde{u}$ continues to satisfy (1.19), and other phenomena discussed above can be treated in this extra generality.

Indeed, in the case we have considered, of a completely confined, incompressible flow of a fluid of uniform density, adding such a conservative force field has no effect on the velocity field u, just on the pressure, though in other situations such a force field could have more pronounced effects.

Exercises

1. Using the divergence theorem, show that whenever $\operatorname{div} u = 0$, u tangent to $\partial\Omega$, $\overline{\Omega}$ compact, and $f \in C^\infty(\overline{\Omega})$, we have

$$\int_\Omega \mathcal{L}_u f \, dV = 0.$$

Hence show that, for any smooth vector field X on $\overline{\Omega}$,

$$\int_{\Omega} \langle \nabla_u X, X \rangle \, dV = 0.$$

From this, conclude that any (sufficiently smooth) u solving (1.7)–(1.9) satisfies the conservation of energy law

$$\frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)} = 0. \tag{1.38}$$

2. When $\dim \Omega = n = 3$, show that the vorticity field w is divergence free. (*Hint*: div curl.)
3. If u, v are vector fields, $\tilde{u}$ the 1-form associated to u, it is generally true that $\nabla_v \tilde{u} = \widetilde{\nabla_v u}$, but not that $\mathcal{L}_v \tilde{u} = \widetilde{\mathcal{L}_v u}$. Why is that?
4. A fluid flow is called *stationary* provided u is independent of t. Establish *Bernoulli's law*, that for a stationary solution of Euler's equations (1.7)–(1.9), the function $(1/2)|u|^2 + p$ is constant along any streamline (i.e., an integral curve of u).
   In the nonstationary case, show that

$$\frac{1}{2}\Big(\frac{\partial}{\partial t} - \mathcal{L}_u\Big)|u|^2 = -\mathcal{L}_u p.$$

   (*Hint*: Use Euler's equation in the form (1.16); take the inner product of both sides with u.)
5. Suppose $\dim \Omega = 3$. Recall from the auxiliary exercise set after § 8 in Chap. 5 the characterization

$$u \times v = X \Longleftrightarrow \tilde{X} = *(\tilde{u} \wedge \tilde{v}).$$

   Show that the form (1.16) of Euler's equation is equivalent to

$$\frac{\partial u}{\partial t} + (\operatorname{curl} u) \times u = -\operatorname{grad}\Big(\frac{1}{2}|u|^2 + p\Big). \tag{1.39}$$

   Also, if $\Omega \subset \mathbb{R}^3$, deduce this from (1.11) together with the identity

$$\operatorname{grad}(u \cdot v) = u \cdot \nabla v + v \cdot \nabla u + u \times \operatorname{curl} v + v \times \operatorname{curl} u,$$

   which is derived in (8.63) of Chap. 5.
6. Deduce the 3-D vorticity equation (1.30) by applying curl to both sides of (1.39) and using the identity

$$\operatorname{curl}(u \times v) = v \cdot \nabla u - u \cdot \nabla v + (\operatorname{div} v)u - (\operatorname{div} u)v,$$

   which is derived in (8.62) of Chap. 5. Also show that the vorticity equation can be written as

$$w_t + \nabla_u w = (\operatorname{Def} u)w, \qquad \operatorname{Def} v = \frac{1}{2}(\nabla v + \nabla v^t). \tag{1.40}$$

   (*Hint*: $w \times w = 0$.)
7. In the setting of Exercise 5, show that, for a stationary flow, $(1/2)|u|^2 + p$ is constant along both any streamline and any vortex line (i.e., an integral curve of $w = \operatorname{curl} u$).

8. For $\dim \Omega = 3$, note that (1.29) implies $[u, w] = 0$ for a stationary flow, with $w = \operatorname{curl} u$. What does Frobenius's theorem imply about this?
9. Suppose u is a (sufficiently smooth) solution to the Euler equation (1.11), also satisfying (1.9), namely, u is tangent to $\partial\Omega$. Show that if $u(0)$ has vanishing divergence, then $u(t)$ has vanishing divergence for all t. (*Hint*: Use the Hodge decomposition discussed between (1.10) and (1.11).)
10. Suppose $\tilde{u}$, the 1-form associated to u, and a 2-form $\tilde{w}$ satisfy the coupled system

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} + \tilde{w} \rfloor u &= -d\Phi, \\ \frac{\partial \tilde{w}}{\partial t} + d(\tilde{w} \rfloor u) &= 0. \end{aligned} \tag{1.41}$$

Show that if $\tilde{w}(0) = d\tilde{u}(0)$, then $\tilde{w}(t) = d\tilde{u}(t)$ for all t. (*Hint*: Set $\tilde{W}(t) = d\tilde{u}(t)$, so by the first half of (1.41), $\partial \tilde{W}/\partial t + d(\tilde{w} \rfloor u) = 0$. Subtract this from the second equation of the pair (1.41).)
11. If u generates a 1-parameter group of isometries of Ω, show that u provides a stationary solution to the Euler equations. (*Hint*: Show that $\operatorname{Def} u = 0 \Rightarrow \nabla_u u = (1/2) \operatorname{grad} |u|^2$.)
12. A flow is called *irrotational* if the vorticity $\tilde{w}$ vanishes. Show that if $\tilde{w}(0) = 0$, then $\tilde{w}(t) = 0$ for all t, under the hypotheses of this section.
13. If a flow is both stationary and irrotational, show that Bernoulli's law can be strengthened to

$$\frac{1}{2}|u|^2 + p \text{ is constant on } \Omega.$$

The common statement of this is that higher fluid velocity means lower pressure (within the limited set of circumstances for which this law holds). (*Hint*: Use (1.16).)
14. Suppose $\overline{\Omega}$ is compact. Show that the space of 1-forms $\tilde{u}$ on $\overline{\Omega}$ satisfying

$$\delta\tilde{u} = 0, \quad d\tilde{u} = 0 \text{ on } \Omega, \quad \langle \nu, u \rangle = 0 \text{ on } \partial\Omega,$$

is the finite-dimensional space of harmonic 1-forms $\mathcal{H}_1^A$, with absolute boundary conditions, figuring into the Hodge decomposition, introduced in (9.36) of Chap. 5. Show that, for $\tilde{u}(0) \in \mathcal{H}_1^A$, Euler's equation becomes the finite-dimensional system

$$\frac{\partial \tilde{u}}{\partial t} + P_h^A(\nabla_u \tilde{u}) = 0, \tag{1.42}$$

where P_h^A is the orthogonal projection of $L^2(\Omega, \Lambda^1)$ onto $\mathcal{H}_1^A$.
15. In the context of Exercise 14, show that an irrotational Euler flow must be stationary, that is, the flow described by (1.42) is trivial. (*Hint*: By (1.16), $\partial\tilde{u}/\partial t = -d\big((1/2)|u|^2 + p\big)$, which is orthogonal to $\mathcal{H}_1^A$.)
16. Suppose Ω is a bounded region in $\mathbb{R}^2$, with $k + 1$ (smooth) boundary components γ_j. Show that $\mathcal{H}_1^A$ is the k-dimensional space

$$\mathcal{H}_1^A = \{*df : f \in C^\infty(\overline{\Omega}), \Delta f = 0 \text{ on } \Omega, f = c_j \text{ on } \gamma_j\},$$

where the c_j are arbitrary constants. Show that a holomorphic diffeomorphism $F : \overline{\Omega} \to \overline{\mathcal{O}}$ takes $\mathcal{H}_1^A(\Omega)$ to $\mathcal{H}_1^A(\mathcal{O})$.

17. If Ω is a planar region as in Exercise 16, show that the space V_σ of velocity fields for Euler flows defined by (1.6) can be characterized as

$$V_\sigma = \{u : \tilde{u} = *df,\ f \in C^\infty(\overline{\Omega}),\ f = c_j \text{ on } \gamma_j\}.$$

Given u in this space, an associated f is called a *stream function*. Show that it is constant along each streamline of u.

18. In the context of Exercise 17, note that $w = \operatorname{rot} u = -\Delta f$. Show that $u \cdot \nabla w = 0$, hence $\partial w/\partial t = 0$, whenever f satisfies a PDE of the form

$$\Delta f = \Phi(f) \text{ on } \Omega, \quad f = c_j \text{ on } \gamma_j. \tag{1.43}$$

19. When $\tilde{u}(0) = *df$ for f satisfying (1.43), show that the resulting flow is stationary, that is, $\partial\tilde{u}/\partial t = 0$, not merely $\partial w/\partial t = 0$. (*Hint*: In this case, $\tilde{u}$ satisfies the linear evolution equation

$$\frac{\partial \tilde{u}}{\partial t} + P_h^A(w * \tilde{u}) = 0,$$

as a consequence of (1.16). It suffices to show that $P_h^A(w * \tilde{u}(0)) = 0$, but indeed $w * \tilde{u}(0) = -\Phi(f)df = d\psi(f)$.)

Note: When Ω is simply connected, the argument simplifies.

20. Let $\overline{\Omega}$ be a compact Riemannian manifold, u a solution to (1.7)–(1.9), with associated vorticity $\tilde{w}$. When does

$$\frac{\partial \tilde{w}}{\partial t} = 0, \text{ for all } t \Longrightarrow \frac{\partial u}{\partial t} = 0?$$

Begin by considering the following cases:
(a) $\mathcal{H}^1(\overline{\Omega}) = 0$. (*Hint*: Use Hodge theory.)
(b) $\dim \mathcal{H}^1(\overline{\Omega}) = 1$. (*Hint*: Use conservation of energy.)
(c) $\overline{\Omega} \subset\subset \mathbb{R}^2$. (*Hint*: Generalize Exercise 19.)

21. Using the exercises on "spaces of gradient and divergence-free vector fields" in § 9 of Chap. 5, show that if we identify vector fields and 1-forms, the Leray projection P is given by

$$Pu = P_\delta^A u + P_h^A u, \quad \text{i.e.,} \quad (I - P)u = P_d^A u = d\delta G^A u. \tag{1.44}$$

22. Let Ω be a smooth, bounded region in $\mathbb{R}^3$ and u a solution to the Euler equation on $I \times \Omega$, where I is a t-interval containing 0. Assume the vorticity w vanishes on $\partial\Omega$ at $t = 0$.
(a) Show that $w = 0$ on $\partial\Omega$, for all $t \in I$.
(b) Show that the quantity

$$h(t) = \int_\Omega u(t,x) \cdot w(t,x)\, dx, \tag{1.45}$$

is independent of t. This is called the *helicity*. (*Hint*: Use formulas for the adjoint of ∇_u when $\operatorname{div} u = 0$; ditto for ∇_w; recall Exercise 2.)
(c) Show that the quantities

$$I(t) = \int_\Omega x \times w(t,x)\, dx, \quad A(t) = \int_\Omega |x|^2 w(t,x)\, dx \tag{1.46}$$

are independent of t. These are called the *impulse* and the *angular impulse*, respectively.
Consider these questions when the hypothesis on w is relaxed to w tangent to $\partial\Omega$ at $t = 0$.

23. Extend results on the conservation of helicity to other 3-manifolds Ω, via a computation of

$$(\partial_t + \mathcal{L}_u)(\tilde{u} \wedge \tilde{w}). \tag{1.47}$$

24. If we consider the motion of an incompressible fluid of *variable* density $\rho(t, x)$, the Euler equations are modified to

$$\rho(u_t + \nabla_u u) = -\text{grad } p, \quad \rho_t + \nabla_u \rho = 0, \tag{1.48}$$

and, as before, div $u = 0$, u tangent to $\partial\Omega$. Show that, in this case, the vorticity $\tilde{w} = d\tilde{u}$ satisfies

$$\partial_t \tilde{w} + \mathcal{L}_u \tilde{w} = \rho^{-2} d\rho \wedge dp. \tag{1.49}$$

(Results in subsequent sections will not apply to this case.)

2. Existence of solutions to the Euler equations

In this section we will examine the existence of solutions to the initial value problem for the Euler equation:

$$\frac{\partial u}{\partial t} + P\nabla_u u = 0, \qquad u(0) = u_0, \tag{2.1}$$

given div $u_0 = 0$, where P is the orthogonal projection of $L^2(M, TM)$ onto the space V_σ of divergence-free vector fields. We suppose M is compact without boundary; regions with boundary will be treated in the next section.

We take an approach very similar to that used for symmetric hyperbolic equations in § 2 of Chap. 16. Thus, with J_ε a Friedrichs mollifier such as used there, we consider the approximating equations

$$\frac{\partial u_\varepsilon}{\partial t} + PJ_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon = 0, \qquad u_\varepsilon(0) = u_0. \tag{2.2}$$

As in that case, we want to show that u_ε exists on an interval independent of ε, and we want to obtain uniform estimates that allow us to pass to the limit $\varepsilon \to 0$. We begin by estimating the L^2-norm. Noting that $u_\varepsilon(t) = Pu_\varepsilon(t)$, we have

$$\begin{aligned} \frac{d}{dt} \|u_\varepsilon(t)\|_{L^2}^2 &= -2(PJ_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, u_\varepsilon) \\ &= -2(\nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, J_\varepsilon u_\varepsilon). \end{aligned} \tag{2.3}$$

Now, generally, we have

$$\nabla_v^* w = -\nabla_v w - (\operatorname{div} v)w, \tag{2.4}$$

as shown in § 3 of Chap. 2. Consequently, when $\operatorname{div} v = 0$, we have

$$(\nabla_v w, w) = -(\nabla_v w, w) = 0. \tag{2.5}$$

Thus (2.3) yields $(d/dt)\|u_\varepsilon(t)\|_{L^2}^2 = 0$, or

$$\|u_\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{2.6}$$

It follows that (2.2) is solvable for all $t \in \mathbb{R}$, when $\varepsilon > 0$.

The next step, to estimate higher-order derivatives of u_ε, is accomplished in almost exact parallel with the analysis (1.8) of Chap. 16, for symmetric hyperbolic systems. Again, to make things simple, let us suppose $M = \mathbb{T}^n$; modifications for the more general case will be sketched below. Then P and J_ε can be taken to be convolution operators, so P, J_ε, and D^α all commute. Then

$$\begin{aligned}\frac{d}{dt}\,\|D^\alpha u_\varepsilon(t)\|_{L^2}^2 &= -2(D^\alpha P J_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, D^\alpha u_\varepsilon) \\ &= -2(D^\alpha L_\varepsilon J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon),\end{aligned} \tag{2.7}$$

where we have set

$$L_\varepsilon w = L(u_\varepsilon, D)w, \tag{2.8}$$

with

$$L(v, D)w = \nabla_v w, \tag{2.9}$$

a first-order differential operator on w whose coefficients $L_j(v)$ depend linearly on v. By (2.4),

$$L_\varepsilon + L_\varepsilon^* = 0, \tag{2.10}$$

since $\operatorname{div} u_\varepsilon = 0$, so (2.7) yields

$$\frac{d}{dt}\,\|D^\alpha u_\varepsilon(t)\|_{L^2}^2 = 2([L_\varepsilon, D^\alpha] J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon). \tag{2.11}$$

Now, just as in (1.13) of Chap. 16, the Moser estimates from § 3 of Chap. 13 yield

$$\begin{aligned} &\|[L_\varepsilon, D^\alpha]w\|_{L^2} \\ &\le C \sum_j \Bigl(\|L_j(u_\varepsilon)\|_{H^k} \|\partial_j w\|_{L^\infty} + \|\nabla L_j(u_\varepsilon)\|_{L^\infty} \|\partial_j w\|_{H^{k-1}} \Bigr). \end{aligned} \tag{2.12}$$

Keep in mind that $L_j(u_\varepsilon)$ is linear in u_ε. Applying this with $w = J_\varepsilon u_\varepsilon$, and summing over $|\alpha| \le k$, we have the basic estimate

$$\frac{d}{dt} \|u_\varepsilon(t)\|_{H^k}^2 \le C \|u_\varepsilon(t)\|_{C^1} \|u_\varepsilon(t)\|_{H^k}^2, \tag{2.13}$$

parallel to the estimate (1.15) in Chap. 16, but with a more precise dependence on $\|u_\varepsilon(t)\|_{C^1}$, which will be useful later on. From here, the elementary arguments used to prove Theorem 1.2 in Chap. 16 extend without change to yield the following:

Theorem 2.1. *Given $u_0 \in H^k(M)$, $k > n/2 + 1$, with div $u_0 = 0$, there is a solution u to (2.1) on an interval I about* 0*, with*

$$u \in L^\infty(I, H^k(M)) \cap Lip(I, H^{k-1}(M)). \tag{2.14}$$

We can also establish the uniqueness, and treat the stability and rate of convergence of u_ε to u, just as was done in Chap. 16, § 1. Thus, with $\varepsilon \in [0, 1]$, we compare a solution u to (2.1) to a solution u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} + P J_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon = 0, \quad u_\varepsilon(0) = v_0. \tag{2.15}$$

Setting $v = u - u_\varepsilon$, we can form an equation for v analogous to (1.25) in Chap. 16, and the analysis (1.25)–(1.36) there goes through without change, to give

$$\|v(t)\|_{L^2}^2 \le K_0(t) \Bigl(\|u_0 - v_0\|_{L^2}^2 + K_2(t) \|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1}, L^2)}^2 \Bigr). \tag{2.16}$$

Thus we have

Proposition 2.2. *Given $k > n/2 + 1$, solutions to (2.1) satisfying (2.14) are unique. They are limits of solutions u_ε to (2.2), and for $t \in I$,*

$$\|u(t) - u_\varepsilon(t)\|_{L^2} \le K_1(t) \|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1}, L^2)}. \tag{2.17}$$

Continuing to follow Chap. 16, we can next look at

$$\begin{aligned} \frac{d}{dt} \|D^\alpha J_\varepsilon u(t)\|_{L^2}^2 &= -2(D^\alpha J_\varepsilon P \nabla_u u, D^\alpha J_\varepsilon u) \\ &= -2(D^\alpha J_\varepsilon L(u, D)u, D^\alpha J_\varepsilon u), \end{aligned} \tag{2.18}$$

given the commutativity of P with $D^\alpha J_\varepsilon$, and then we can follow the analysis of (1.40)–(1.45) given there without any change, to get

$$\frac{d}{dt}\,\|J_\varepsilon u(t)\|^2_{H^k} \le C\big(1+\|u(t)\|_{C^1}\big)\|u(t)\|^2_{H^k}, \tag{2.19}$$

for the solution u to (2.1) constructed above. Now, as in the proof of Proposition 5.1 in Chap. 16, we can note that (2.19) is equivalent to an integral inequality, and pass to the limit $\varepsilon \to 0$, to deduce

$$\frac{d}{dt}\,\|u(t)\|^2_{H^k} \le C\big(1+\|u(t)\|_{C^1}\big)\|u(t)\|^2_{H^k}, \tag{2.20}$$

parallel to (1.46) of Chap. 16, but with a significantly more precise dependence on $\|u(t)\|_{C^1}$. Consequently, as in Proposition 1.4 of Chap. 16, we can sharpen the first part of (2.14) to

$$u \in C(I, H^k(M)).$$

Furthermore, we can deduce that if $u \in C(I, H^k(M))$ solves the Euler equation, $I = (-a, b)$, then u continues beyond the endpoints unless $\|u(t)\|_{C^1}$ blows up at an endpoint. However, for the Euler equations, there is the following important sharpening, due to Beale–Kato–Majda [BKM]:

Proposition 2.3. *If $u \in C(I, H^k(M))$ solves the Euler equations, $k > n/2 + 1$, and if*

$$\sup_{t\in I} \|w(t)\|_{L^\infty} \le K < \infty, \tag{2.21}$$

where w is the vorticity, then the solution u continues to an interval I', containing $\overline{I}$ in its interior, $u \in C(I', H^k(M))$.

For the proof, recall that if $\tilde{u}(t)$ and $\tilde{w}(t)$ are the 1-form and 2-form on M, associated to u and w, then

$$\tilde{w} = d\tilde{u}, \qquad \delta\tilde{u} = 0. \tag{2.22}$$

Hence $\delta\tilde{w} = \delta d\tilde{u} + d\delta\tilde{u} = \Delta\tilde{u}$, where Δ is the Hodge Laplacian, so

$$\tilde{u} = G\delta\tilde{w} + P_0\tilde{u}, \tag{2.23}$$

where P_0 is a projection onto the space of harmonic 1-forms on M, which is a finite-dimensional space of C^∞-forms. Now $G\delta$ is a pseudodifferential operator of order -1:

$$G\delta = A \in OPS^{-1}(M). \tag{2.24}$$

Consequently, $\|G\delta\tilde{w}\|_{H^{1,p}} \le C_p\|w\|_{L^p}$ for any $p \in (1,\infty)$. This breaks down for $p = \infty$, but, as we show below, we have, for any $s > n/2$,

$$\|A\tilde{w}\|_{C^1} \le C\big(1 + \log^+ \|\tilde{w}\|_{H^s}\big)\|\tilde{w}\|_{L^\infty} + C. \tag{2.25}$$

Therefore, under the hypothesis (2.21), we obtain an estimate

$$\|u(t)\|_{C^1} \le C\big(1 + \log^+ \|u\|^2_{H^k}\big), \tag{2.26}$$

provided $k > n/2 + 1$, using (2.23) and the facts that $\|\tilde{w}\|_{H^{k-1}} \le c\|u\|_{H^k}$ and that $\|u(t)\|_{L^2}$ is constant. Thus (2.20) yields the differential inequality

$$\frac{dy}{dt} \le C(1 + \log^+ y)y, \quad y(t) = \|u(t)\|^2_{H^k}. \tag{2.27}$$

Now one form of Gronwall's inequality (cf. Chap. 1, (5.19)–(5.21)) states that if $Y(t)$ solves

$$\frac{dY}{dt} = F(t, Y), \quad Y(0) = y(0), \tag{2.28}$$

while $dy/dt \le F(t, y)$, and if $\partial F/\partial y \ge 0$, then $y(t) \le Y(t)$ for $t \ge 0$. We apply this to $F(t, Y) = C(1 + \log^+ Y)Y$, so (2.28) gives

$$\int \frac{dY}{(1 + \log^+ Y)Y} = Ct + C_1. \tag{2.29}$$

Since

$$\int_1^\infty \frac{dY}{(1 + \log^+ Y)Y} = \infty, \tag{2.30}$$

we see that $Y(t)$ exists for all $t \in [0,\infty)$ in this case. This provides an upper bound

$$\|u(t)\|^2_{H^k} \le Y(t), \tag{2.31}$$

as long as (2.21) holds. Thus Proposition 2.3 will be proved once we establish the estimate (2.25). We will establish a general result, which contains (2.25).

Lemma 2.4. *If $P \in OPS^0_{1,0}$, $s > n/2$, then*

$$\|Pu\|_{L^\infty} \le C\|u\|_{L^\infty} \cdot \Big[1 + \log\Big(\frac{\|u\|_{H^s}}{\|u\|_{L^\infty}}\Big)\Big]. \tag{2.32}$$

We suppose the norms are arranged to satisfy $\|u\|_{L^\infty} \le \|u\|_{H^s}$. Another way to write the result is in the form

$$\|Pu\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Big(\log \frac{1}{\varepsilon}\Big)\|u\|_{L^\infty}, \tag{2.33}$$

for $0 < \varepsilon \le 1$, with C independent of ε. Then, letting $\varepsilon^\delta = \|u\|_{L^\infty}/\|u\|_{H^s}$ yields (2.32). The estimate (2.33) is valid when $s > n/2 + \delta$. We will derive (2.33) from an estimate relating the L^∞-, H^s-, and C^0_*-norms. The Zygmund spaces C^r_* are defined in § 8 of Chap. 13.

It suffices to prove (2.33) with P replaced by $P + cI$, where c is greater than the L^2-operator norm of P; hence we can assume $P \in OPS^0_{1,0}$ is elliptic and invertible, with inverse $Q \in OPS^0_{1,0}$. Then (2.33) is equivalent to

$$\|u\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Big(\log \frac{1}{\varepsilon}\Big)\|Qu\|_{L^\infty}. \tag{2.34}$$

Now since $Q : C^0_* \to C^0_*$, with inverse P, and the C^0_*-norm is weaker than the L^∞-norm, this estimate is a consequence of

$$\|u\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Big(\log \frac{1}{\varepsilon}\Big)\|u\|_{C^0_*}, \tag{2.35}$$

for $s > n/2 + \delta$. This result is proved in Chap. 13, § 8; see Proposition 8.11 there.

We now have (2.25), so the proof of Proposition 2.3 is complete. One consequence of Proposition 2.3 is the following classical result.

Proposition 2.5. *If* $\dim M = 2$, $u_0 \in H^k(M)$, $k > 2$, *and* $\operatorname{div} u_0 = 0$, *then the solution to the Euler equation (2.1) exists for all* $t \in \mathbb{R}$; $u \in C(\mathbb{R}, H^k(M))$.

Proof. Recall that in this case w is a scalar field and the vorticity equation is

$$\frac{\partial w}{\partial t} + \nabla_u w = 0, \tag{2.36}$$

which implies that, as long as $u \in C(I, H^k(M))$, $t \in I$,

$$\|w(t)\|_{L^\infty} = \|w(0)\|_{L^\infty}. \tag{2.37}$$

Thus the hypothesis (2.21) is fulfilled.

When $\dim M \ge 3$, the vorticity equation takes a more complicated form, which does not lead to (2.37). It remains a major outstanding problem to decide whether smooth solutions to the Euler equation (2.1) persist in this case. There are numerical studies of three-dimensional Euler flows, with particular attention to the evolution of the vorticity, such as [BM].

Having discussed details in the case $M = \mathbb{T}^n$, we now describe modifications when M is a more general compact Riemannian manifold without boundary. One modification is to estimate, instead of (2.7),

$$\begin{aligned} \frac{d}{dt}\|\Delta^\ell u_\varepsilon(t)\|_{L^2}^2 &= -2(\Delta^\ell P J_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, \Delta^\ell u_\varepsilon) \\ &= -2(\Delta^\ell P L_\varepsilon J_\varepsilon u_\varepsilon, \Delta^\ell J_\varepsilon u_\varepsilon), \end{aligned} \tag{2.38}$$

the latter identity holding provided Δ, P, and J_ε all commute. This can be arranged by taking $J_\varepsilon = e^{\varepsilon\Delta}$; P and Δ automatically commute here. In this case, with D^α replaced by Δ^ℓ, (2.11)–(2.12) go through, to yield the basic estimate (2.13), provided $k = 2\ell > n/2 + 1$. When $[n/2]$ is even, this gives again the results of Theorem 2.1–Proposition 2.5. When $[n/2]$ is odd, the results obtained this way are slightly weaker, if ℓ is restricted to be an integer.

An alternative approach, which fully recovers Theorem 2.1–Proposition 2.5, is the following. Let $\{X_j\}$ be a finite collection of vector fields on M, spanning T_xM at each x, and for $J = (j_1, \dots, j_k)$, let $X^J = \nabla_{X_{j_1}} \cdots \nabla_{X_{j_k}}$, a differential operator of order $k = |J|$. We estimate

$$\frac{d}{dt}\|X^J u_\varepsilon(t)\|_{L^2}^2 = -2(X^J P J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, X^J u_\varepsilon). \tag{2.39}$$

We can still arrange that P and J_ε commute, and write this as

$$\begin{aligned} &-2(L_\varepsilon X^J J_\varepsilon u_\varepsilon, X^J J_\varepsilon u_\varepsilon) - 2([X^J, L_\varepsilon] J_\varepsilon u_\varepsilon, X^J J_\varepsilon u_\varepsilon) \\ &\quad - 2(X^J L_\varepsilon J_\varepsilon u_\varepsilon, [X^J, P J_\varepsilon] u_\varepsilon) - 2([X^J, P J_\varepsilon] L_\varepsilon J_\varepsilon u_\varepsilon, X^J u_\varepsilon). \end{aligned} \tag{2.40}$$

Of these four terms, the first is analyzed as before, due to (2.10). For the second term we have the same type of Moser estimate as in (2.12). The new terms to analyze are the last two terms in (2.40). In both cases the key is to see that, for $\varepsilon \in (0, 1]$,

$$[X^J, P J_\varepsilon] \text{ is bounded in } OPS^{k-1}_{1,0}(M) \quad \text{if } |J| = k, \tag{2.41}$$

which follows from the containment $P \in OPS^0_{1,0}(M)$ and the boundedness of J_ε in $OPS^0_{1,0}(M)$. If we push one factor X_{j_1} in X^J from the left side to the right side of the third inner product in (2.40), we dominate each of the last two terms by

$$C\|L_\varepsilon J_\varepsilon u_\varepsilon\|_{H^{k-1}} \cdot \|u_\varepsilon\|_{H^k} \tag{2.42}$$

if $|J| = k$. To complete the estimate, we use the identity

$$\operatorname{div}(u \otimes v) = (\operatorname{div} v)u + \nabla_v u, \tag{2.43}$$

which yields

$$L_\varepsilon J_\varepsilon u_\varepsilon = \operatorname{div}(J_\varepsilon u_\varepsilon \otimes u_\varepsilon). \tag{2.44}$$

Now, by the Moser estimates, we have

$$\|L_\varepsilon J_\varepsilon u_\varepsilon\|_{H^{k-1}} \le C\,\|J_\varepsilon u_\varepsilon \otimes u_\varepsilon\|_{H^k} \le C\,\|u_\varepsilon\|_{L^\infty}\|u_\varepsilon\|_{H^k}. \tag{2.45}$$

Consequently, we again obtain the estimate (2.13), and hence the proofs of Theorem 2.1–Proposition 2.5 again go through.

So far in this section we have discussed strong solutions to the Euler equations, for which there is a uniqueness result known. We now give a result of [DM], on the existence of weak solutions to the two-dimensional Euler equations, with initial data less regular than in Proposition 2.5.

Proposition 2.6. *If* $\dim M = 2$ *and* $u_0 \in H^{1,p}(M)$*, for some* $p > 1$*, then there exists a weak solution to (2.1):*

$$u \in L^\infty\big(\mathbb{R}^+, H^{1,p}(M)\big) \cap C\big(\mathbb{R}^+, L^2(M)\big). \tag{2.46}$$

Proof. Take $f_j \in C^\infty(M),\ f_j \to u_0$ in $H^{1,p}(M)$, and let $v_j \in C^\infty(\mathbb{R}^+ \times M)$ solve

$$\frac{\partial v_j}{\partial t} + P\,\operatorname{div}(v_j \otimes v_j) = 0, \quad \operatorname{div} v_j = 0, \quad v_j(0) = f_j. \tag{2.47}$$

Here we have used (2.43) to write $\nabla_{v_j} v_j = \operatorname{div}(v_j \otimes v_j)$. Let $w_j = \operatorname{rot} v_j$, so $w_j(0) \to \operatorname{rot} u_0$ in $L^p(M)$. Hence $\|w_j(0)\|_{L^p}$ is bounded in j, and the vorticity equation implies

$$\|w_j(t)\|_{L^p} \le C, \quad \forall\, t, j. \tag{2.48}$$

Also $\|v_j(0)\|_{L^2}$ is bounded and hence $\|v_j(t)\|_{L^2}$ is bounded, so

$$\|v_j(t)\|_{H^{1,p}} \le C. \tag{2.49}$$

The Sobolev imbedding theorem gives $H^{1,p}(M) \subset L^{2+2\delta}(M),\ \delta > 0$, when $\dim M = 2$, so

$$\|v_j(t) \otimes v_j(t)\|_{L^{1+\delta}} \le C. \tag{2.50}$$

Hence, by (2.47),

$$\|\partial_t v_j(t)\|_{H^{-1,1+\delta}} \le C. \tag{2.51}$$

An interpolation of (2.49) and (2.51) gives

$$v_j \text{ bounded in } C^r\big([0,\infty), L^s(M)\big), \tag{2.52}$$

for some $r > 0,\ s > 2$. Together with (2.49), this implies

$$\|v_j\| \text{ compact in } C\big([0,T], L^2(M)\big), \tag{2.53}$$

for any $T < \infty$. Thus we can choose a subsequence v_{j_ν} such that

$$v_{j_\nu} \longrightarrow u \text{ in } C\big([0,T], L^2(M)\big), \quad \forall\, T < \infty, \tag{2.54}$$

the convergence being in norm. Hence

$$v_{j_\nu} \otimes v_{j_\nu} \longrightarrow u \otimes u \quad \text{in } C\big(\mathbb{R}^+, L^1(M)\big), \tag{2.55}$$

so

$$P \operatorname{div}(v_{j_\nu} \otimes v_{j_\nu}) \longrightarrow P \operatorname{div}(u \otimes u) \quad \text{in } C\big(\mathbb{R}^+, \mathcal{D}'(M)\big), \tag{2.56}$$

so the limit satisfies (2.1).

The question of the uniqueness of a weak solution obtained in Proposition 2.6 is open.

It is of interest to consider the case when rot $u_0 = w_0$ is not in $L^p(M)$ for some $p > 1$, but just in $L^1(M)$, or more generally, let w_0 be a finite measure on M. This problem was addressed in [DM], which produced a "measure-valued solution" (i.e., a "fuzzy solution," in the terminology used in Chap. 13, § 11). In [Del] it was shown that if w_0 is a *positive* measure (and $M = \mathbb{R}^2$), then there is a global weak solution; see also [Mj5]. Other work, with particular attention to cases where rot u_0 is a linear combination of delta functions, is discussed in [MP]; see also [Cho].

We also mention the extension of Proposition 2.6 in [Cha], to the case $w_0 \in L(\log L)$.

The following provides extra information on the limiting case $p = \infty$ of Proposition 2.6:

Proposition 2.7. *If* dim $M = 2$, *rot* $u_0 \in L^\infty(M)$, *and u is a weak solution to (2.1) given by Proposition 2.6, then*

$$u \in C(\mathbb{R}^+ \times M), \tag{2.57}$$

and, for each $t \in \mathbb{R}^+$, in any local coordinate chart on M, if $|x - y| \le 1/2$,

$$|u(t,x) - u(t,y)| \le C|x-y| \,\log \frac{1}{|x-y|}\, \|rot\ u_0\|_{L^\infty}. \tag{2.58}$$

Furthermore, u generates a flow, consisting of homeomorphisms $\mathcal{F}^t : M \to M$.

Proof. The continuity in (2.57) holds whenever $u_0 \in H^{1,p}(M)$ with $p > 2$, as can be deduced from (2.46), its corollary

$$\partial_t u \in L^\infty\big(\mathbb{R}^+, L^p(M)\big), \quad p > 2, \tag{2.59}$$

and interpolation. In fact, this gives a Hölder estimate on u. Next, we have

$$\|\mathrm{rot}\, u(t)\|_{L^\infty} \le \|\mathrm{rot}\, u_0\|_{L^\infty}, \quad \forall\, t \ge 0. \tag{2.60}$$

Since $u(t)$ is obtained from $\mathrm{rot}\, u(t)$ via (2.23), the estimate (2.58) is a consequence of the fact that

$$A \in OPS^{-1}(M) \Longrightarrow A : L^\infty(M) \to \mathrm{LLip}(M), \tag{2.61}$$

where, with $\delta(x,y) = \mathrm{dist}(x,y),\ \lambda(\delta) = \delta \log(1/\delta)$,

$$\mathrm{LLip}(M) = \big\{f \in C(M) : |f(x) - f(y)| \le C\lambda\big(\delta(x,y)\big)\big\}. \tag{2.62}$$

The result (2.61) can be established directly from integral kernel estimates. Alternatively, (2.61) follows from the inclusion

$$C^1_*(M) \subset \mathrm{LLip}(M), \tag{2.63}$$

since we know that $A \in OPS^{-1}(M) \Rightarrow A : L^\infty(M) \to C^1_*(M)$. In turn, the inclusion (2.63) is a consequence of the following characterization of LLip, due to [BaC]:

Let $\Psi_0 \in C_0^\infty(\mathbb{R}^n)$ satisfy $\Psi_0(\xi) = 1$ for $|\xi| \le 1$, and set $\Psi_k(\xi) = \Psi_0(2^{-k}\xi)$. Recall that, with $\psi_0 = \Psi_0,\ \psi_k = \Psi_k - \Psi_{k-1}$ for $k \ge 1$,

$$f \in C^0_*(\mathbb{R}^n) \Longleftrightarrow \|\psi_k(D) f\|_{L^\infty} \le C.$$

It follows that, for any $u \in \mathcal{E}'(\mathbb{R}^n)$,

$$u \in C^1_*(\mathbb{R}^n) \Longleftrightarrow \|\nabla \psi_k(D) u\|_{L^\infty} \le C. \tag{2.64}$$

By comparison, we have the following:

Lemma 2.8. *Given $u \in \mathcal{E}'(\mathbb{R}^n)$, we have*

$$u \in LLip(\mathbb{R}^n) \Longleftrightarrow \|\nabla \Psi_k(D) u\|_{L^\infty} \le C(k+1). \tag{2.65}$$

We leave the details of either of these approaches to (2.61) as an exercise. Now, for t-dependent vector fields satisfying (2.57)–(2.58), the existence and uniqueness of solutions of the associated ODEs, and continuous dependence on initial data, are established in Appendix A of Chap. 1, and the rest of Proposition 2.7 follows.

We mention that uniqueness has been established for solutions to (2.1) described by Proposition 2.7; see [Kt1] and [Yud]. A special case of Proposition 2.7 is that for which rot u_0 is piecewise constant. One says these are "vortex patches." There has been considerable interest in properties of the evolution of such vortex patches; see [Che3] and also [BeC].

Exercises

1. Refine the estimate (2.13) to

$$\frac{d}{dt}\,\|u_\varepsilon(t)\|^2_{H^k} \le C\,\|\nabla u_\varepsilon\|_{L^\infty}\,\|u_\varepsilon(t)\|^2_{H^k}, \tag{2.66}$$

   for $k > n/2 + 1$.
2. Using interpolation inequalities, show that if $k = s + r,\ s = n/2 + 1 + \delta$, then

$$\frac{d}{dt}\,\|u_\varepsilon(t)\|^2_{H^k} \le C\,\|u_\varepsilon(t)\|^{2(1+\gamma)}_{H^k}, \quad \gamma = \frac{s}{2k}.$$

3. Give a treatment of the Euler equation with an external force term:

$$\frac{\partial u}{\partial t} + \nabla_u u = -\text{grad } p + f, \quad \text{div } u = 0. \tag{2.67}$$

4. The enstrophy of an smooth Euler flow is defined by

$$\text{Ens}(t) = \|w(t)\|^2_{L^2(M)}, \quad w = \text{ vorticity}. \tag{2.68}$$

   If u is a smooth solution to (2.1) on $I \times M,\ t \in I$, and $\dim M = 3$, show that

$$\frac{d}{dt}\|w(t)\|^2_{L^2} = 2\big(\nabla_w u, w\big)_{L^2}. \tag{2.69}$$

5. Recall the deformation tensor associated to a vector field u,

$$\text{Def}(u) = \frac{1}{2}\big(\nabla u + \nabla u^t\big), \tag{2.70}$$

   which measures the degree to which the flow of u distorts the metric tensor g. Denote by ϑ_u the associated second-order, symmetric covariant tensor field (i.e., $\vartheta_u = (1/2)\mathcal{L}_u g$). Show that when $\dim M = 3$, (2.69) is equivalent to

$$\frac{d}{dt}\|w(t)\|^2_{L^2} = 2\int_M \vartheta_u(w, w)\, dV. \tag{2.71}$$

6. Show that the estimate (2.32) can be generalized and sharpened to

$$\|Pu\|_{L^\infty} \le C\,\|u\|_{C^0_*} \cdot \Big[1 + \log\Big(\frac{\|u\|_{H^{s,p}}}{\|u\|_{C^0_*}}\Big)\Big], \quad P \in OPS^0_{1,\delta}, \tag{2.72}$$

   given $\delta \in [0, 1),\ p \in (1, \infty)$, and $s > n/p$.
7. Prove Lemma 2.8, and hence deduce (2.61).

3. Euler flows on bounded regions

Having discussed the existence of solutions to the Euler equations for flows on a compact manifold without boundary in § 2, we now consider the case of a compact manifold $\overline{M}$ with boundary ∂M (and interior M). We want to solve the PDE

$$\frac{\partial u}{\partial t} + P\nabla_u u = 0, \quad \operatorname{div} u = 0, \tag{3.1}$$

with boundary condition

$$\nu \cdot u = 0 \quad \text{on } \partial M, \tag{3.2}$$

where ν is the normal to ∂M, and initial condition

$$u(0) = u_0. \tag{3.3}$$

We work on the spaces

$$V^k = \{u \in H^k(M, TM) : \operatorname{div} u = 0,\ \nu \cdot u\big|_{\partial M} = 0\}. \tag{3.4}$$

As shown in the third problem set in § 9 of Chap. 5 (see (9.79), V^0 is the closure of V_σ (given by (1.6)) in $L^2(M, TM)$. Hence the Leray projection P is the orthogonal projection of $L^2(M, TM)$ onto V^0. This result uses the Hodge decomposition, and results on the Hodge Laplacian with absolute boundary conditions, which also imply that

$$P : H^k(M, TM) \longrightarrow V^k. \tag{3.5}$$

Furthermore, the Hodge decomposition yields the characterization

$$(I - P)v = -\operatorname{grad} p, \tag{3.6}$$

where p is uniquely defined up to an additive constant by

$$-\Delta p = \operatorname{div} v \text{ on } M, \quad -\frac{\partial p}{\partial \nu} = \nu \cdot v \text{ on } \partial M. \tag{3.7}$$

See also Exercises 1–2 at the end of this section.

The following estimates will play a central role in our analysis of the Euler equations.

Proposition 3.1. *Let u and v be C^1-vector fields in $\overline{M}$. Assume $u \in V^k$. If $v \in H^{k+1}(M)$, then*

$$\left|(\nabla_u v, v)_{H^k}\right| \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big)\|v\|_{H^k}, \tag{3.8}$$

while if $v \in V^k$, *then*

$$\big\|(1-P)\nabla_u v\big\|_{H^k} \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big). \tag{3.9}$$

Proof. We begin with the $k = 0$ case of (3.8). Indeed, Green's formula gives

$$(\nabla_u v, w)_{L^2} = -(v, \nabla_u w)_{L^2} - \big(v, (\operatorname{div} u)w\big)_{L^2} + \int_{\partial M} \langle \nu, u\rangle\, \langle v, w\rangle\, dS. \tag{3.10}$$

If div $u = 0$ and $\nu \cdot u\big|_{\partial M} = 0$, the last two terms vanish, so the $k = 0$ case of (3.8) is sharpened to

$$(\nabla_u v, v)_{L^2} = 0 \quad \text{if } u \in V^0 \tag{3.11}$$

and v is C^1 on $\overline{M}$. This also holds if $u \in V^0 \cap C(\overline{M}, T)$ and $v \in H^1$.

To treat (3.8) for $k \ge 1$, we use the following inner product on $H^k(M,T)$. Pick a finite set of smooth vector fields $\{X_j\}$, spanning $T_x\overline{M}$ for each $x \in \overline{M}$, and set

$$(u, v)_{H^k} = \sum_{|J|\le k} (X^J u, X^J v)_{L^2}, \tag{3.12}$$

where $X^J = \nabla_{X_{j_1}} \cdots \nabla_{X_{j_\ell}}$ are as in (2.39), $|J| = \ell$. Now, we have

$$(X^J \nabla_u v, X^J v)_{L^2} = (\nabla_u X^J v, X^J v)_{L^2} + ([X^J, \nabla_u]v, X^J v)_{L^2}. \tag{3.13}$$

The first term on the right vanishes, by (3.11). As for the second, as in (2.12) we have the Moser estimate

$$\big\|[X^J, \nabla_u]v\big\|_{L^2} \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big). \tag{3.14}$$

This proves (3.8).

In order to establish (3.9), it is useful to calculate div $\nabla_u v$. In index notation $X = \nabla_u v$ is given by $X^j = v^j{}_{;k}u^k$, so div $X = X^j{}_{;j}$ yields

$$\operatorname{div} \nabla_u v = v^j{}_{;k;j}u^k + v^j{}_{;k}u^k{}_{;j}. \tag{3.15}$$

If M is flat, we can simply change the order of derivatives of v; more generally, using the Riemann curvature tensor R,

$$v^j{}_{;k;j} = v^j{}_{;j;k} + R^j{}_{\ell jk}v^\ell. \tag{3.16}$$

Noting that $R^j{}_{\ell jk} = \mathrm{Ric}_{\ell k}$ is the Ricci tensor, we have

$$\text{(3.17)} \qquad \operatorname{div} \nabla_u v = \nabla_u(\operatorname{div} v) + \mathrm{Ric}(u, v) + \mathrm{Tr}\big((\nabla u)(\nabla v)\big),$$

where ∇u and ∇v are regarded as tensor fields of type $(1, 1)$. When $\operatorname{div} v = 0$, of course the first term on the right side of (3.17) disappears, so

$$\text{(3.18)} \qquad \operatorname{div} v = 0 \Longrightarrow \operatorname{div} \nabla_u v = \mathrm{Tr}\big((\nabla u)(\nabla v)\big) + \mathrm{Ric}(u, v).$$

Note that only first-order derivatives of v appear on the right. Thus P acts on $\nabla_u v$ more like the identity than it might at first appear.

To proceed further, we use (3.6) to write

$$\text{(3.19)} \qquad (1 - P)\nabla_u v = -\operatorname{grad} \varphi,$$

where, parallel to (3.7), φ satisfies

$$\text{(3.20)} \qquad -\Delta\varphi = \operatorname{div} \nabla_u v \text{ on } M, \quad -\frac{\partial\varphi}{\partial\nu} = \nu \cdot \big(\nabla_u v\big) \text{ on } \partial M.$$

The computation of $\operatorname{div} \nabla_u v$ follows from (3.18). To analyze the boundary value in (3.20), we use the identity $\langle \nu, \nabla_u v\rangle = \nabla_u\langle \nu, v\rangle - \langle \nabla_u \nu, v\rangle$, and note that when u and v are tangent to ∂M, the first term on the right vanishes. Hence,

$$\text{(3.21)} \qquad \langle \nu, \nabla_u v\rangle = -\langle \nabla_u \nu, v\rangle = \widetilde{II}(u, v),$$

where $\widetilde{II}$ is the second fundamental form of ∂M. Thus (3.20) can be rewritten as

$$\text{(3.22)} \qquad -\Delta\varphi = \mathrm{Tr}\big((\nabla u)(\nabla v)\big) + \mathrm{Ric}(u, v) \text{ on } M, \quad \frac{\partial\varphi}{\partial\nu} = -\widetilde{II}(u, v).$$

Note that in the last expression for $\partial\varphi/\partial\nu$ there are no derivatives of v. Now, by (3.22) and the estimates for the Neumann problem derived in Chap. 5, we have

$$\text{(3.23)} \qquad \|\nabla\varphi\|_{H^k} \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big),$$

which proves (3.9).

Note that (3.8)–(3.9) yield the estimate

$$\text{(3.24)} \qquad \big|(P\nabla_u v, v)_{H^k}\big| \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big)\|v\|_{H^k},$$

given $u \in V^k, v \in V^{k+1}$.

In order to solve (3.1)–(3.3), we use a Galerkin-type method, following [Tem2]. Fix $k > n/2 + 1$, where $n = \dim M$, and take $u_0 \in V^k$. We use the inner product on V^k, derived from (3.12). Now there is an isomorphism $B_0 : V^k \to (V^k)'$, defined by $\langle B_0 v, w\rangle = (v, w)_{V^k}$. Using $V^k \subset V^0 \subset (V^k)'$, we define an unbounded, self-adjoint operator B on V^0 by

$$\mathcal{D}(B) = \{v \in V^k : B_0 v \in V^0\}, \quad B = B_0\big|_{\mathcal{D}(B)}. \tag{3.25}$$

This is a special case of the Friedrichs extension method, discussed in general in Appendix A, § 8. It follows from the compactness of the inclusion $V^k \hookrightarrow V^0$ that B^{-1} is compact, so V^0 has an orthonormal basis $\{w_j : j = 1, 2, \dots\}$ such that $Bw_j = \lambda_j w_j$, $\lambda_j \nearrow \infty$. Let P_j be the orthogonal projection of V^0 onto the span of $\{w_1, \dots, w_j\}$. It is useful to note that

$$(P_j u, v)_{V^0} = (u, P_j v)_{V^0} \quad \textit{and} \quad (P_j u, v)_{V^k} = (u, P_j v)_{V^k}. \tag{3.26}$$

Our approximating equation will be

$$\frac{\partial u_j}{\partial t} + P_j \nabla_{u_j} u_j = 0, \quad u_j(0) = P_j u_0. \tag{3.27}$$

Here, we extend P_j to be the orthogonal projection of $L^2(M, TM)$ onto the span of $\{w_1, \dots, w_j\}$.

We first estimate the V^0-norm (i.e., the L^2-norm) of u_j, using

$$\begin{aligned} \frac{d}{dt} \|u_j(t)\|_{V^0}^2 &= -2(P_j \nabla_{u_j} u_j, u_j)_{V^0} \\ &= -2(\nabla_{u_j} u_j, u_j)_{L^2}. \end{aligned} \tag{3.28}$$

By (3.11), $(\nabla_{u_j} u_j, u_j)_{L^2} = 0$, so

$$\|u_j(t)\|_{V^0} = \|P_j u_0\|_{L^2}. \tag{3.29}$$

Hence solutions to (3.27) exist for all $t \in \mathbb{R}$, for each j.

Our next goal is to estimate higher-order derivatives of u_j, so that we can pass to the limit $j \to \infty$. We have

$$\frac{d}{dt} \|u_j(t)\|_{V^k}^2 = -2(P_j \nabla_{u_j} u_j, u_j)_{V^k} = -2(P \nabla_{u_j} u_j, u_j)_{V^k}, \tag{3.30}$$

using (3.26). We can estimate this by (3.24), so we obtain the basic estimate:

$$\frac{d}{dt} \|u_j(t)\|_{V^k}^2 \le C \|u_j\|_{C^1} \|u_j\|_{V^k}^2. \tag{3.31}$$

This is parallel to (2.13), so what is by now a familiar argument yields our existence result:

Theorem 3.2. *Given $u_0 \in V^k,\ k > n/2 + 1$, there is a solution to (3.1)–(3.3) for t in an interval I about* 0*, with*

$$u \in L^\infty(I, V^k) \cap Lip(I, V^{k-1}). \tag{3.32}$$

The solution is unique, in this class of functions.

The last statement, about uniqueness, as well as results on stability and rate of convergence as $j \to \infty$, follow as in Proposition 2.2.

If u is a solution to (3.1)–(3.3) satisfying (3.32) with initial data $u_0 \in V^k$, we want to estimate the rate of change of $\|u(t)\|^2_{H^k}$, as was done in (2.18)–(2.20). Things will be a little more complicated, due to the presence of a boundary ∂M. Following [KL], we define the smoothing operators J_ε on $H^k(M, TM)$ as follows. Assume M is an open subset (with closure $\overline{M}$) of the compact Riemannian manifold $\widetilde{M}$ without boundary, and let

$$E : H^\ell(M, T) \longrightarrow H^\ell(\widetilde{M}, T), \quad 0 \le \ell \le k + 1,$$

be an extension operator, such as we constructed in Chap. 4. Let $R : H^\ell(\widetilde{M}, T) \to H^\ell(M, T)$ be the restriction operator, and set

$$J_\varepsilon u = R\widetilde{J}_\varepsilon E u, \tag{3.33}$$

where $\widetilde{J}_\varepsilon$ is a Friedrichs mollifier on $\widetilde{M}$. If we apply J_ε to the solution $u(t)$ of current interest, we have

$$\begin{aligned} \frac{d}{dt}\|J_\varepsilon u(t)\|^2_{H^k} &= -2(J_\varepsilon P \nabla_u u, J_\varepsilon u)_{H^k} \\ &= -2(J_\varepsilon \nabla_u u, J_\varepsilon u)_{H^k} + 2\big(J_\varepsilon(1-P)\nabla_u u, J_\varepsilon u\big)_{H^k}. \end{aligned} \tag{3.34}$$

Using (3.9), we estimate the last term by

$$\begin{aligned} &2\big|\big(J_\varepsilon(1-P)\nabla_u u, J_\varepsilon u\big)_{H^k}\big| \\ &\qquad \le C\big\|(1-P)\nabla_u u\big\|_{H^k} \cdot \|u\|_{H^k} \le C\|u(t)\|_{C^1}\|u(t)\|^2_{H^k}. \end{aligned} \tag{3.35}$$

To analyze the rest of the right side of (3.34), write

$$(J_\varepsilon \nabla_u u, J_\varepsilon u)_{H^k} = \sum_{|J| \le k} \big(X^J J_\varepsilon \nabla_u u, X^J J_\varepsilon u\big)_{L^2}, \tag{3.36}$$

using (3.12). Now we have

$$X^J J_\varepsilon \nabla_u u = X^J [J_\varepsilon, \nabla_u] u + [X^J, \nabla_u] J_\varepsilon u + \nabla_u (X^J J_\varepsilon u). \tag{3.37}$$

We look at these three terms successively. First, by (3.14),

$$\big\|[X^J, \nabla_u] J_\varepsilon u\big\|_{L^2} \le C\, \|u(t)\|_{C^1(\overline{M})} \|u(t)\|_{H^k(M)}. \tag{3.38}$$

Next, as in (1.44)–(1.45) of Chap. 16 on hyperbolic PDE, we claim to have an estimate

$$\big\|[J_\varepsilon, \nabla_u] u\big\|_{H^k(M)} \le C\, \|u(t)\|_{C^1(\overline{M})} \|u\|_{H^k(M)}. \tag{3.39}$$

To obtain this, we can use a Friedrichs mollifier $\widetilde{J}_\varepsilon$ on $\widetilde{M}$ with the property that

$$\operatorname{supp} w \subset K \Rightarrow \operatorname{supp} \widetilde{J}_\varepsilon w \subset K, \quad K = \widetilde{M} \setminus M. \tag{3.40}$$

In that case, if $\widetilde{u} = Eu$ and $\widetilde{w} = Ew$, then

$$[J_\varepsilon, \nabla_u] w = R[\widetilde{J}_\varepsilon, \nabla_{\widetilde{u}}] \widetilde{w}. \tag{3.41}$$

Thus (3.39) follows from known estimates for $\widetilde{J}_\varepsilon$.

Finally, the $L^2(M)$-inner product of the last term in (3.37) with $X^J J_\varepsilon u$ is zero. Thus we have a bound

$$\big|(J_\varepsilon \nabla_u u, J_\varepsilon u)_{H^k}\big| \le C\, \|u(t)\|_{C^1} \|u(t)\|_{H^k}^2, \tag{3.42}$$

and hence

$$\frac{d}{dt} \big\| J_\varepsilon u(t)\big\|_{H^k}^2 \le C\, \|u(t)\|_{C^1} \|u(t)\|_{H^k}^2. \tag{3.43}$$

As before, we can convert this to an integral inequality and take $\varepsilon \to 0$, obtaining

$$\|u(t)\|_{H^k}^2 \le \|u_0\|_{H^k}^2 + C \int_0^t \|u(s)\|_{C^1(\overline{M})} \|u(s)\|_{H^k}^2\, ds. \tag{3.44}$$

As with the exploitation of (2.19)–(2.20), we have

Proposition 3.3. *If $k > n/2 + 1$, $u_0 \in V^k$, the solution u to (3.1)–(3.3) given by Theorem 3.2 satisfies*

$$u \in C(I, V^k). \tag{3.45}$$

Furthermore, if I is an open interval on which (3.45) holds, u solving (3.1)–(3.3), and if

$$\sup_{t \in I} \|u(t)\|_{C^1(\overline{M})} \le K < \infty, \tag{3.46}$$

then the solution u continues to an interval I', containing $\overline{I}$ in its interior, $u \in C(I', V^k)$.

We will now extend the result of [BKM], Proposition 2.3, to the Euler flow on a region with boundary. Our analysis follows [Fer] in outline, except that, as in § 2, we make use of some of the Zygmund space analysis developed in § 8 of Chap. 13.

Proposition 3.4. *If $u \in C(I, V^k)$ solves the Euler equation, with $k > n/2 + 1$, $I = (-a, b)$, and if the vorticity w satisfies*

$$\sup_{t \in I} \|w(t)\|_{L^\infty} \le K < \infty, \tag{3.47}$$

then the solution u continues to an interval I', containing $\overline{I}$ in its interior, $u \in C(I', V^k)$.

To start the proof, we need a result parallel to (2.23), relating u to w.

Lemma 3.5. *If $\tilde{u}$ and $\tilde{w}$ are the 1-form and 2-form on $\overline{M}$, associated to u and w, then*

$$\tilde{u} = \delta G^A \tilde{w} + P_h^A \tilde{u}, \tag{3.48}$$

where G^A is the Green operator for Δ, with absolute boundary conditions, and P_h^A the orthogonal projection onto the space of harmonic 1-forms with absolute boundary conditions.

Proof. We know that

$$d\tilde{u} = \tilde{w}, \quad \delta \tilde{u} = 0, \quad \iota_n \tilde{u} = 0. \tag{3.49}$$

In particular, $\tilde{u} \in H_A^1(M, \Lambda^1)$, defined by (9.11) of Chap. 5. Thus we can write the Hodge decomposition of $\tilde{u}$ as

$$\tilde{u} = (d + \delta) G^A (d + \delta) \tilde{u} + P_h^A \tilde{u}. \tag{3.50}$$

See Exercise 2 in the first exercise set of § 9, Chap. 5. By (3.49), this gives (3.48).

Now since G^A is the solution operator to a regular elliptic boundary problem, it follows from Theorem 8.9 (complemented by (8.54)–(8.55)) of Chap. 13 that

$$G^A : C^0(\overline{M}, \Lambda^2) \longrightarrow C_*^2(\overline{M}, \Lambda^2), \tag{3.51}$$

where $C_*^2(\overline{M})$ is a Zygmund space, defined by (8.37)–(8.41) of Chap. 13. Hence, from (3.48), we have

$$\|\tilde{u}(t)\|_{C_*^1} \le C \|\tilde{w}(t)\|_{L^\infty} + C \|\tilde{u}(t)\|_{L^2}. \tag{3.52}$$

Of course, the last term is equal to $C\|\tilde{u}(0)\|_{L^2}$. Thus, under the hypothesis (3.47), we have

$$\|u(t)\|_{C^1_*} \le K' < \infty, \quad t \in I. \tag{3.53}$$

Now the estimate (8.53) of Chap. 13 gives

$$\|u(t)\|_{C^1} \le C\Big[1 + \log^+\big(\|u(t)\|_{H^k}\big)\Big], \tag{3.54}$$

for any $k > n/2 + 1$, parallel to (2.26).

To prove Proposition 3.4, we can exploit (3.43) in the same way we did (2.19), to obtain, via (3.54), the estimate

$$\frac{dy}{dt} \le C\big(1 + \log^+ y\big)y, \qquad y(t) = \|u(t)\|^2_{H^k}. \tag{3.55}$$

A use of Gronwall's inequality exactly as in (2.27)–(2.31) finishes the proof.

As in § 2, one consequence of Proposition 3.4 is the classical global existence result when $\dim M = 2$.

Proposition 3.6. *If dim $M = 2$ and $u_0 \in V^k$, $k > 2$, then the solution to the Euler equations (3.1)–(3.3) exists for all $t \in \mathbb{R}$; $u \in C(\mathbb{R}, V^k)$.*

Proof. As in (2.36), the vorticity w is a scalar field, satisfying

$$\frac{\partial w}{\partial t} + \nabla_u w = 0.$$

Since u is tangent to ∂M, this again yields

$$\|w(t)\|_{L^\infty} = \|w(0)\|_{L^\infty}.$$

Exercises

1. Show that if $u \in L^2(M, TM)$ and $\operatorname{div} u = 0$, then $\nu \cdot u\big|_{\partial M}$ is well defined in $H^{-1}(\partial M)$. Hence (3.4) is well defined for $k = 0$.
2. Show that the result (3.6)–(3.7) specifying $(I - P)v$ follows from (1.44). (*Hint*: Take $p = -\delta G^A \tilde{v}$.)
3. Show that the result (3.5) that $P : H^k(M, TM) \to V^k$ follows from (1.44). Show that V^k is dense in V^ℓ, for $0 \le \ell < k$.
4. For $s \in [0, \infty)$, define V^s by (3.4) with $s = k$, not necessarily an integer. Equivalently,

$$V^s = V^0 \cap H^s(M, TM).$$

Demonstrate the interpolation property

$$[V^0, V^k]_\theta = V^{k\theta}, \quad 0 < \theta < 1.$$

(*Hint*: Show that $P : H^s(M, TM) \to V^s$, and make use of this fact.)

5. Let u be a 1-form on M. Show that $d^*du = v$, where, in index notation,

$$-v_j = u_{j;k}{}^{;k} - u_{k;j}{}^{;k}.$$

In analogy with (3.15)–(3.16), reorder the derivatives in the last term to deduce that $d^*du = \nabla^*\nabla u - dd^*u + \mathrm{Ric}(u)$, or equivalently,

$$(d^*d + dd^*)u = \nabla^*\nabla u + \mathrm{Ric}(u), \tag{3.56}$$

which is a special case of the Weitzenbock formula. Compare with (4.16) of Chap. 10.

6. Construct a Friedrichs mollifier on $\widetilde{M}$, a compact manifold without boundary, having the property (3.40). (*Hint*: In the model case $\mathbb{R}^n$, consider convolution by $\varepsilon^{-n}\varphi(x/\varepsilon)$, where we require $\int \varphi(x)\,dx = 1$, and $\varphi \in C_0^\infty(\mathbb{R}^n)$ is supported on $|x - e_1| \le 1/2,\ e_1 = (1, 0, \dots, 0)$.)

4. Navier–Stokes equations

We study here the Navier–Stokes equations for the viscous incompressible flow of a fluid on a compact Riemannian manifold M. The equations take the form

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \mathcal{L} u - \text{ grad } p, \quad \text{div } u = 0, \quad u(0) = u_0. \tag{4.1}$$

for the velocity field u, where p is the pressure, which is eliminated from (4.1) by applying P, the orthogonal projection of $L^2(M, TM)$ onto the kernel of the divergence operator. In (4.1), ∇ is the covariant derivative. For divergence-free fields u, one has the identity

$$\nabla_u u = \mathrm{div}(u \otimes u), \tag{4.2}$$

the right side being the divergence of a second-order tensor field. This is a special case of the general identity $\mathrm{div}(u \otimes v) = \nabla_v u + (\mathrm{div}\, v)u$, which arose in (2.43). The quantity ν in (4.1) is a positive constant. If $M = \mathbb{R}^n$, $\mathcal{L}$ is the Laplace operator Δ, acting on the separate components of the velocity field u.

Now, if M is not flat, there are at least two candidates for the role of the Laplace operator, the Hodge Laplacian

$$\Delta = -(d^*d + dd^*),$$

or rather its conjugate upon identifying vector fields and 1-forms via the Riemannian metric ("lowering indices"), and the Bochner Laplacian

$$\mathcal{L}_B = -\nabla^*\nabla,$$

where $\nabla : C^\infty(M, TM) \to C^\infty(M, T^* \otimes T)$ arises from the covariant derivative. In order to see what $\mathcal{L}$ is in (4.1), we record another form of (4.1), namely

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \operatorname{div} S - \operatorname{grad} p, \quad \operatorname{div} u = 0, \tag{4.3}$$

where S is the "stress tensor"

$$S = \nabla u + \nabla u^t = 2 \operatorname{Def} u,$$

also called the "deformation tensor." This tensor was introduced in Chap. 2, § 3; cf. (3.35). In index notation, $S^{jk} = u^{j;k} + u^{k;j}$, and the vector field div S is given by

$$S^{jk}{}_{;k} = u^{j;k}{}_{;k} + u^{k;j}{}_{;k}.$$

The first term on the right is $-\nabla^*\nabla u$. The second term can be written (as in (3.16)) as

$$u^k{}_{;k}{}^{;j} + R^k{}_{\ell k}{}^j u^\ell = \big(\operatorname{grad} \operatorname{div} u + \operatorname{Ric}(u)\big)^j.$$

Thus, as long as div $u = 0$,

$$\operatorname{div} S = -\nabla^*\nabla u + \operatorname{Ric}(u).$$

By comparison, a special case of the Weitzenbock formula, derivable in a similar fashion (see Exercise 5 in the previous section), is

$$\Delta u = -\nabla^*\nabla u - \operatorname{Ric}(u)$$

when u is a 1-form. In other words, on ker div,

$$\mathcal{L}u = \Delta u + 2 \operatorname{Ric}(u). \tag{4.4}$$

The Hodge Laplacian Δ has the property of commuting with the projection P onto ker div, as long as M has no boundary. For simplicity of exposition, we will restrict attention throughout the rest of this section to the case of Riemannian manifolds M for which Ric is a constant scalar multiple c_0 of the identity, so

$$\mathcal{L} = \Delta + 2c_0 \quad \text{on ker div}, \tag{4.5}$$

and the right side also commutes with P. Then we can rewrite (4.1) as

$$\frac{\partial u}{\partial t} = \nu \mathcal{L}u - P\nabla_u u, \quad u(0) = u_0, \tag{4.6}$$

where, as above, the vector field u_0 is assumed to have divergence zero. Let us note that, in any case,

$$\mathcal{L} = -2 \operatorname{Def}^* \operatorname{Def}$$

is a negative-semidefinite operator.

We will perform an analysis similar to that of § 2; in this situation we will obtain estimates independent of ν, and we will be in a position to pass to the limit $\nu \to 0$. We begin with the approximating equation

$$\frac{\partial u_\varepsilon}{\partial t} + PJ_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon = \nu J_\varepsilon \mathcal{L} J_\varepsilon u_\varepsilon, \quad u_\varepsilon(0) = u_0, \tag{4.7}$$

parallel to (2.2), using a Friedrichs mollifier J_ε. Arguing as in (2.3)–(2.6), we obtain

$$\frac{d}{dt} \|u_\varepsilon(t)\|^2_{L^2} = -4\nu \|\mathrm{Def}\, J_\varepsilon u_\varepsilon(t)\|^2_{L^2} \le 0, \tag{4.8}$$

hence

$$\|u_\varepsilon(t)\|_{L^2} \le \|u_0\|_{L^2}. \tag{4.9}$$

Thus it follows that (4.7) is solvable for all $t \in \mathbb{R}$ whenever $\nu \ge 0$ and $\varepsilon > 0$.

We next estimate higher-order derivatives of u_ε, as in § 2. For example, if $M = \mathbb{T}^n$, following (2.7)–(2.13), we obtain now

$$\begin{aligned} \frac{d}{dt} \|u_\varepsilon(t)\|^2_{H^k} &\le C \|u_\varepsilon(t)\|_{C^1} \|u_\varepsilon(t)\|^2_{H^k} - 4\nu \|\mathrm{Def}\, J_\varepsilon u_\varepsilon(t)\|^2_{H^k} \\ &\le C \|u_\varepsilon(t)\|_{C^1} \|u_\varepsilon(t)\|^2_{H^k}, \end{aligned} \tag{4.10}$$

for $\nu \ge 0$. For more general M, one has similar results parallel to analyses of (2.34) and (2.35). Note that the factor C is independent of ν. As in Theorem 2.1 (see also Theorem 1.2 of Chap. 16), these estimates are sufficient to establish a local existence result, for a limit point of u_ε as $\varepsilon \to 0$, which we denote by u_ν.

Theorem 4.1. *Given $u_0 \in H^k(M)$, $k > n/2 + 1$, with div $u_0 = 0$, there is a solution u_ν on an interval $I = [0, A)$ to (4.6), satisfying*

$$u_\nu \in L^\infty(I, H^k(M)) \cap Lip(I, H^{k-2}(M)). \tag{4.11}$$

The interval I and the estimate of u_ν in $L^\infty(I, H^k(M))$ can be taken independent of $\nu \ge 0$.

We can also establish the uniqueness, and treat the stability and rate of convergence of u_ε to $u = u_\nu$ as before. Thus, with $\varepsilon \in [0, 1]$, we compare a solution $u = u_\nu$ to (4.6) to a solution $u_{\nu\varepsilon} = w$ to

$$\frac{\partial w}{\partial t} + PJ_\varepsilon \nabla_w J_\varepsilon w = \nu J_\varepsilon \mathcal{L} J_\varepsilon w, \quad w(0) = w_0. \tag{4.12}$$

Setting $v = u_\nu - u_{\nu\varepsilon}$, we have again an estimate of the form (2.16), hence:

Proposition 4.2. *Given $k > n/2 + 1$, solutions to (4.6) satisfying (4.11) are unique. They are limits of solutions $u_{\nu\varepsilon}$ to (4.7), and, for $t \in I$,*

$$\|u_\nu(t) - u_{\nu\varepsilon}(t)\|_{L^2} \le K_1(t)\|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}, \tag{4.13}$$

the quantity on the right being independent of $\nu \in [0, \infty)$.

Continuing to follow § 2, we can next look at

$$\begin{aligned} \frac{d}{dt} \|D^\alpha J_\varepsilon u_\nu(t)\|_{L^2}^2 = & -2\big(D^\alpha J_\varepsilon L(u_\nu, D)u_\nu, D^\alpha J_\varepsilon u_\nu\big) \\ & - 2\nu\|\mathrm{Def}\ D^\alpha J_\varepsilon u_\nu(t)\|_{L^2}^2, \end{aligned} \tag{4.14}$$

parallel to (2.18), and as in (2.19)–(2.20) deduce

$$\begin{aligned} \frac{d}{dt} \|u_\nu(t)\|_{H^k}^2 &\le C\|u_\nu(t)\|_{C^1}\|u_\nu(t)\|_{H^k}^2 - 4\nu\|\mathrm{Def}\ u_\nu(t)\|_{H^k}^2 \\ &\le C\|u_\nu(t)\|_{C^1}\|u_\nu(t)\|_{H^k}^2. \end{aligned} \tag{4.15}$$

This time, the argument leading to $u \in C(I, H^k(M))$, in the case of the solution to a hyperbolic equation or the Euler equation (2.1), gives for u_ν solving (4.6) with $u_0 \in H^k(M)$,

$$u_\nu \text{ is continuous in } t \text{ with values in } H^k(M), \quad \text{at } t = 0, \tag{4.16}$$

provided $k > n/2 + 1$. At other points $t \in I$, one has right continuity in t. This argument does not give left continuity since the evolution equation (4.6) is not well posed backward in time. However, a much stronger result holds for positive $t \in I$, as will be seen in (4.17) below.

Having considered results with estimates independent of $\nu \ge 0$, we now look at results for fixed $\nu > 0$ (or which at least require ν to be bounded away from 0). Then (4.6) behaves like a semilinear parabolic equation, and we will establish the following analogue of Proposition 1.3 of Chap. 15. We assume $n \ge 2$.

Proposition 4.3. *If div $u_0 = 0$ and $u_0 \in L^p(M)$, with $p > n = \dim M$, and if $\nu > 0$, then (4.6) has a unique short-time solution on an interval $I = [0, T]$:*

$$u = u_\nu \in C(I, L^p(M)) \cap C^\infty((0, T) \times M). \tag{4.17}$$

Proof. It is useful to rewrite (4.6) as

$$\frac{\partial u}{\partial t} + P\ \mathrm{div}(u \otimes u) = \nu\mathcal{L}u, \quad u(0) = u_0, \tag{4.18}$$

using the identity (4.2). In this form, the parallel with (1.16) of Chap. 15, namely,

$$\frac{\partial u}{\partial t} = \nu \Delta u + \sum \partial_j F_j(u),$$

is evident. The proof is done in the same way as the results on semilinear parabolic equations there. We write (4.18) as an integral equation

$$u(t) = e^{t\nu\mathcal{L}} u_0 - \int_0^t e^{(t-s)\nu\mathcal{L}} P \operatorname{div}\bigl(u(s) \otimes u(s)\bigr)\, ds = \Psi u(t), \tag{4.19}$$

and look for a fixed point of

$$\Psi : C(I, X) \to C(I, X), \quad X = L^p(M) \cap \ker \operatorname{div}. \tag{4.20}$$

As in the proof of Propositions 1.1 and 1.3 in Chap. 15, we fix $\alpha > 0$, set

$$Z = \{u \in C([0, T], X) : u(0) = u_0,\ \|u(t) - u_0\|_X \le \alpha\}, \tag{4.21}$$

and show that if $T > 0$ is small enough, then $\Psi : Z \to Z$ is a contraction map. For that, we need a Banach space Y such that

$$\Phi : X \to Y \text{ is Lipschitz, uniformly on bounded sets,} \tag{4.22}$$

$$e^{t\mathcal{L}} : Y \to X, \quad \text{for } t > 0, \tag{4.23}$$

and, for some $\gamma < 1$,

$$\|e^{t\mathcal{L}}\|_{\mathcal{L}(Y,X)} \le C t^{-\gamma}, \quad \text{for } t \in (0, 1]. \tag{4.24}$$

The map Φ in (4.22) is

$$\Phi(u) = P \operatorname{div}(u \otimes u). \tag{4.25}$$

We set

$$Y = H^{-1,p/2}(M) \cap \ker \operatorname{div},$$

and these conditions are all seen to hold, as long as $p > n$; to check (4.24), use (1.15) of Chap. 15. Thus we have the solution u_ν to (4.6), belonging to $C([0, T], L^p(M))$. To obtain the smoothness stated in (4.17), the proof of smoothness in Proposition 1.3 of Chap. 15 applies essentially verbatim.

Local existence with initial data $u_0 \in L^n(M)$ was established in [Kt4]. We also mention results on local existence when u_0 belongs to certain Morrey spaces, given in [Fed, Kt5, T2].

Note that the length of the interval I on which u_ν is produced in Proposition 4.3 depends only on $\|u_0\|_{L^p}$ (given M and ν). Hence one can get global existence provided one can bound $\|u(t)\|_{L^p(M)}$, for some $p > n$. In view of this we have the following variant of Proposition 2.3 (with a much simpler proof):

Proposition 4.4. *Given $\nu > 0$, $p > n$, if $u \in C([0,T), L^p(M))$ solves (4.6), and if the vorticity w satisfies*

$$\sup_{t\in[0,T)} \|w(t)\|_{L^q} \le K < \infty, \quad q = \frac{np}{n+p}, \tag{4.26}$$

then the solution u continues to an interval $[0,T')$, for some $T' > T$,

$$u \in C([0,T'), L^p(M)) \cap C^\infty((0,T') \times M),$$

solving (4.6).

Proof. As in the proof of Proposition 2.3, we have

$$u = Aw + P_0 u,$$

where P_0 is a projection onto a finite-dimensional space of smooth fields, $A \in OPS^{-1}(M)$. Since we know that $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$ and since $A : L^q \to H^{1,q} \subset L^p$, we have an L^p-bound on $u(t)$ as $t \nearrow T$, as needed to prove the proposition.

Note that we require on q precisely that $q > n/2$, in order for the corresponding p to exceed n.

Note also that when $\dim M = 2$, the vorticity w is scalar and satisfies the PDE

$$\frac{\partial w}{\partial t} + \nabla_u w = \nu(\Delta + 2c_0)w; \tag{4.27}$$

as long as (4.5) holds, generalizing the $\nu = 0$ case, we have $\|w(t)\|_{L^\infty} \le e^{2\nu c_0 t}\|w(0)\|_{L^\infty}$ (this time by the maximum principle), and consequently global existence.

When $\dim M = 3$, w is a vector field and (as long as (4.5) holds) the vorticity equation is

$$\frac{\partial w}{\partial t} + \nabla_u w - \nabla_w u = \nu \mathcal{L} w. \tag{4.28}$$

It remains an open problem whether (4.1) has global solutions in the space $C^\infty((0,\infty) \times M)$ when $\dim M \ge 3$, despite the fact that one thinks this should be easier for $\nu > 0$ than in the case of the Euler equation. We describe here a couple of results that are known in the case $\nu > 0$.

Proposition 4.5. *Let $k > n/2 + 1$, $\nu > 0$. If $\|u_0\|_{H^k}$ is small enough, then (4.6) has a global solution in $C([0,\infty), H^k) \cap C^\infty((0,\infty) \times M)$.*

What "small enough" means will arise in the course of the proof, which will be a consequence of the first part of the estimate (4.15). To proceed from this, we can pick positive constants A and B such that

$$\|\mathrm{Def}\, u\|_{H^k}^2 \geq A\|u\|_{H^k}^2 - B\|u\|_{L^2}^2,$$

so (4.15) yields

$$\frac{d}{dt}\,\|u(t)\|_{H^k}^2 \leq \big\{C\|u(t)\|_{C^1} - 2\nu A\big\}\|u\|_{H^k}^2 + 2\nu B\|u(t)\|_{L^2}^2.$$

Now suppose

$$\|u_0\|_{L^2}^2 \leq \delta \quad \text{and} \quad \|u_0\|_{H^k}^2 \leq L\delta;$$

L will be specified below. We require $L\delta$ to be so small that

$$\|v\|_{H^k}^2 \leq 2L\delta \Longrightarrow \|v\|_{C^1} \leq \frac{\nu A}{C}. \tag{4.29}$$

Recall that $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$. Consequently, as long as $\|u(t)\|_{H^k}^2 \leq 2L\delta$, we have

$$\frac{dy}{dt} \leq -\nu A y + 2\nu B\delta, \quad y(t) = \|u(t)\|_{H^k}^2.$$

Such a differential inequality implies

$$y(t) \leq \max\big\{y(t_0), 2BA^{-1}\delta\big\}, \;\; \text{for } t \geq t_0. \tag{4.30}$$

Consequently, if we take $L = 2B/A$ and pick δ so small that (4.29) holds, we have a global bound $\|u(t)\|_{H^k}^2 \leq L\delta$, and corresponding global existence.

A substantially sharper result of this nature is given in Exercises 4–9 at the end of this section.

We next prove the famous Hopf theorem, on the existence of global *weak* solutions to (4.6), given $\nu > 0$, for initial data $u_0 \in L^2(M)$. The proof is parallel to that of Proposition 1.7 in Chap. 15. In order to make the arguments given here resemble those for viscous flow on Euclidean space most closely, we will assume throughout the rest of this section that (4.5) holds with $c_0 = 0$ (i.e., that Ric $= 0$).

Theorem 4.6. *Given $u_0 \in L^2(M)$, div $u_0 = 0$, $\nu > 0$, the (4.6) has a weak solution for $t \in (0,\infty)$,*

$$\begin{aligned} u \in L^\infty\big(\mathbb{R}^+, L^2(M)\big) \cap L^2_{loc}\big(\mathbb{R}^+, H^1(M)\big) \\ \cap\; Lip_{loc}\big(\mathbb{R}^+, H^{-2}(M) + H^{-1,1}(M)\big). \end{aligned} \tag{4.31}$$

We will produce u as a limit point of solutions u_ε to a slight modification of (4.7), namely we require each J_ε to be a projection; for example, take

$J_\varepsilon = \chi(\varepsilon\Delta)$, where $\chi(\lambda)$ is the characteristic function of $[-1, 1]$. Then J_ε commutes with Δ and with P. We also require $u_\varepsilon(0) = J_\varepsilon u_0$; then $u_\varepsilon(t) = J_\varepsilon u_\varepsilon(t)$. Now from (4.9), which holds here also, we have

$$\{u_\varepsilon : \varepsilon \in (0, 1]\} \text{ is bounded in } L^\infty(\mathbb{R}^+, L^2). \tag{4.32}$$

This follows from (4.8), further use of which yields

$$4\nu \int_0^T \|\mathrm{Def}\, u_\varepsilon(t)\|_{L^2}^2\, dt = \|J_\varepsilon u_0\|_{L^2}^2 - \|u_\varepsilon(T)\|_{L^2}^2, \tag{4.33}$$

as in (1.39) of Chap. 15. Hence, for each bounded interval $I = [0, T]$,

$$\{u_\varepsilon\} \text{ is bounded in } L^2(I, H^1(M)). \tag{4.34}$$

Now, as in (4.18), we write our PDE for u_ε as

$$\frac{\partial u_\varepsilon}{\partial t} + PJ_\varepsilon \operatorname{div}(u_\varepsilon \otimes u_\varepsilon) = \nu\Delta u_\varepsilon, \tag{4.35}$$

since $J_\varepsilon \Delta J_\varepsilon u_\varepsilon = \Delta u_\varepsilon$. From (4.32) we see that

$$\{u_\varepsilon \otimes u_\varepsilon : \varepsilon \in (0, 1]\} \text{ is bounded in } L^\infty(\mathbb{R}^+, L^1(M)). \tag{4.36}$$

We use the inclusion $L^1(M) \subset H^{-n/2-\delta}(M)$. Hence, by (4.35), for each $\delta > 0$,

$$\{\partial_t u_\varepsilon\} \text{ is bounded in } L^2(I, H^{-n/2-1-\delta}(M)), \tag{4.37}$$

so

$$\{u_\varepsilon\} \text{ is bounded in } H^1(I, H^{-n/2-1-\delta}(M)). \tag{4.38}$$

As in the proof of Proposition 1.7 in Chap. 15, we now interpolate between (4.34) and (4.38), to obtain

$$\{u_\varepsilon\} \text{ is bounded in } H^s(I, H^{1-s-s(n/2+1+\delta)}(M)), \tag{4.39}$$

and hence, as in (1.45) there,

$$\{u_\varepsilon\} \text{ is compact in } L^2(I, H^{1-\gamma}(M)), \tag{4.40}$$

for all $\gamma > 0$.

Now the rest of the argument is easy. We can pick a sequence $u_k = u_{\varepsilon_k}$ $(\varepsilon_k \to 0)$ such that

$$u_k \to u \quad \text{in } L^2([0,T], H^{1-\gamma}(M)), \text{ in norm,} \tag{4.41}$$

arranging that this hold for all $T < \infty$, and from this it is easy to deduce that u is a desired weak solution to (4.6).

Solutions of (4.6) obtained as limits of u_ε as in the proof of Theorem 4.6 are called Leray–Hopf solutions to the Navier–Stokes equations. The uniqueness and smothness of a Leray–Hopf solution so constructed remain open problems if dim $M \geq 3$. We next show that when dim $M = 3$, such a solution is smooth except for at most a fairly small exceptional set.

Proposition 4.7. *If dim $M = 3$ and u is a Leray–Hopf solution of (4.6), then there is an open dense subset $\mathcal{J}$ of $(0, \infty)$ such that $\mathbb{R}^+ \setminus \mathcal{J}$ has Lebesgue measure zero and*

$$u \in C^\infty(\mathcal{J} \times M). \tag{4.42}$$

Proof. For $T > 0$ arbitrary, $I = [0, T]$, use (4.40). With $u_k = u_{\varepsilon_k}$, passing to a subsequence, we can suppose

$$\|u_{k+1} - u_k\|_E \leq 2^{-k}, \quad E = L^2(I, H^{1-\gamma}(M)). \tag{4.43}$$

Now if we set

$$\Gamma(t) = \sup_k \|u_k(t)\|_{H^{1-\gamma}}, \tag{4.44}$$

we have

$$\Gamma(t) \leq \|u_1(t)\|_{H^{1-\gamma}} + \sum_{k=1}^{\infty} \|u_{k+1}(t) - u_k(t)\|_{H^{1-\gamma}}, \tag{4.45}$$

hence

$$\Gamma \in L^2(I). \tag{4.46}$$

In particular, $\Gamma(t)$ is finite *almost everywhere*. Let

$$S = \{t \in I : \Gamma(t) < \infty\}. \tag{4.47}$$

For small $\gamma > 0$, $H^{1-\gamma}(M) \subset L^p(M)$ with p close to 6 when dim $M = 3$, and products of two elements in $H^{1-\gamma}(M)$ belong to $H^{1/2-\gamma'}(M)$, with $\gamma' > 0$ small. Recalling that u_ε satisfies (4.35), we now apply the analysis used in the proof of Proposition 4.3 to u_k, concluding that, for each $t_0 \in S$, there exists $T(t_0) > 0$, depending only on $\Gamma(t_0)$, such that, for small $\gamma' > 0$, we have

$$\{u_k\} \text{ bounded in } C\big([t_0, t_0 + T(t_0)], H^{1-\gamma}(M)\big) \cap C^\infty\big((t_0, t_0 + T(t_0)) \times M\big).$$

Consequently, if we form the open set

$$\mathcal{J}_T = \bigcup_{t_0 \in S} \big(t_0, t_0 + T(t_0)\big), \tag{4.48}$$

then any weak limit u of $\{u_k\}$ has the property that $u \in C^\infty(\mathcal{J}_T \times M)$. It remains only to show that $I \setminus \mathcal{J}_T$ has Lebesgue measure zero; the denseness of $\mathcal{J}_T$ in I will automatically follow. To see this, fix $\delta_1 > 0$. Since $\text{meas}(I \setminus S) = 0$, there exists $\delta_2 > 0$ such that if $S_{\delta_2} = \{t \in S : T(t) \geq \delta_2\}$, then $\text{meas}(I \setminus S_{\delta_2}) < \delta_1$. But $\mathcal{J}_T$ contains the translate of S_{δ_2} by $\delta_2/2$, so $\text{meas}(I \setminus \mathcal{J}_T) \leq \delta_1 + \delta_2/2$. This completes the proof.

There are more precise results than this. As shown in [CKN], when $M = \mathbb{R}^3$, the subset of $\mathbb{R}^+ \times M$ on which a certain type of Leray–Hopf solution, called "admissible," is not smooth, must have vanishing one-dimensional Hausdorff measure. In [CKN] it is shown that admissible Leray–Hopf solutions exist.

We now discuss some results regarding the uniqueness of weak solutions to the Navier–Stokes equations (4.6). Thus, let $I = [0, T]$, and suppose

$$u_j \in L^\infty\big(I, L^2(M)\big) \cap L^2\big(I, H^1(M)\big), \quad j = 1, 2, \tag{4.49}$$

are two weak solutions to

$$\frac{\partial u_j}{\partial t} + P \operatorname{div}(u_j \otimes u_j) = \nu \Delta u_j, \qquad u_j(0) = u_0, \tag{4.50}$$

where $u_0 \in L^2(M),\ \operatorname{div} u_0 = 0$. Then $v = u_1 - u_2$ satisfies

$$\frac{\partial v}{\partial t} + P \operatorname{div}(u_1 \otimes v + v \otimes u_2) = \nu \Delta v, \quad v(0) = 0. \tag{4.51}$$

We will estimate the rate of change of $\|v(t)\|_{L^2}^2$, using the following:

Lemma 4.8. *Provided*

$$v \in L^2(I, H^1(M)) \quad \textit{and} \quad \frac{\partial v}{\partial t} \in L^2(I, H^{-1}(M)), \tag{4.52}$$

then $\|v(t)\|_{L^2}^2$ *is absolutely continuous and*

$$\frac{d}{dt} \|v(t)\|_{L^2}^2 = 2(v_t, v)_{L^2} \in L^1.$$

Furthermore, $v \in C(I, L^2)$.

Proof. The identity is clear for smooth v, and the rest follows by approximation.

By hypothesis (4.49), the functions u_j satisfy the first part of (4.52). By (4.50), the second part of (4.52) is satisfied provided $u_j \otimes u_j \in L^2(I \times M)$, that is, provided

$$u_j \in L^4(I \times M). \tag{4.53}$$

We now proceed to investigate the L^2-norm of v, solving (4.51). If u_j satisfy both (4.49) and (4.53), we have

$$\begin{aligned} \frac{d}{dt}\,\|v(t)\|_{L^2}^2 &= -2(\nabla_v u_1, v) - 2(\nabla_{u_2} v, v) - 2\nu\|\nabla v\|_{L^2}^2 \\ &= 2(u_1, \nabla_v v) - 2\nu\|\nabla v\|_{L^2}^2, \end{aligned} \tag{4.54}$$

since $\nabla_v^* = -\nabla_v$ and $\nabla_{u_2}^* = -\nabla_{u_2}$ for these two divergence-free vector fields. Consequently, we have

$$\frac{d}{dt}\,\|v(t)\|_{L^2}^2 \le 2\|u_1\|_{L^4} \cdot \|v\|_{L^4} \cdot \|\nabla v\|_{L^2} - 2\nu\|\nabla v\|_{L^2}^2. \tag{4.55}$$

Our goal is to get a differential inequality implying $\|v(t)\|_{L^2} = 0$; this requires estimating $\|v(t)\|_{L^4}$ in terms of $\|v(t)\|_{L^2}$ and $\|\nabla v\|_{L^2}$. Since $H^{1/2}(M^2) \subset L^4(M^2)$ and $H^1(M^3) \subset L^6(M^3)$, we can use the following estimates when $\dim M = 2$ or 3:

$$\begin{aligned} \|v\|_{L^4} &\le C\|v\|_{L^2}^{1/2} \cdot \|\nabla v\|_{L^2}^{1/2} + C\|v\|_{L^2}, \quad \dim M = 2, \\ \|v\|_{L^4} &\le C\|v\|_{L^2}^{1/4} \cdot \|\nabla v\|_{L^2}^{3/4} + C\|v\|_{L^2}, \quad \dim M = 3. \end{aligned} \tag{4.56}$$

With these estimates, we are prepared to prove the following uniqueness result:

Proposition 4.9. *Let u_1 and u_2 be weak solutions to (4.6), satisfying (4.49) and (4.53). Suppose $\dim M = 2$ or 3; if $\dim M = 3$, suppose furthermore that*

$$u_1 \in L^8\big(I, L^4(M)\big). \tag{4.57}$$

If $u_1(0) = u_2(0)$, then $u_1 = u_2$ on $I \times M$.

Proof. For $v = u_1 - u_2$, we have the estimate (4.55). Using (4.56), we have

$$\begin{aligned} 2\|u_1\|_{L^4}\|v\|_{L^4}\|\nabla v\|_{L^2} \le \nu\|\nabla v\|_{L^2}^2 &+ C\nu^{-3}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^4 \\ &+ C\nu^{-1}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^2 \end{aligned} \tag{4.58}$$

when $\dim M = 2$, and

$$\begin{aligned} 2\|u_1\|_{L^4}\|v\|_{L^4}\|\nabla v\|_{L^2} \le \nu\|\nabla v\|_{L^2}^2 &+ C\nu^{-7}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^8 \\ &+ C\nu^{-1}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^2 \end{aligned} \tag{4.59}$$

when $\dim M = 3$. Consequently,

$$\frac{d}{dt}\,\|v(t)\|_{L^2}^2 \le C_n(\nu)\|v(t)\|_{L^2}^2\Big(\|u_1\|_{L^4}^p + \|u_1\|_{L^4}^2\Big), \tag{4.60}$$

where $p = 4$ if $\dim M = 2$ and $p = 8$ if $\dim M = 3$. Then Gronwall's inequality gives

$$\|v(t)\|_{L^2}^2 \le \|u_1(0) - u_2(0)\|_{L^2}^2 \exp\Big\{C_n(\nu)\int_0^t \Big(\|u_1(s)\|_{L^4}^p + \|u_1(s)\|_{L^4}^2\Big)\,ds\Big\},$$

proving the proposition.

We compare the properties of the last proposition with properties that Leray–Hopf solutions can be shown to have:

Proposition 4.10. *If u is a Leray–Hopf solution to (4.1) and $I = [0,T]$, then*

$$u \in L^4(I \times M) \quad \text{if } \dim M = 2, \tag{4.61}$$

and

$$u \in L^{8/3}\big(I, L^4(M)\big) \quad \text{if } \dim M = 3. \tag{4.62}$$

Also,

$$u \in L^2\big(I, L^4(M)\big) \quad \text{if } \dim M = 4. \tag{4.63}$$

Proof. Since $u \in L^\infty(I, L^2) \cap L^2(I, H^1)$, (4.61) follows from the first part of (4.56), and (4.62) follows from the second part. Similarly, (4.63) follows from the inclusion

$$H^1(M^4) \subset L^4(M^4).$$

In particular, the hypotheses of Proposition 4.9 are seen to hold for Leray–Hopf solutions when $\dim M = 2$, so there is a uniqueness result in that case. On the other hand, there is a gap between the conclusion (4.62) and the hypothesis (4.57) when $\dim M = 3$.

Exercises

In the exercises below, assume for simplicity that $\mathrm{Ric} = 0$, so (4.5) holds with $c_0 = 0$.

1. One place dissipated energy can go is into heat. Suppose a "temperature" function $T = T(t,x)$ satisfies a PDE

$$\frac{\partial T}{\partial t} + \nabla_u T = \alpha\Delta T + 4\nu|\mathrm{Def}\,u|^2, \tag{4.64}$$

coupled to (4.6), where α is a positive constant. Show that the total energy

$$E(t) = \int_M \Big\{ |u(t,x)|^2 + T(t,x) \Big\}\, dx$$

is conserved, provided u and T possess sufficient smoothness. Discuss local existence of solutions to the coupled equations (4.1) and (4.64).

2. Show that under the hypotheses of Theorem 4.1,

$$u_\nu \to v, \quad \text{as } \nu \to 0,$$

v being the solution to the Euler equation (i.e., the solution to the $\nu = 0$ case of (4.6)). In what topology can you demonstrate this convergence?

3. Give the details of the interpolation argument yielding (4.39).
4. Combining Propositions 4.3 and 4.5, show that if div $u_0 = 0$, $p > n$, and $\|u_0\|_{L^p}$ is small enough, then (4.6) has a global solution

$$u \in C\big([0,\infty), L^p\big) \cap C^\infty\big((0,\infty) \times M\big).$$

In Exercises 5–10, suppose dim $M = 3$. Let u solve (4.6), with vorticity w.

5. Show that the vorticity satisfies

$$\frac{d}{dt}\, \|w(t)\|_{L^2}^2 = 2(\nabla_w u, w) - 2\nu \|\nabla w\|_{L^2}^2. \tag{4.65}$$

6. Using $(\nabla_w u, w) = -(u, \nabla_w w) - \big(u, (\text{div } w) w\big)$, deduce that

$$\frac{d}{dt}\, \|w(t)\|_{L^2}^2 \le C \|u\|_{L^3} \cdot \|w\|_{L^6} \cdot \|\nabla w\|_{L^2} - 2\nu \|\nabla w\|_{L^2}^2.$$

Show that

$$\|w\|_{L^6} \le C \|\nabla w\|_{L^2} + C \|u\|_{L^2}, \tag{4.66}$$

and hence

$$\frac{d}{dt}\, \|w(t)\|_{L^2}^2 \le C \|u\|_{L^3} \Big(\|\nabla w\|_{L^2}^2 + \|u\|_{L^2}^2 \Big) - 2\nu \|\nabla w\|_{L^2}^2.$$

7. Show that

$$\|u\|_{L^3} \le C \|u\|_{L^2}^{1/2} \cdot \|w\|_{L^2}^{1/2} + C \|u\|_{L^2},$$

and hence, if $\|u_0\|_{L^2} = \beta$,

$$\frac{d}{dt}\, \|w(t)\|_{L^2}^2 \le C \big(\beta^{1/2} \|w\|_{L^2}^{1/2} + \beta \big) \big(\|\nabla w\|_{L^2}^2 + \beta^2 \big) - 2\nu \|\nabla w\|_{L^2}^2.$$

8. Show that there exist constants $A, B \in (0,\infty)$, depending on M, such that

$$\|\nabla w\|_{L^2}^2 \ge A \|w\|_{L^2}^2 - B\beta^2, \tag{4.67}$$

and hence that $y(t) = \|w(t)\|_{L^2}^2$ satisfies

$$\frac{dy}{dt} \le C\big(\beta^{1/2} y^{1/4} + \beta\big)\beta^2 - \nu A y + \nu B \beta^2$$

as long as

$$C\left(\beta^{1/2} y^{1/4} + \beta\right) < \nu. \tag{4.68}$$

9. As long as (4.68) holds, $dy/dt \le \nu\beta^2(1+B) - \nu A y$. As in (4.30), this gives

$$y(t) \le \max\big\{y(t_0), \beta^2(1+B)A^{-1}\big\},, \quad \text{for } t \ge t_0.$$

Thus (4.68) persists as long as $C\big(\beta(1+B)^{1/4} A^{-1/4} + \beta\big) < \nu$. Deduce a global existence result for the Navier–Stokes equations (4.1) when $\dim M = 3$ and

$$\begin{gathered} C\left(\|u_0\|_{L^2}^{1/2} \|w(0)\|_{L^2}^{1/2} + \|u_0\|_{L^2}\right) < \nu, \\ C\|u_0\|_{L^2}\left(1 + (1+B)^{1/4} A^{-1/4}\right) < \nu. \end{gathered} \tag{4.69}$$

For other global existence results, see [Bon] and [Che1].

10. Deduce from (4.65) that

$$\frac{d}{dt}\,\|w(t)\|_{L^2}^2 \le C\big(\|w\|_{L^3}^3 + \|w\|_{L^3}^2 \|u\|_{L^2}\big) - 2\nu \|\nabla w\|_{L^2}^2.$$

Work on this, applying

$$\|w\|_{L^3} \le C\|w\|_{L^2}^{1/2} \cdot \|w\|_{L^6}^{1/2},$$

in concert with (4.66).

11. Generalize results of this section to the case where no extra hypotheses are made on Ric. Consider also cases where *some* assumptions are made (e.g. Ric ≥ 0, or Ric ≤ 0). (*Hint*: Instead of (4.6) or (4.18), we have

$$\frac{\partial u}{\partial t} = \nu \Delta u - P \operatorname{div}(u \otimes u) + PBu, \quad Bu = 2\nu \operatorname{Ric}(u).)$$

12. Assume u is a Killing field on M, that is, u generates a group of isometries of M. According to Exercise 11 of § 1, u provides a steady solution to the Euler equation (1.11). Show that u also provides a steady solution to the Navier–Stokes equation (4.1), provided $\mathcal{L}$ is given by (4.4). If $M = S^2$ or S^3, with its standard metric, show that such u (if not zero) does not give a steady solution to (4.1) if $\mathcal{L}$ is taken to be either the Hodge Laplacian Δ or the Bochner Laplacian $\nabla^*\nabla$. Physically, would you expect such a vector field u to give rise to a viscous force?

13. Show that a t-dependent vector field $u(t)$ on $[0, T) \times M$ satisfying

$$u \in L^1\big([0, T), \mathrm{Lip}^1(M)\big)$$

generates a well-defined flow consisting of homeomorphisms.

14. Let u be a solution to (4.1) with $u_0 \in L^p(M),\ p > n$, as in Proposition 4.3. Show that, given $s \in (0, 2]$,

$$\|u(t)\|_{H^{s,p}} \le C t^{-s/2}, \quad 0 < t < T.$$

Taking $s \in (1+n/p, 2)$, deduce from Exercise 13 that u generates a well-defined flow consisting of homeomorphisms.
For further results on flows generated by solutions to the Navier–Stokes equations, see [ChL] and [FGT].

5. Viscous flows on bounded regions

In this section we let $\overline{\Omega}$ be a compact manifold with boundary and consider the Navier–Stokes equations on $\mathbb{R}^+ \times \Omega$,

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \mathcal{L} u - \operatorname{grad} p, \quad \operatorname{div} u = 0. \tag{5.1}$$

We will assume for simplicity that Ω is flat, or more generally, $\mathrm{Ric} = 0$ on Ω, so, by (4.4), $\mathcal{L} = \Delta$. When $\partial\Omega \neq \emptyset$, we impose the "no-slip" boundary condition

$$u = 0, \ \text{ for } x \in \partial\Omega. \tag{5.2}$$

We also set an initial condition

$$u(0) = u_0. \tag{5.3}$$

We consider the following spaces of vector fields on Ω, which should be compared to the spaces V_σ of (1.6) and V^k of (3.4). First, set

$$\mathcal{V} = \{u \in C_0^\infty(\Omega, T\Omega) : \operatorname{div} u = 0\}. \tag{5.4}$$

Then set

$$W^k = \text{ closure of } \mathcal{V} \text{ in } H^k(\Omega, T), \quad k = 0, 1. \tag{5.5}$$

Lemma 5.1. *We have $W^0 = V^0$ and*

$$W^1 = \{u \in H_0^1(\Omega, T) : \mathit{div}\, u = 0\}. \tag{5.6}$$

Proof. Clearly, $W^0 \subset V^0$. As noted in § 1, it follows from (9.79)–(9.80) of Chap. 5 that

$$(V^0)^\perp = \{\nabla p : p \in H^1(\Omega)\}, \tag{5.7}$$

the orthogonal complement taken in $L^2(\Omega, T)$. To show that $\mathcal{V}$ is dense in V^0, suppose $u \in L^2(\Omega, T)$ and $(u, v) = 0$ for all $v \in \mathcal{V}$. We need to conclude that $u = \nabla p$ for some $p \in H^1(\Omega)$. To accomplish this, let us make note of the following simple facts. First, (5.8) $\nabla : H^1(\Omega) \to L^2(\Omega, T)$ has closed range $\mathcal{R}_0$; $\quad \mathcal{R}_0^\perp = \ker \nabla^* = V^0.$

The last identity follows from (5.7). Second, and more directly useful,

$$\begin{aligned} &\nabla : L^2(\Omega) \to H^{-1}(\Omega, T) \text{ has closed range } \mathcal{R}_1, \\ &\mathcal{R}_1^\perp = \ker \nabla^* = \{u \in H_0^1(\Omega, T) : \operatorname{div} u = 0\} = W_\Omega^{(1)}, \end{aligned} \tag{5.9}$$

the last identity defining $W_\Omega^{(1)}$.

Now write Ω as an increasing union $\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \nearrow \Omega$, each Ω_j having smooth boundary. We claim $u_j = u|_{\Omega_j}$ is orthogonal to $W_{\Omega_j}^{(1)}$, defined as in (5.9). Indeed, if $v \in W_{\Omega_j}^{(1)}$ (and you extend v to be 0 on $\Omega \setminus \Omega_j$), then $\rho(\varepsilon\sqrt{-\Delta})v = v_\varepsilon$ belongs to $\mathcal{V}$ if $\hat{\rho} \in C_0^\infty(\mathbb{R})$ and ε is small, and $v_\varepsilon \to v$ in H^1-norm if $\rho(0) = 1$, so $(u, v) = \lim (u, v_\varepsilon) = 0$. From (5.9) it follows that there exist $p_j \in L^2(\Omega_j)$ such that $u = \nabla p_j$ on Ω_j; p_j is uniquely determined up to an additive constant (if Ω_j is connected) so we can make all the p_j fit together, giving $u = \nabla p$. If $u \in L^2(\Omega, T)$, p must belong to $H^1(\Omega)$.

The same argument works if $u \in H^{-1}(\Omega, T)$ is orthogonal to $\mathcal{V}$; we obtain $u = \nabla p$ with $p \in L^2(\Omega)$; one final application of (5.9) then yields (5.6), finishing off the lemma.

Thus, if $u_0 \in W^1$, we can rephrase (5.1), demanding that

$$\frac{d}{dt} (u, v)_{W^0} + (\nabla_u u, v)_{W^0} = -\nu (u, v)_{W^1}, \quad \text{for all } v \in \mathcal{V}. \tag{5.10}$$

Alternatively, we can rewrite the PDE as

$$\frac{\partial u}{\partial t} + P\, \nabla_u u = -\nu A u. \tag{5.11}$$

Here, P is the orthogonal projection of $L^2(\Omega, T)$ onto $W^0 = V^0$, namely, the same P as in (1.10) and (3.1), hence described by (3.5)–(3.6). The operator A is an unbounded, positive, self-adjoint operator on W^0, defined via the Friedrichs extension method, as follows. We have $A_0 : W^1 \to (W^1)^*$ given by

$$\langle A_0 u, v \rangle = (u, v)_{W^1} = (du, dv)_{L^2}, \tag{5.12}$$

the last identity holding because $\operatorname{div} u = \operatorname{div} v = 0$. Then set

$$\mathcal{D}(A) = \{u \in W^1 : A_0 u \in W^0\}, \quad A = A_0\big|_{\mathcal{D}(A)}, \tag{5.13}$$

using $W^1 \subset W^0 \subset (W^1)^*$. Automatically, $\mathcal{D}(A^{1/2}) = W^1$. The operator A is called the Stokes operator. The following result is fundamental to the analysis of (5.1)–(5.2):

Proposition 5.2. *$\mathcal{D}(A) \subset H^2(\Omega, T)$. In fact, $\mathcal{D}(A) = H^2(\Omega, T) \cap W^1$.*

In fact, if $u \in \mathcal{D}(A)$ and $Au = f \in W^0$, then $(f, v)_{L^2} = (-\Delta u, v)_{L^2}$, for all $v \in \mathcal{V}$. We know $\Delta u \in H^{-1}$, so from Lemma 5.1 and (5.9) we conclude that there exists $p \in L^2(\Omega)$ such that

$$-\Delta u = f + \nabla p. \tag{5.14}$$

Also we know that div $u = 0$ and $u \in H_0^1(\Omega, T)$. We want to conclude that $u \in H^2$ and $p \in H^1$. Let us identify vector fields and 1-forms, so

$$-\Delta u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0. \tag{5.15}$$

In order not to interrupt the flow of the analysis of (5.1)–(5.2), we will show in Appendix A at the end of this chapter that solutions to (5.15) possess appropriate regularity.

We will define

$$W^s = \mathcal{D}(A^{s/2}), \quad s \geq 0. \tag{5.16}$$

Note that this is consistent with (5.5), for $s = k = 0$ or 1.

We now construct a local solution to the initial-value problem for the Navier–Stokes equation, by converting (5.11) into an integral equation:

$$u(t) = e^{-t\nu A}\, u_0 - \int_0^t e^{(s-t)\nu A}\, P \operatorname{div}\big(u(s) \otimes u(s)\big)\, ds = \Psi u(t). \tag{5.17}$$

We want to find a fixed point of Ψ on $C(I, X)$, for $I = [0, T]$, with some $T > 0$, and X an appropriate Banach space. We take X to be of the form

$$X = W^s = \mathcal{D}(A^{s/2}), \tag{5.18}$$

for a value of s to be specified below. As in the construction in § 4, we need a Banach space Y such that

$$\Phi : X \to Y \text{ is Lipschitz, uniformly on bounded sets,} \tag{5.19}$$

where

$$\Phi(u) = P \operatorname{div}(u \otimes u), \tag{5.20}$$

and such that, for some $\gamma < 1$,

$$\|e^{-tA}\|_{\mathcal{L}(Y,X)} \leq C t^{-\gamma}, \tag{5.21}$$

for $t \in (0,1]$. We take

$$Y = W^0. \tag{5.22}$$

As $\|e^{-tA}\|_{\mathcal{L}(W^0,W^s)} \sim Ct^{-s/2}$ for $t \le 1$, the condition (5.21) requires $s \in (0,2)$, in (5.18). We need to verify (5.19). Note that, by Proposition 5.2 and interpolation,

$$W^s \subset H^s(\Omega, T), \quad \text{for } 0 \le s \le 2. \tag{5.23}$$

Thus (5.19) will hold provided

$$M : H^s(\Omega, T) \to H^1(\Omega, T \otimes T), \quad \text{with } M(u) = u \otimes u. \tag{5.24}$$

Lemma 5.3. *Provided dim $\Omega \le 5$, there exists $s_0 < 2$ such that (5.24) holds for all $s > s_0$.*

Proof. If dim $M = n$, one has

$$H^{n/2+\varepsilon} \cdot H^{n/2+\varepsilon} \subset H^{n/2+\varepsilon} \quad \text{and } H^{n/4} \cdot H^{n/4} \subset H^0 = L^2,$$

the latter because $H^{n/4} \subset L^4$. Other inclusions

$$H^r \cdot H^r \subset H^\sigma, \quad r = \frac{n}{4} + \frac{\sigma}{2} + \varepsilon\theta, \quad \sigma = \theta\Bigl(\frac{1}{2}n + \varepsilon\Bigr), \tag{5.25}$$

follow by a straightforward interpolation. One sees that (5.24) holds for $s > s_0$ with

$$s_0 = \frac{n}{4} + \frac{1}{2} \quad (\text{ if } n \ge 2). \tag{5.26}$$

For $2 \le n \le 5,\ s_0$ increases from 1 to $7/4$; for $n = 6,\ s_0 = 2$.

Thus we have an existence result:

Proposition 5.4. *Suppose dim $\Omega \le 5$. If s_0 is given by (5.26) and $u_0 \in W^s$ for some $s \in (s_0, 2)$, then there exists $T > 0$ such that (5.17) has a unique solution*

$$u \in C\bigl([0,T], W^s\bigr). \tag{5.27}$$

We can extend the last result a bit once the following is established:

Proposition 5.5. *Set $V^s = V^0 \cap H^s(\Omega, T)$, for $0 \le s \le 1$. We have*

$$W^s = V^s, \ \textit{for } 0 \le s < \frac{1}{2}, \tag{5.28}$$

and hence

$$P : H^s(\Omega, T) \longrightarrow W^s, \tag{5.29}$$

for such s.

Proof. To deduce (5.29) from (5.28), note that, by (3.5), $P : H^s(\Omega, T) \to H^s(\Omega, T)$ for $s = 0, 1$, hence, by interpolation, for all $s \in [0, 1]$, so $P : H^s(\Omega, T) \to V^s$, for $s \in [0, 1]$.

To establish (5.28), recall that $W^1 = \mathcal{D}(A^{1/2}) = V^0 \cap H_0^1(\Omega, T)$. We hence have

$$W^s = [V^0, V^0 \cap H_0^1(\Omega, T)]_s, \quad \text{for } 0 \le s \le 1.$$

Thus (5.28) will follow from the identity

$$[V^0, V^0 \cap H_0^1(\Omega, T)]_s = V^0 \cap [L^2(\Omega, T), H_0^1(\Omega, T)]_s, \quad 0 \le s \le 1, \tag{5.30}$$

since, as seen in (5.37) of Chap. 4,

$$[L^2(\Omega), H_0^1(\Omega)]_s = H^s(\Omega), \quad \text{for } 0 \le s < \frac{1}{2}. \tag{5.31}$$

Following [FM], we make use of the following result to establish (5.30):

Lemma 5.6. *There is a continuous projection Q from $L^2(\Omega, T)$ onto V^0 such that Q maps $H^2(\Omega, T) \cap H_0^1(\Omega, T) = \mathcal{D}(\Delta)$ to $H^2(\Omega, T) \cap W^1 = \mathcal{D}(A)$.*

Here Δ is the Laplace operator on Ω, with Dirichlet boundary condition. We know that

$$[L^2(\Omega, T), H^2(\Omega, T) \cap H_0^1(\Omega, T)]_{1/2} = \mathcal{D}\big((-\Delta)^{1/2}\big) = H_0^1(\Omega, T), \tag{5.32}$$

so the lemma implies that the projection Q has the property

$$Q : H_0^1(\Omega, T) \longrightarrow W^1 = V^0 \cap H_0^1(\Omega, T), \tag{5.33}$$

and (5.30) is a straightforward consequence of this result.

Proof of lemma. We define the continuous operator $Q_0 : \mathcal{D}(\Delta) \to \mathcal{D}(A)$ by

$$Q_0 u = -A^{-1} P \Delta u, \quad u \in \mathcal{D}(\Delta). \tag{5.34}$$

Since $Q_0 u = u$ for $u \in \mathcal{D}(A) = \mathcal{D}(\Delta) \cap V^0$ and since $\mathcal{D}(A)$ is dense in V^0, it suffices to show that Q_0 can be extended to a bounded operator from $L^2(\Omega, T)$ to V^0. Indeed, by the self-adjointness of A and Δ, we have, for the adjoint, mapping V^0 to $L^2(\Omega, T)$,

$$Q_0^* = -\Delta\iota A^{-1}, \quad \iota : V^0 \hookrightarrow L^2(\Omega, T), \tag{5.35}$$

which is a bounded operator from V^0 to $L^2(\Omega, T)$, since the inclusion ι maps $\mathcal{D}(A)$ into $\mathcal{D}(\Delta)$. This proves the lemma, so Proposition 5.5 is established.

We now return to the integral equation (5.17), replacing $Y = W^0$ in (5.22) by

$$Y = W^\sigma = V^\sigma, \quad \sigma \in \Big[0, \frac{1}{2}\Big). \tag{5.36}$$

We take $X = W^s$, as in (5.18), and this time we need $s - \sigma \in (0, 2)$ in order for (5.21) to hold with $\gamma < 1$. Higher regularity for the Stokes operator gives

$$W^s \subset H^s(\Omega, T), \ \text{ for } s \in \mathbb{R}^+, \tag{5.37}$$

extending (5.23). Thus (5.19) will hold provided we extend (5.24) to

$$M : H^s(\Omega, T) \longrightarrow H^{1+\sigma}(\Omega, T \otimes T), \quad M(u) = u \otimes u. \tag{5.38}$$

Let us write

$$\sigma = \frac{1}{2} - \delta, \quad s = 2 + \sigma - \delta = \frac{5}{2} - 2\delta.$$

By the arguments used in Lemma 5.3, we have the following:

Lemma 5.7. *Provided* $\dim \Omega \le 6$, *if* $\delta \in (0, 1/2)$ *is small enough, and* $\sigma = 1/2 - \delta$, *there exists* $s_0 \in (\sigma, 2 + \sigma)$ *such that (5.38) holds for all* $s > s_0$.

Proof. If $n \le 4$, then $H^s(\Omega)$ is an algebra for $s = 5/2 - 2\delta$ if δ is small enough. If $n \ge 5$, we can take $s_0 = (n + 3)/4$.

Thus we have the following complement to Proposition 5.4:

Proposition 5.8. *Suppose* $\dim \Omega \le 6$. *If* $\delta > 0$ *is small enough,* $s = 5/2 - 2\delta$, *and* $u_0 \in W^s$, *then there exists* $T > 0$ *such that (5.17) has a unique solution in* $C\big([0, T], W^s\big)$.

There are results on higher regularity of strong solutions, for $0 < t < T$. We refer to [Tem3] for a discussion of this.

Having treated strong solutions, we next establish the Hopf theorem on the global existence of weak solutions to the Navier–Stokes equations, in the case of domains with boundary.

Theorem 5.9. *Assume* $\dim \Omega \le 3$. *Given* $u_0 \in W^0, \nu > 0$, *the system (5.1)–(5.3) has a weak solution for* $t \in (0, \infty)$,

$$u \in L^\infty(\mathbb{R}^+, W^0) \cap L^2_{loc}(\mathbb{R}^+, W^1). \tag{5.39}$$

The proof is basically parallel to that of Theorem 4.6. We sketch the argument. As above, we assume for simplicity that Ω is Ricci flat. We have the Stokes operator A, a self-adjoint operator on W^0, defined by (5.12)–(5.13). As in the proof of Theorem 4.6, we consider the family of projections $J_\varepsilon = \chi(\varepsilon A)$, where $\chi(\lambda)$ is the characteristic function of $[-1, 1]$. We approximate the solution u by u_ε, solving

$$\frac{\partial u_\varepsilon}{\partial t} + J_\varepsilon P \operatorname{div}(u_\varepsilon \otimes u_\varepsilon) = -\nu A u_\varepsilon, \quad u_\varepsilon(0) = J_\varepsilon u_0. \tag{5.40}$$

This has a global solution, $u_\varepsilon \in C^\infty\big([0,\infty), \text{Range } J_\varepsilon\big)$. As in (4.32), $\{u_\varepsilon\}$ is bounded in $L^\infty(\mathbb{R}^+, L^2(\Omega))$. Also, as in (4.33),

$$2\nu \int_0^T \|\nabla u_\varepsilon(t)\|_{L^2}^2 \, dt = \|J_\varepsilon u_0\|_{L^2}^2 - \|u_\varepsilon(T)\|_{L^2}^2, \tag{5.41}$$

for each $T \in \mathbb{R}^+$. Thus, parallel to (4.34), for any bounded interval $I = [0, T]$,

$$\{u_\varepsilon\} \text{ is bounded in } L^2(I, W^1). \tag{5.42}$$

Instead of paralleling (4.36)–(4.39), we prefer to use (5.42) to write

$$\{\nabla_{u_\varepsilon} u_\varepsilon\} \text{ bounded in } L^1\big(I, L^{3/2}(\Omega)\big), \tag{5.43}$$

provided $\dim \Omega \le 3$. In such a case, we also have

$$P : W^1 \to H^1(\Omega) \subset L^3(\Omega),$$

and hence

$$P : L^{3/2}(\Omega) \longrightarrow (W^1)^*. \tag{5.44}$$

Also $\{J_\varepsilon\}$ is uniformly bounded on W^1 and its dual $(W^1)^*$, and $A : W^1 \to (W^1)^*$. Thus, in place of (4.37), we have

$$\{\partial_t u_\varepsilon\} \text{ bounded in } L^1(I, (W^1)^*), \tag{5.45}$$

so

$$\{u_\varepsilon\} \text{ is bounded in } H^s(I, (W^1)^*), \quad \forall\, s \in \Big(0, \frac{1}{2}\Big). \tag{5.46}$$

Now we interpolate this with (5.42), to get, for all $\delta > 0$,

$$\{u_\varepsilon\} \text{ bounded in } H^s(I, H^{1-\delta}(\Omega)), \quad s = s(\delta) > 0, \tag{5.47}$$

hence, parallel to (4.40),

$$\{u_\varepsilon\} \text{ is compact in } L^2(I, H^{1-\delta}(\Omega)), \quad \forall\, \delta > 0. \tag{5.48}$$

The rest of the argument follows as in the proof of Theorem 4.6.

We also have results parallel to Propositions 4.9–4.10:

Proposition 5.10. *Let u_1 and u_2 be weak solutions to (5.11), satisfying*

$$u_j \in L^\infty(I, W^0) \cap L^2(I, W^1), \quad u_j \in L^4(I \times \Omega). \tag{5.49}$$

Suppose $\dim \Omega = 2$ *or* 3*; if* $\dim \Omega = 3$*, suppose furthermore that*

$$u_1 \in L^8\big(I, L^4(\Omega)\big). \tag{5.50}$$

If $u_1(0) = u_2(0)$*, then* $u_1 = u_2$ *on* $I \times \Omega$*.*

The proof of both this result and the following are by the same arguments as used in § 4.

Proposition 5.11. *If u is a Leray–Hopf solution and $I = [0, T]$, then*

$$u \in L^4(I \times \Omega) \quad \textit{if } \dim \Omega = 2 \tag{5.51}$$

and

$$u \in L^{8/3}\big(I, L^4(\Omega)\big) \quad \textit{if } \dim \Omega = 3. \tag{5.52}$$

Thus we have uniqueness of Leray–Hopf solutions if $\dim \Omega = 2$. The following result yields extra smoothness if $u_0 \in W^1$:

Proposition 5.12. *If* $\dim \Omega = 2$*, and u is a Leray–Hopf solution to the Navier–Stokes equations, with* $u(0) = u_0 \in W^1$*, then, for any* $I = [0, T]$, $T < \infty$*,*

$$u \in L^\infty(I, W^1) \cap L^2(I, W^2), \tag{5.53}$$

and

$$\frac{\partial u}{\partial t} \in L^2(I, W^0). \tag{5.54}$$

Proof. Let u_j be the approximate solution u_ε defined by (5.40), with $\varepsilon = \varepsilon_j \to 0$. We have

$$\frac{1}{2}\frac{d}{dt}\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 + \nu\|Au_j(t)\|_{L^2}^2 = -\big(\nabla_{u_j}u_j, Au_j\big)_{L^2}, \tag{5.55}$$

upon taking the inner product of (5.40) with Au_ε. Now there is the estimate

$$\big|\big(\nabla_{u_j} u_j, Au_j\big)\big| \le C\|\nabla u_j\|_{L^3}^3. \tag{5.56}$$

To see this, note that since $d \circ (I - P) = 0$, we have, for $u \in W^2$,

$$(\nabla_u u, u)_{V_1} = (d\nabla_u u, du) = \big([d, \nabla_u]u, du\big) + \frac{1}{2}\big([\nabla_u + \nabla_u^*]\, du, du\big),$$

and the absolute value of each of the last two terms is easily bounded by $\int |\nabla u|^3\, dV$.

In order to estimate the right side of (5.56), we use the Sobolev imbedding result

$$H^{1/3}(\Omega) \subset L^3(\Omega), \quad \dim \Omega = 2, \tag{5.57}$$

which implies $\|v\|_{L^3} \le C\|v\|_{L^2}^{2/3}\|v\|_{H^1}^{1/3}$, so

$$\begin{aligned}\|\nabla u_j\|_{L^3}^3 &\le C\|\nabla u_j\|_{L^2}^2 \cdot \|\nabla u_j\|_{H^1} \\ &\le C'(\nu\delta)^{-1}\|\nabla u_j\|_{L^2}^4 + C'\nu\delta\|\nabla u_j\|_{H^1}^2.\end{aligned} \tag{5.58}$$

We have $\|\nabla u_j\|_{H^1}^2 \le C\|Au_j\|_{L^2}^2 + C\|u_j\|_{L^2}^2$, by Proposition 5.2, so if δ is picked small enough, we can absorb the $\|\nabla u_j\|_{H^1}^2$-term into the left side of (5.55). We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 &+ \frac{\nu}{2}\|Au_j(t)\|_{L^2}^2 \\ &\le C\|A^{1/2}u_j(t)\|_{L^2}^4 + C\Big(\|u_j(t)\|_{L^2}^4 + \|u_j(t)\|_{L^2}^2\Big).\end{aligned} \tag{5.59}$$

We want to apply Gronwall's inequality. It is convenient to set

$$\sigma_j(t) = \big\|A^{1/2}u_j(t)\big\|_{L^2}^2, \quad \Phi(\lambda) = \lambda^4 + \lambda^2. \tag{5.60}$$

The boundedness of u_ε in $L^2_{\mathrm{loc}}(\mathbb{R}^+, W^1)$ (noted in (5.42)) implies that, for any $T < \infty$,

$$\int_0^T \sigma_j(t)\, dt \le K(T) < \infty, \tag{5.61}$$

with $K(T)$ independent of j. If we drop the term $(\nu/2)\|Au_j(t)\|_{L^2}^2$ from (5.59), we obtain

$$\frac{d}{dt}\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 \le C\sigma_j(t)\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 + C\Phi\Big(\|u_j(t)\|_{L^2}\Big), \tag{5.62}$$

and Gronwall's inequality yields

$$\|A^{1/2}u_j(t)\|_{L^2}^2 \le e^{CK(t)}\,\|A^{1/2}u_0\|_{L^2}^2 + Ce^{CK(t)}\int_0^t \Phi\Big(\|u_j(s)\|_{L^2}\Big)\,ds. \tag{5.63}$$

This implies that u_j is bounded in $L^\infty(I, W^1)$, and then integrating (5.59) implies u_j is bounded in $L^2(I, W^2)$. The conclusions (5.53) and (5.54) follow.

The argument used to prove Proposition 5.12 does not extend to the case in which $\dim \Omega = 3$. In fact, if $\dim \Omega = 3$, then (5.57) must be replaced by

$$H^{1/2}(\Omega) \subset L^3(\Omega), \quad \dim \Omega = 3, \tag{5.64}$$

which implies $\|v\|_{L^3} \le C\|v\|_{L^2}^{1/2}\|v\|_{H^1}^{1/2}$, and hence (5.58) is replaced by

$$\begin{aligned} \|\nabla u_j\|_{L^3}^3 &\le C\|\nabla u_j\|_{L^2}^{3/2} \cdot \|\nabla u_j\|_{H^1}^{3/2} \\ &\le C(\nu\delta)^{-3}\|\nabla u_j\|_{L^2}^6 + C\nu\delta\|\nabla u_j\|_{H^1}^2. \end{aligned} \tag{5.65}$$

Unfortunately, the power 6 of $\|\nabla u_j\|_{L^2}$ on the right side of (5.65) is too large in this case for an analogue of (5.60)–(5.63) to work, so such an approach fails if $\dim \Omega = 3$.

On the other hand, when $\dim \Omega = 3$, we do have the inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A^{1/2}u_j(t)\|_{L^2}^2 &+ \frac{\nu}{2}\|Au_j(t)\|_{L^2}^2 \\ &\le C\|A^{1/2}u_j(t)\|_{L^2}^6 + C\big(\|u_j(t)\|_{L^2}^6 + \|u_j(t)\|_{L^2}^2\big). \end{aligned} \tag{5.66}$$

We have an estimate $\|u_j(t)\|_{L^2} \le K$, so we can apply Gronwall's inequality to the differential inequality

$$\frac{1}{2}\frac{d}{dt}Y_j(t) \le CY_j(t)^3 + C(K^6 + K^2)$$

to get a uniform bound on $Y_j(t) = \|A^{1/2}u_j(t)\|_{L^2}^2$, at least on some interval $[0, T_0]$. Thus we have the following result.

Proposition 5.13. *If $\dim \Omega = 3$, and u is a Leray–Hopf solution to the Navier–Stokes equations, with $u(0) = u_0 \in W^1$, then there exists $T_0 = T_0(\|u_0\|_{W^1}) > 0$ such that*

$$u \in L^\infty\big([0, T_0], W^1\big) \cap L^2\big([0, T_0], W^2\big), \tag{5.67}$$

and

$$\frac{\partial u}{\partial t} \in L^2\big([0, T_0], W^0\big). \tag{5.68}$$

Note that the properties of the solution u on $[0, T_0] \times \Omega$ in (5.67) are stronger than the properties (5.49)–(5.50) required for uniqueness in Proposition 5.10. Hence we have the following:

Corollary 5.14. *If* $\dim \Omega = 3$ *and* u_1 *and* u_2 *are Leray–Hopf solutions to the Navier–Stokes equations, with* $u_0(0) = u_2(0) = u_0 \in W^1$*, then there exists* $T_0 = T_0(\|u_0\|_{W^1}) > 0$ *such that* $u_1(t) = u_2(t)$ *for* $0 \le t \le T_0$.

Furthermore, if $u_0 \in W^s$ *with* $s \in (s_0, 2)$ *as in Proposition 5.4, then the strong solution* $u \in C([0, T], W^s)$ *provided by Proposition 5.4 agrees with any Leray–Hopf solution, for* $0 \le t \le \min(T, T_0)$.

As we have seen, a number of results presented in § 4 for viscous fluid flows on domains without boundary extend to the case of domains with boundary. We now mention some phenomena that differ in the two cases.

The role of the vorticity equation is altered when $\partial\Omega \neq \emptyset$. One still has the PDE for $w = \operatorname{curl} u$, for example,

$$\begin{aligned} \frac{\partial w}{\partial t} + \nabla_u w &= \nu \Delta w \qquad (\dim \Omega = 2), \\ \frac{\partial w}{\partial t} + \nabla_u w - \nabla_w u &= \nu \Delta w \qquad (\dim \Omega = 3), \end{aligned} \tag{5.69}$$

but when $\partial\Omega \neq \emptyset$, the initial value $w(0)$ alone does not serve to determine $w(t)$ for $t > 0$ from such a PDE, and a good boundary condition to impose on $w(t, x)$ is not available. This is not a problem in the $\nu = 0$ case, since u itself is tangent to the boundary. For $\nu > 0$, one result is that one can have $w(0) = 0$ but $w(t) \neq 0$ for $t > 0$. In other words, for $\nu > 0$, interaction of the fluid with the boundary can create vorticity.

The most crucial effect a boundary has lies in complicating the behavior of solutions u_ν in the limit $\nu \to 0$. There is no analogue of the ν-independent estimates of Propositions 4.1 and 4.2 when $\partial\Omega \neq \emptyset$. This is connected to the change of boundary condition, from $u_\nu|_{\partial\Omega} = 0$ for ν positive (however small) to $n \cdot u|_{\partial\Omega} = 0$ when $\nu = 0$, n being the normal to $\partial\Omega$. Study of the small-ν limit is important because it arises naturally. In many cases flow of air can be modeled as an incompressible fluid flow with $\nu \approx 10^{-5}$. However, after more than a century of investigation, this remains an extremely mysterious problem. See the next section for further discussion of these matters.

Exercises

1. Show that $\mathcal{D}(A^k) \subset H^{2k}(\Omega, T)$, for $k \in \mathbb{Z}^+$. Hence establish (5.37).
2. Extend the L^2-Sobolev space results of this section to L^p-Sobolev space results.
3. Work out results parallel to those of this section for the Navier–Stokes equations, when the no-slip boundary condition (5.2) is replaced by the “slip” boundary condition:

$$2\nu \operatorname{Def}(u) N - pN = 0 \quad \text{on } \partial\Omega, \tag{5.70}$$

where N is a unit normal field to $\partial\Omega$ and $\operatorname{Def}(u)$ is a tensor field of type $(1,1)$, given by (2.60). Relate (5.70) to the identity

$$(\nu\mathcal{L}u - \nabla p, v) = -2\nu(\operatorname{Def} u, \operatorname{Def} v) \quad \text{whenever } \operatorname{div} v = 0.$$

6. Vanishing viscosity limits

In this section we consider some classes of solutions to the Navier–Stokes equations

$$\frac{\partial u^\nu}{\partial t} + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu\Delta u^\nu + F^\nu, \quad \operatorname{div} u^\nu = 0, \tag{6.1}$$

on a bounded domain, or a compact Riemannian manifold, $\overline{\Omega}$ (with a flat metric), with boundary $\partial\Omega$, satisfying the no-slip boundary condition

$$u^\nu\big|_{\mathbb{R}^+\times\partial\Omega} = 0, \tag{6.2}$$

and initial condition

$$u^\nu(0) = u_0, \tag{6.3}$$

and investigate convergence as $\nu \to 0$ to the solution to the Euler equation

$$\frac{\partial u^0}{\partial t} + \nabla_{u^0} u^0 + \nabla p^0 = F^0, \quad \operatorname{div} u^0 = 0, \tag{6.4}$$

with boundary condition

$$u^0 \parallel \partial\Omega, \tag{6.5}$$

and initial condition as in (6.3). We assume

$$\operatorname{div} u_0, \quad u_0 \parallel \partial\Omega, \tag{6.6}$$

but do not assume $u_0 = 0$ on $\partial\Omega$.

When $\partial\Omega \neq \emptyset$, the problem of convergence $u^\nu \to u^0$ is very difficult, and there are not many positive results, though there is a large literature. The enduring monograph [Sch] contains a great deal of formal work, much stimulated by ideas of L. Prandtl. More modern mathematical progress includes a result of [Kt7], that $u^\nu(t) \to u^0(t)$ in L^2-norm, uniformly in $t \in [0,T]$, provided one has an estimate

$$\nu \int_0^T \int\limits_{\Gamma_{c\nu}} |\nabla u^\nu(t,x)|^2 \, dx\, dt \longrightarrow 0, \ \text{ as } \nu \to 0, \tag{6.7}$$

where $\Gamma_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\}$. Unfortunately, this condition is not amenable to checking. In [W] there is a variant, namely that such convergence holds provided

$$\nu \int_0^T \int_{\Gamma_{\eta(\nu)}} |\nabla_T u^\nu(t,x)|^2 \, dx \, dt \longrightarrow 0, \quad \text{as } \nu \to 0, \tag{6.8}$$

with $\eta(\nu)/\nu \to \infty$ as $\nu \to 0$, where ∇_T denotes the derivative tangent to $\partial\Omega$.

Here we confine attention to two classes of examples. The first is the class of circularly symmetric flows on the disk in 2D. The second is a class of circular pipe flows, in 3D, which will be described in more detail below. Both of these classes are mentioned in [W] as classes to which the results there apply. However, we will seek more detailed information on the nature of the convergence $u^\nu \to u^0$. Our analysis follows techniques developed in [LMNT, MT1, MT2]. See also [Mat, BW, LMN] for other work in the 2D case. Most of these papers also treated moving boundaries, but for simplicity we treat only stationary boundaries here.

We start with circularly symmetric flows on the disk $\Omega = D = \{x \in \mathbb{R}^2 : |x| < 1\}$. Here, we take $F^\nu \equiv 0$. By definition, a vector field u_0 on D is circularly symmetric provided

$$u_0(R_\theta x) = R_\theta u_0(x), \quad \forall\, x \in D, \tag{6.9}$$

for each $\theta \in [0, 2\pi]$, where R_θ is counterclockwise rotation by θ. The general vector field satisfying (6.9) has the form

$$s_0(|x|)x^\perp + s_1(|x|)x, \tag{6.10}$$

with s_j scalar and $x^\perp = Jx$, where $J = R_{\pi/2}$, but the condition $\operatorname{div} u_0 = 0$ together with the condition $u_0 \parallel \partial D$, forces $s_1 = 0$, so the type of initial data we consider is characterized by

$$u_0(x) = s_0(|x|)x^\perp. \tag{6.11}$$

It is easy to see that $\operatorname{div} u_0 = 0$ for each such u_0. Another characterization of vector fields of the form (6.11) is the following. For each unit vector $\omega \in S^1 \subset \mathbb{R}^2$, let $\Phi_\omega : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection across the line generated by ω, i.e., $\Phi(a\omega + bJ\omega) = a\omega - bJ\omega$. Then a vector field u_0 on D has the form (6.11) if and only if

$$u_0(\Phi_\omega x) = -\Phi_\omega u_0(x), \quad \forall\, \omega \in S^1,\ x \in D. \tag{6.12}$$

A vector field u_0 of the form (6.11) is a steady solution to the 2D Euler equation (with $F^0 = 0$). In fact, a calculation gives

$$\nabla_{u_0} u_0 = -s_0(|x|)^2 x = -\nabla p_0(x), \tag{6.13}$$

with

$$p_0(x) = \tilde{p}_0(|x|), \quad \tilde{p}_0(r) = -\int_r^1 s_0(\rho)^2 \rho \, d\rho. \tag{6.14}$$

Consequently, in this case the vanishing viscosity problem is to show that the solution u^ν to (6.1)–(6.3) satisfies $u^\nu(t) \to u_0$ as $\nu \to 0$. The following is the key to the analysis of the solution u^ν.

Proposition 6.1. *Given that u_0 has the form (6.11), the solution u^ν to (6.1)–(6.3) (with $F^\nu \equiv 0$) is circularly symmetric for each $t > 0$, of the form*

$$u^\nu(t,x) = s^\nu(t,|x|)x^\perp, \tag{6.15}$$

and it coincides with the solution to the linear PDE

$$\frac{\partial u^\nu}{\partial t} = \nu \Delta u^\nu, \tag{6.16}$$

with boundary condition (6.2) and initial condition (6.3).

Proof. Let u^ν solve (6.16), (6.2), and (6.3), with u_0 as in (6.11). We claim (6.15) holds. In fact, for each unit vector $\omega \in \mathbb{R}^2$, $-\Phi_\omega u^\nu(t, \Phi_\omega x)$ also solves (6.16), with the same initial data and boundary conditions as u^ν, so these functions must coincide, and (6.15) follows. Hence $\operatorname{div} u^\nu = 0$ for each $t > 0$. Also we have an analogue of (6.13)–(6.14):

$$\begin{gathered} \nabla_{u^\nu} u^\nu = -\nabla p^\nu, \quad p^\nu(t,x) = \tilde{p}^\nu(t,|x|), \\ \tilde{p}^\nu(t,r) = -\int_r^1 s^\nu(t,\rho)^2 \rho \, d\rho. \end{gathered} \tag{6.17}$$

Hence this u^ν is the solution to (6.1)–(6.3).

To restate matters, for $\Omega = D$, the solution to (6.1)–(6.3) is in this case simply

$$u^\nu(t,x) = e^{\nu t \Delta} u_0(x). \tag{6.18}$$

The following is a simple consequence.

Proposition 6.2. *Assume u_0, of the form (6.11), belongs to a Banach space $\mathfrak{X}$ of $\mathbb{R}^2$-valued functions on D. If $\{e^{t\Delta} : t \ge 0\}$ is a strongly continuous semigroup on $\mathfrak{X}$, then $u^\nu(t,\cdot) \to u_0$ in $\mathfrak{X}$ as $\nu \to 0$, locally uniformly in $t \in [0,\infty)$.*

As seen in Chap. 6, $\{e^{t\Delta} : t \ge 0\}$ is strongly continuous on the following spaces:

$$L^p(D), \ 1 \le p < \infty, \quad C_o(D) = \{f \in C(\overline{D}) : f = 0 \text{ on } \partial D\}. \tag{6.19}$$

Also, it is strongly continuous on $\mathcal{D}_s = \mathcal{D}((-\Delta)^{s/2})$ for all $s \in \mathbb{R}^+$. We recall from Chap. 5 that

$$\mathcal{D}_2 = H^2(D) \cap H^1_0(D), \quad \mathcal{D}_1 = H^1_0(D), \tag{6.20}$$

and

$$\mathcal{D}_s = [L^2(D), H^1_0(D)]_s, \quad 0 < s < 1. \tag{6.21}$$

In particular, by interpolation results given in Chap. 4,

$$\mathcal{D}_s = H^s(D), \quad 0 < s < \frac{1}{2}. \tag{6.22}$$

We also mention Proposition 7.4 of Chap. 13, which implies this heat semigroup is strongly continuous on

$$C^1_b(\overline{D}) = \{f \in C^1(\overline{D}) : f = 0 \text{ on } \partial D\}. \tag{6.23}$$

On the other hand, if $u_0 \in C(\overline{D})$ but does not vanish on ∂D, then $e^{t\Delta}u_0$ does not converge uniformly to u_0 on $\overline{D}$, as $t \to 0$, though as shown in Corollary 8.2 of Chap. 6, we do have convergence of $e^{t\Delta}u_0$ to u_0 uniformly on compact subsets of D. Thus there is a boundary layer attached to ∂D where uniform convergence fails. We recall Proposition 8.3 of Chap. 6 in the current context.

Proposition 6.3. *Given $u_0 \in C^\infty(\overline{D})$, we have, as $\nu \searrow 0$, locally uniformly in $t \in \mathbb{R}^+$,*

$$\begin{aligned} e^{\nu t\Delta}u_0 \sim u_0(x) &+ \sum_{k\ge1} \frac{(\nu t)^k}{k!}\Delta^k u_0(x) \\ &- \sum_{j\ge0} 2b_j(x)(4\nu t)^{j/2} E_j\Bigl(\frac{\varphi(x)}{\sqrt{4\nu t}}\Bigr). \end{aligned} \tag{6.24}$$

Here, $b_j \in C^\infty(\overline{D})$, $\varphi(x) = \operatorname{dist}(x, \partial D) = 1 - |x|$, and the special functions $E_j(y)$ are given by

$$E_j(y) = \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2}(s-y)^j \, ds. \tag{6.25}$$

We mention that $b_0 = u_0$ on ∂D and $b_j|_{\partial D} = 0$ for j odd. Also, $E_0(0) = 1/2$. The primary "boundary layer" term is

$$-2u_0(x)E_0\Bigl(\frac{1-|x|}{\sqrt{4\nu t}}\Bigr), \tag{6.26}$$

and we see the boundary layer thickness is $\sim \sqrt{4\nu t}$.

We pass from this class of 2D problems to the following class of 3D problems. We look for solutions to (6.1)–(6.3) with $u^\nu = u^\nu(t,x,z)$, $p^\nu = p^\nu(t,x,z)$, $(t,x,z) \in \mathbb{R}^+ \times \Omega$, where $\Omega = D \times \mathbb{R}$, D being the 2D disk as above. Thus Ω is an infinitely long circular pipe. In this case, we consider external force fields of the form

$$F^\nu(t,x,z) = (0, f^\nu(t)), \tag{6.27}$$

so F^ν is parallel to the z-axis, with z-component $f^\nu(t)$. We take initial data of the following form:

$$u^\nu(0,x,z) = u_0(x) = (v_0(x), w_0(x)), \tag{6.28}$$

where v_0 is a vector field on D and w_0 is the z-component of u_0. We require the conditions

$$\operatorname{div} u_0 = 0, \ \ u_0 \parallel \partial\Omega, \quad i.e., \quad \operatorname{div} v_0 = 0, \ \ v_0 \parallel \partial D, \tag{6.29}$$

and we require the vector field v_0 on D to be circularly symmetric, so, as in (6.11), $v_0(x) = s_0(|x|)x^\perp$, hence

$$u_0(x) = (s_0(|x|)x^\perp, w_0(x)). \tag{6.30}$$

The fact that Ω is infinite is inconvenient. To get the theoretical treatment started, it is convenient to modify the set-up by requiring that solutions be periodic (say of period L) in z, so we replace Ω by $\Omega_L = D \times (\mathbb{R}/L\mathbb{Z})$. In such a case, results of § 5 imply that, for each $\nu > 0$, (6.1)–(6.3) has a unique strong, short time solution, given mild regularity hypotheses on $v_0(x)$ and $w_0(x)$ (a solution that, as we will shortly see, persists for all time $t > 0$ under the current hypotheses), and the solution is z-translation invariant, i.e.,

$$u^\nu = (v^\nu(t,x), w^\nu(t,x)), \quad p^\nu = p^\nu(t,x). \tag{6.31}$$

Consequently,

$$\nabla_{u^\nu} u^\nu = (\nabla_{v^\nu} v^\nu, \nabla_{v^\nu} w^\nu), \quad \operatorname{div} u^\nu = \operatorname{div} v^\nu. \tag{6.32}$$

Hence, in the current setting, (6.1) is equivalent to the following system of equations on $\mathbb{R}^+ \times D$:

$$\frac{\partial v^\nu}{\partial t} + \nabla_{v^\nu} v^\nu + \nabla p^\nu = \nu \Delta v^\nu, \quad \operatorname{div} v^\nu = 0, \tag{6.33}$$

$$\frac{\partial w^\nu}{\partial t} + \nabla_{v^\nu} w^\nu = \nu \Delta w^\nu + f^\nu. \tag{6.34}$$

Note that (6.33) is the Navier–Stokes equation for flow on D, which we have just treated. Given initial data satisfying (6.30), we have

$$v^\nu(t,x) = e^{\nu t\Delta} v_0(x), \tag{6.35}$$

where Δ is the Laplace operator on D, with Dirichlet boundary condition. The results of Proposition 6.2, complemented by (6.19)–(6.23) apply, as do those of Proposition 6.3, taking care of (6.33).

It remains to investigate (6.34). For this, we have the initial and boundary conditions

$$w^\nu(0) = w_0, \quad w^\nu\big|_{\mathbb{R}^+\times\partial D} = 0, \tag{6.36}$$

and we ask whether, as $\nu \searrow 0$, w^ν converges to w^0, solving

$$\frac{\partial w^0}{\partial t} + \nabla_{v^0} w^0 = f^0(t), \quad w^0(0) = w_0. \tag{6.37}$$

We impose no boundary condition on w^0, which is natural since $v^0 = v_0$ is tangent to ∂D.

Before pursuing this convergence question, we pause to observe a class of steady solutions to (6.33)–(6.34) known as *Poiseuille flows*. Namely, given $\alpha \in \mathbb{R} \setminus 0$,

$$u_0(x) = \alpha(0, 1 - |x|^2) \tag{6.38}$$

is such a steady solution, with

$$p^\nu(t,x) = 0, \quad f^\nu(t) = (0, 4\nu\alpha). \tag{6.39}$$

An alternative description is to set

$$p^\nu(t,x,z) = -4\nu\alpha z, \quad f^\nu(t) = 0. \tag{6.40}$$

This latter is a common presentation, and one refers to Poiseuille flow as "pressure driven" However, this presentation does not fit into our set-up, since we passed from the infinite pipe $D\times\mathbb{R}$ to the periodized pipe $D\times(\mathbb{R}/L\mathbb{Z})$, and p^ν in (6.40) is not periodic in z. These Poiseuille flows do fit into our set-up, but we need to represent the force that maintains the flow as an external force.

We return to the convergence problem. For notational convenience, we set

$$X_\nu = \nabla_{v^\nu} = s^\nu(t,|x|)\frac{\partial}{\partial\theta}, \quad X = \nabla_{v^0} = s_0(|x|)\frac{\partial}{\partial\theta}. \tag{6.41}$$

Thus we examine solutions to

$$\tag{6.42} \begin{aligned} \frac{\partial w^\nu}{\partial t} &= \nu \Delta w^\nu - X_\nu w^\nu + f^\nu(t), \\ w^\nu(0,x) &= w_0(x), \quad w^\nu\big|_{\mathbb{R}^+ \times \partial D} = 0, \end{aligned}$$

compared to solutions to

$$\tag{6.43} \frac{\partial w^0}{\partial t} = -X w^0 + f^0(t), \quad w^0(0,x) = w_0(x).$$

We do not assume $w_0|_{\partial D} = 0$. In order to separate the two phenomena that make (6.42) a singular perturbation (6.43), namely the appearance of $\nu\Delta$ on the one hand and the replacement of X by X_ν on the other hand, we rewrite (6.42) as

$$\tag{6.44} \frac{\partial w^\nu}{\partial t} = (\nu\Delta - X) w^\nu + (X - X_\nu) w^\nu + f^\nu(t),$$

and apply Duhamel's formula to get

$$\tag{6.45} w^\nu(t) = e^{t(\nu\Delta - X)} w_0 + \int_0^t e^{(t-s)(\nu\Delta - X)} \big[(X - X_\nu) w^\nu(s) + f^\nu(s)\big]\, ds.$$

By comparison, we can write the solution to (6.43) as

$$\tag{6.46} w^\nu(t) = e^{-tX} w_0 + \int_0^t f^0(s)\, ds.$$

Consequently,

$$\tag{6.47} w^\nu(t,x) - w^0(t,x) = R_1(\nu,t,x) + R_2(\nu,t,x) + R_3(\nu,t,x),$$

where

$$\tag{6.48} \begin{aligned} R_1(\nu,t,x) &= e^{t(\nu\Delta - X)} w_0 - e^{-tX} w_0, \\ R_2(\nu,t,x) &= \int_0^t \Big[f^\nu(s) e^{(t-s)(\nu\Delta - X)} 1 - f^0(s) \Big]\, ds, \\ R_3(\nu,t,x) &= \int_0^t e^{(t-s)(\nu\Delta - X)} (s_0 - s^\nu) \frac{\partial w^\nu}{\partial \theta}\, ds. \end{aligned}$$

The term R_2 is the easiest to treat. By radial symmetry,

$$e^{(t-s)(\nu\Delta - X)} 1 = e^{(t-s)\nu\Delta} 1,$$

and we can write

$$R_2(\nu,t,x) = \int_0^t \bigl[f^\nu(s) - f^0(s)\bigr]\,ds + \int_0^t f^\nu(s)\Bigl[e^{(t-s)\nu\Delta}1 - 1\Bigr]\,ds. \tag{6.49}$$

The uniform asymptotic expansion of the last integrand is a special case of (6.24):

$$e^{(t-s)\nu\Delta}1 - 1 \sim -\sum_{j\ge 0} 2b_j(x)(\nu(t-s))^{j/2} E_j\Bigl(\frac{1-|x|}{\sqrt{4\nu(t-s)}}\Bigr), \tag{6.50}$$

with $b_j \in C^\infty(\overline{D})$, $b_0|_{\partial D} = 1$, and E_j as in (6.25). The principal contribution giving the boundary layer effect for the term $R_2(\nu,t,x)$ is

$$-2\int_0^t f^\nu(s) E_0\Bigl(\frac{1-|x|}{\sqrt{4\nu(t-s)}}\Bigr)\,ds. \tag{6.51}$$

Methods initiated in [MT1] and carried out for this case in [MT2] produce a uniform asymptotic expansion for $R_1(\nu,t,x)$ almost as explicit as that given above for R_2, but with much greater effort. Here we will be content to present simpler estimates on R_1. Our analysis of

$$W^\nu(t,x) = e^{t(\nu\Delta - X)} w_0(x) \tag{6.52}$$

starts with the following. Recall that X is divergence free and tangent to ∂D.

Lemma 6.4. *Given $\nu > 0$,*

$$\mathcal{D}((\nu\Delta - X)^j) = \mathcal{D}(\Delta^j), \quad j = 1, 2. \tag{6.53}$$

Proof. We have, for $\nu > 0$,

$$\mathcal{D}(\nu\Delta - X) = \{f \in H^2(D) : f\bigr|_{\partial D} = 0\}, \tag{6.54}$$

and

$$\mathcal{D}((\nu\Delta - X)^2) = \{f \in H^4(D) : f\bigr|_{\partial D} = \nu\Delta f - Xf\bigr|_{\partial D} = 0\}. \tag{6.55}$$

The first space is clearly equal to $\mathcal{D}(\Delta)$. Since X is tangent to ∂D, $f|_{\partial D} = 0 \Rightarrow Xf|_{\partial D} = 0$, so the second space coincides with $\{f \in H^4(D) : f|_{\partial D} = \Delta f|_{\partial D} = 0\}$, which is $\mathcal{D}(\Delta^2)$.

Remark: The analogous identity of domains typically fails for larger j.

To proceed, since W^ν in (6.52) satisfies $\partial_t W^\nu = -XW^\nu + \nu\Delta W^\nu$, we can use Duhamel's formula to write

$$W^\nu(t) = e^{-tX} w_0 + \nu \int_0^t e^{-(t-s)X} \Delta W^\nu(s)\, ds, \tag{6.56}$$

hence

$$\|e^{t(\nu\Delta - X)} w_0 - e^{-tX} w_0\|_{L^p} \le \nu \int_0^t \|\Delta W^\nu(s)\|_{L^p}\, ds. \tag{6.57}$$

The following provides a useful estimate on the right side of (6.57) when $p = 2$.

Lemma 6.5. *Take $w_0 \in \mathcal{D}(\Delta^2) = \mathcal{D}((\nu\Delta - X)^2)$, and construct W^ν as in (6.52). Then there exists $K \in (0,\infty)$, independent of $\nu > 0$, such that*

$$\|\Delta W^\nu(t)\|_{L^2}^2 \le e^{2Kt} \|\Delta w_0\|_{L^2}^2. \tag{6.58}$$

Proof. We have

$$\begin{aligned} \frac{d}{dt}\|\Delta W^\nu(t)\|_{L^2}^2 &= 2\,\mathrm{Re}(\Delta\partial_t W^\nu, \Delta W^\nu) \\ &= 2\,\mathrm{Re}(\nu\Delta^2 W^\nu, \Delta W^\nu) - 2\,\mathrm{Re}(\Delta X W^\nu, \Delta W^\nu) \\ &\le -2\,\mathrm{Re}(\Delta X W^\nu, \Delta W^\nu) \\ &= -2\,\mathrm{Re}(X\Delta W^\nu, \Delta W^\nu) - 2\,\mathrm{Re}([\Delta, X]W^\nu, \Delta W^\nu) \\ &\le 2K\|\Delta W^\nu\|_{L^2}^2, \end{aligned} \tag{6.59}$$

with K independent of ν. The last estimate holds because

$$g \in \mathcal{D}(\Delta) \Rightarrow |(Xg, g)| \le K_1 \|g\|_{L^2}^2, \tag{6.60}$$

and

$$\begin{aligned} &W^\nu(t) \in \mathcal{D}(\Delta^2) \Rightarrow [\Delta, X]W^\nu(t) \in L^2(D), \quad \text{and} \\ &\qquad \|[\Delta, X]W^\nu(t)\|_{L^2} \le \widetilde{K}_2 \|W^\nu(t)\|_{H^2} \le K_2 \|\Delta W^\nu(t)\|_{L^2}. \end{aligned} \tag{6.61}$$

The estimate (6.58) follows.

We can now prove the following.

Proposition 6.6. *Given $p \in [1,\infty)$, $w_0 \in L^p(D)$, we have*

$$e^{t(\nu\Delta - X)} w_0 \longrightarrow e^{-tX} w_0, \quad \text{as } \nu \searrow 0, \tag{6.62}$$

with convergence in L^p-norm.

Proof. We know that $e^{t\nu\Delta}$ is a contraction semigroup on $L^p(D)$ and e^{-tX} is a group of isometries on $L^p(D)$, and we have the Trotter product formula:

$$e^{t(\nu\Delta-X)}w_0 = \lim_{n\to\infty}\Big(e^{(t/n)\nu\Delta}e^{-(t/n)X}\Big)^n w_0, \tag{6.63}$$

in L^p-norm, hence $e^{t(\nu\Delta-X)}$ is a contraction semigroup on $L^p(D)$. By (6.58) and (6.57), we have L^2 convergence for $w_0 \in \mathcal{D}(\Delta^2)$, which is dense in $L^2(D)$. This gives (6.62) for $p = 2$, by the standard approximation argument, a second use of which gives (6.62) for all $p \in [1,2]$.

Suppose next that $p \in (2,\infty)$, with dual exponent $p' \in (1,2)$. The previous results work with X replaced by $-X$, yielding $e^{t(\nu\Delta+x)}g \to e^{tX}g$, as $\nu \searrow 0$, in $L^{p'}$-norm, for all $g \in L^{p'}(D)$. This implies that for $w_0 \in L^p(D)$, convergence in (6.62) holds in the weak* topology of $L^p(D)$. Now, since e^{-tX} is an isometry on $L^p(D)$, we have

$$\|e^{-tX}w_0\|_{L^p} \ge \limsup_{\nu\to 0}\, \|e^{t(\nu\Delta-X)}w_0\|_{L^p}, \tag{6.64}$$

for each $w_0 \in L^p(D)$. Since $L^p(D)$ is a uniformly convex Banach space for such p, this yields L^p-norm convergence in (6.62).

To produce higher order Sobolev estimates, we have from (6.58) the estimate

$$\|e^{t(\nu\Delta-X)}w_0\|_{\mathcal{D}(\Delta)} \le e^{Kt}\|w_0\|_{\mathcal{D}(\Delta)}, \tag{6.65}$$

first for each $w_0 \in \mathcal{D}(\Delta^2)$, hence for each $w_0 \in \mathcal{D}(\Delta)$. Interpolation with the L^2-estimate then gives

$$\|e^{t(\nu\Delta-X)}w_0\|_{\mathcal{D}((-\Delta)^{s/2})} \le e^{Kt}\|w_0\|_{\mathcal{D}((-\Delta)^{s/2})}, \tag{6.66}$$

for each $s \in [0,2],\ w_0 \in \mathcal{D}((-\Delta)^{s/2}) = \mathcal{D}_s$. As noted in (6.22), $\mathcal{D}_s = H^s(D)$ for $0 \le s < 1/2$, so we have

$$\|e^{t(\nu\Delta-X)}w_0\|_{H^s(D)} \le Ce^{Kt}\|w_0\|_{H^s(D)}, \quad 0 \le s < \frac{1}{2}, \tag{6.67}$$

with C and K independent of $\nu \in (0,1]$. We can interpolate the estimate (6.67) with

$$\|e^{t(\nu\Delta-X)}w_0\|_{L^p(D)} \le \|w_0\|_{L^p(D)}, \quad 1 \le p < \infty. \tag{6.68}$$

Using

$$[H^s(D), L^p(D)]_\theta = H^{(1-\theta)s,q(\theta)}(D), \qquad \frac{1}{q(\theta)} = \frac{1-\theta}{2} + \frac{\theta}{p}, \tag{6.69}$$

which follows from material in Chap. 13, § 6, we have

$$\|e^{t(\nu\Delta - X)} w_0\|_{H^{\sigma,q}(D)} \le C_{\sigma,q} e^{Kt} \|w\|_{H^{\sigma,q}(D)}, \tag{6.70}$$

valid for

$$2 \le q < \infty, \quad \sigma q \in [0,1). \tag{6.71}$$

Similar arguments give such operator bounds on e^{-tX}. We have the following convergence result.

Proposition 6.7. *Let σ, q satisfy (6.71). Then, for each $t \in (0,\infty)$,*

$$w_0 \in H^{\sigma,q}(D) \Longrightarrow \lim_{\nu\to 0} e^{t(\nu\Delta - X)} w_0 = e^{-tX} w_0, \tag{6.72}$$

in $H^{\sigma,q}$-norm.

Proof. Given $w_0 \in H^{\sigma,q}(D)$, (6.70) implies $\{e^{t(\nu\Delta - X)} w_0 : \nu \in (0,1]\}$ is bounded in $H^{\sigma,q}(D)$ for each $t \in (0,\infty)$, so there is a weak* limit point. But Proposition 6.6 yields convergence to $e^{-tX} w_0$ in L^q-norm, so $e^{-tX} w_0$ is the only possible weak* limit point. Norm convergence in $H^{\tau,q}(D)$, for each $\tau < \sigma$, then follows from the compactness of the inclusion $H^{\sigma,q}(D) \hookrightarrow H^{\tau,q}(D)$. Taking $\sigma' > \sigma$ such that $\sigma' q < 1$, the argument above yields $e^{t(\nu\Delta - X)} w_0 \to e^{-tX} w_0$ in $H^{\sigma,q}$-norm for each $w_0 \in H^{\sigma',q}(D)$. The conclusion follows by denseness of $H^{\sigma',q}(D)$ in $H^{\sigma,q}(D)$, plus the uniform operator bound (6.70).

This concludes our treatment of $R_1(\nu, t, x)$. As mentioned, more precise results, including boundary layer analyses, are given in [MT1] and [MT2].

We move to an analysis of $R_3(\nu, t, x)$ in (6.48), i.e.,

$$R_3(\nu, t, x) = \int_0^t e^{(t-s)(\nu\Delta - X)} (s_0 - s^\nu) \frac{\partial w^\nu}{\partial \theta}\, ds, \tag{6.73}$$

where w^ν solves (6.34) and (6.36). Note that $\partial/\partial\theta$ commutes with X, X_ν, and Δ, so $z^\nu(t,x) = \partial w^\nu/\partial\theta$ solves

$$\frac{\partial z^\nu}{\partial t} = (\nu\Delta - X_\nu) z^\nu, \quad z^\nu\Big|_{\mathbb{R}^+ \times \partial D} = 0, \quad z^\nu(0,x) = \frac{\partial w_0}{\partial \theta}. \tag{6.74}$$

The maximum principle gives

$$\Big\| \frac{\partial w^\nu}{\partial \theta}(s) \Big\|_{L^\infty(D)} \le \Big\| \frac{\partial w_0}{\partial \theta} \Big\|_{L^\infty(D)}. \tag{6.75}$$

Since the semigroup $e^{t(\nu\Delta-X)}$ is positivity preserving, we have

$$|R_3(\nu,t,x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{(t-s)(\nu\Delta-X)} \bigl|s_0(|x|)-s^\nu(s,|x|)\bigr|\,ds. \tag{6.76}$$

Also, by radial symmetry,

$$e^{(t-s)(\nu\Delta-X)}|s_0-s^\nu| = e^{\nu(t-s)\Delta}|s_0-s^\nu|, \tag{6.77}$$

so

$$|R_3(\nu,t,x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{\nu(t-s)\Delta}|\tilde{s}_0-\tilde{s}^\nu|\,ds, \tag{6.78}$$

where

$$\tilde{s}_0(x)=s_0(|x|), \quad \tilde{s}^\nu(t,x)=s^\nu(t,|x|), \tag{6.79}$$

and, we recall from (6.41),

$$v^\nu(t,x)=s^\nu(t,|x|)x^\perp, \quad v_0(x)=s_0(|x|)x^\perp. \tag{6.80}$$

Turning these around, we have

$$s^\nu(t,|x|)=\frac{1}{|x|^2}v^\nu(t,x)\cdot x^\perp, \quad s_0(|x|)=\frac{1}{|x|^2}v_0(x)\cdot x^\perp, \tag{6.81}$$

and also, if $\{e_1,e_2\}$ denotes the standard orthonormal basis of $\mathbb{R}^2$,

$$\begin{aligned} s^\nu(t,r) &= \frac{1}{r}v^\nu(t,re_1)\cdot e_2 \\ &= \frac{1}{r}\int_0^1 \frac{d}{d\sigma}v^\nu(t,r\sigma e_1)\cdot e_2\,d\sigma \\ &= \int_0^1 e_2\cdot\nabla_{e_1}v^\nu(t,r\sigma e_1)\,d\sigma, \end{aligned} \tag{6.82}$$

and similarly

$$s_0(r)=\int_0^1 e_2\cdot\nabla_{e_1}v_0(t,r\sigma e_1)\,d\sigma. \tag{6.83}$$

The representation (6.81) is effective away from a neighborhood of $\{x=0\}$, especially near ∂D, where one reads off the uniform convergence of $s^\nu(t,r)$ to $s_0(r)$ except on the boundary layer discussed above in the analysis of $v^\nu(t,\cdot)\to v_0(\cdot)$, given $v_0\in C^\infty(\overline{D})$.

The representation (6.82)–(6.83) is effective on a neighborhood of $\{x = 0\}$, for example the disk $\overline{D}_{1/2} = \{x \in \mathbb{R}^2 : |x| \le 1/2\}$, and it shows that $s^\nu(t,r) \to s_0(r)$ uniformly on $r \le 1/2$ provided $v^\nu(t,\cdot) \to v_0(t,\cdot)$ in $C^1(\overline{D}_{1/2})$. Results from Chap. 6, § 8 (cf. Propositions 8.1–8.2) imply one has such convergence if $v_0 \in C^1(\overline{D})$, and in particular if $v_0 \in C^\infty(\overline{D})$.

Furthermore, the maximum principle implies

$$e^{\nu t\Delta}|\tilde{s}_0 - \tilde{s}^\nu| \le h_{\nu t} * |\tilde{s}_0 - \tilde{s}^\nu|, \tag{6.84}$$

where $h_{\nu t}$ is the free space heat kernel, given (with $n = 2$) by

$$h_{\nu t}(x) = (4\pi\nu t)^{-n/2} e^{-|x|^2/4\nu t}, \tag{6.85}$$

and $|\tilde{s}_0 - \tilde{s}^\nu|$ is extended by 0 outside $\overline{D}$. We hence have the following boundary layer estimates on R_3.

Proposition 6.8. *Assume $v_0, w_0 \in C^\infty(\overline{D})$. Then, given $T \in (0,\infty)$, we have a uniform bound*

$$|R_3(\nu,t,x)| \le C, \tag{6.86}$$

for $t \in [0,T]$, $\nu \in (0,1]$, $x \in \overline{D}$. Furthermore, as $\nu \to 0$,

$$R_3(\nu,t,x) \to 0 \ \text{ uniformly on } \ \overline{D} \setminus \Gamma_{\omega(\nu)}, \tag{6.87}$$

as long as

$$\frac{\omega(\nu)}{\sqrt{\nu}} \to \infty. \tag{6.88}$$

We recall $\Gamma_\delta = \{x \in D : \operatorname{dist}(x, \partial D) \le \delta\}$.

Among other results established in [MT1]–[MT2], we mention one here. For $k \in \mathbb{N}$, set

$$\mathcal{V}^k(D) = \{f \in L^2(D) : X_{j_1} \cdots X_{j_\ell} f \in L^2(D),\ \forall\, \ell \le k,\ X_{j_m} \in \mathfrak{X}^1(D)\}, \tag{6.89}$$

where $\mathfrak{X}^1(D)$ denotes the space of smooth vector felds on $\overline{D}$ that are tangent to ∂D. After establishing that

$$f \in \mathcal{V}^k(D) \Longrightarrow \lim_{t\to 0} e^{t\Delta} f = f, \ \text{ in } \mathcal{V}^k\text{-norm}, \tag{6.90}$$

and

$$f \in \mathcal{V}^k(D) \Longrightarrow \lim_{\nu \searrow 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \ \text{ in } \mathcal{V}^k\text{-norm}, \tag{6.91}$$

these works proved the following (cf. [MT2], Proposition 3.10).

Proposition 6.9. *Assume $v_0 \in C^\infty(\overline{D})$ and $w_0 \in C^1(\overline{D})$. Take $k \in \mathbb{N}$ and also assume $w_0 \in \mathcal{V}^k(D)$. Then, for each $t > 0$, as $\nu \searrow 0$,*

$$v^\nu(t,\cdot) \to v_0 \ \ \text{and} \ \ w^\nu(t,\cdot) \to w^0(t,\cdot), \ \ \text{in} \ \mathcal{V}^k\text{-norm}. \tag{6.92}$$

Such a result is consistent with Prandtl's principle, that in the boundary layer it is normal derivatives of the velocity field, not tangential derivatives, that blow up as $\nu \to 0$. We mention that the convergence of $R_2(\nu, t, x)$ to 0 in $\mathcal{V}^k$-norm follows from the analysis described in (6.50), and the convergence of $R_1(\nu, t, x)$ to 0 in $\mathcal{V}^k$-norm, given $w_0 \in C^\infty(\overline{D})$, follows from a parallel analysis, carried out in [MT2], but not here. The convergence of $R_3(\nu, t, x)$ to 0 in $\mathcal{V}^k$-norm, given $v_0, w_0 \in C^\infty(\overline{D})$, does not follow from the results on R_3 established here; this requires further arguments.

The two cases analyzed above are much simpler than the general cases, which might involve turbulent boundary layers and boundary layer separation. Another issue is loss of stability of a solution as ν decreases. One can read more about such problems in [Bat, Sch, ChM, OO], and references given there. We also mention [VD], which has numerous interesting illustrations of fluid phenomena, at various viscosities.

Exercises

1. Verify the characterization (6.12) of vector fields of the form (6.11).
2. Verify the calculation (6.13)–(6.14).
3. Produce a proof of (6.90), at least for $k = 1$. Try for larger k.

7. From velocity field convergence to flow convergence

In §6 we have given some results on convergence of the solutions u^ν to the Navier–Stokes equations

$$\begin{aligned} &\frac{\partial u^\nu}{\partial t} + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu \quad \text{on } I \times \Omega, \\ &\operatorname{div} u^\nu = 0, \quad u^\nu\big|_{I\times\partial\Omega} = 0, \quad u^\nu(0) = u_0, \end{aligned} \tag{7.1}$$

to the solution u to the Euler equation

$$\begin{aligned} &\frac{\partial u}{\partial t} + \nabla_u u + \nabla p = 0 \quad \text{on } I \times \Omega, \\ &\operatorname{div} u = 0, \quad u \parallel \partial\Omega, \quad u(0) = u_0, \end{aligned} \tag{7.2}$$

as $\nu \searrow 0$ (given $\operatorname{div} u_0 = 0,\ u_0 \parallel \partial\Omega$).

We now tackle the question of what can be said about convergence of fluid flows generated by the t-dependent velocity fields u^ν to the flow generated by u. Given the convergence results of § 6, we are motivated to see what sort of flow convergence can be deduced from fairly weak hypotheses on $u,\ u^\nu$, and the nature of the convergence $u^\nu \to u$. We obtain some such results here; further results can be found in [DL].

We will make the following hypotheses on the t-dependent vector fields u and u^ν.

$$u \in \mathrm{Lip}([0,T]\times\overline{\Omega}), \quad \operatorname{div} u(t) = 0, \quad u(t) \parallel \partial\Omega, \tag{7.3}$$

$$u^\nu \in \mathrm{Lip}([\varepsilon,T]\times\overline{\Omega}),\ \forall\, \varepsilon > 0, \quad \operatorname{div} u^\nu(t) = 0, \quad u^\nu(t) \parallel \partial\Omega, \tag{7.4}$$

$$u^\nu \in L^\infty([0,T]\times\Omega). \tag{7.5}$$

Say $\nu \in (0,1]$. Here $\overline{\Omega}$ is a smoothly bounded domain in $\mathbb{R}^n$, or more generally it could be a compact Riemannian manifold with smooth boundary $\partial\Omega$. We do not assume any uniformity in ν on the estimates associated to (7.4)–(7.5).

The field u defines volume preserving bi-Lipschitz maps

$$\varphi^{t,s} : \overline{\Omega} \longrightarrow \overline{\Omega}, \quad s,t \in [0,T], \tag{7.6}$$

satisfying

$$\frac{\partial}{\partial t}\varphi^{t,s}(x) = u(t,\varphi^{t,s}(x)), \quad \varphi^{s,s}(x) = x. \tag{7.7}$$

Similarly the fields u^ν define volume preserving bi-Lipschitz maps

$$\varphi_\nu^{t,s} : \overline{\Omega} \longrightarrow \overline{\Omega}, \quad s,t \in (0,T], \tag{7.8}$$

satisfying

$$\frac{\partial}{\partial t}\varphi_\nu^{t,s}(x) = u^\nu(t,\varphi_\nu^{t,s}(x)), \quad \varphi_\nu^{s,s}(x) = x. \tag{7.9}$$

Note that

$$\begin{aligned} \varphi^{t,s}\circ\varphi^{s,r} &= \varphi^{t,r}, \quad r,s,t \in [0,T], \\ \varphi_\nu^{t,s}\circ\varphi_\nu^{s,r} &= \varphi_\nu^{t,r}, \quad r,s,t \in (0,T]. \end{aligned} \tag{7.10}$$

Our convergence results will be phrased in terms of strong operator convergence on $L^p(\Omega)$ of operators $\mathcal{S}_\nu^{t,0}$ to $\mathcal{S}^{t,0}$, where

$$\begin{aligned} \mathcal{S}^{t,s} f_0(x) &= f_0(\varphi^{s,t}(x)), \quad s,t \in [0,T], \\ \mathcal{S}_\nu^{t,s} f_0(x) &= f_0(\varphi_\nu^{s,t}(x)), \quad s,t \in (0,T], \end{aligned} \tag{7.11}$$

and $\mathcal{S}_\nu^{t,0}$ (as well as $\varphi_\nu^{0,t}$) will be constructed below. These operators are also characterized as follows. For $f = f(t,x)$ satisfying

$$\frac{\partial f}{\partial t} = -\nabla_{u(t)} f(t), \quad t \in [0,T], \tag{7.12}$$

we set

$$\mathcal{S}^{t,s} f(s) = f(t), \quad s,t \in [0,T]. \tag{7.13}$$

Note that f is advected by the flow generated by $u(t)$. Clearly

$$\mathcal{S}^{t,s} : L^p(\Omega) \longrightarrow L^p(\Omega), \quad \text{isometrically isomorphically}, \ \forall\, s,t \in [0,T]. \tag{7.14}$$

Similarly, for $f^\nu = f^\nu(t,x)$ solving

$$\frac{\partial f^\nu}{\partial t} = -\nabla_{u^\nu(t)} f^\nu(t), \tag{7.15}$$

we set

$$\mathcal{S}_\nu^{t,s} f^\nu(s) = f^\nu(t), \quad s,t \in (0,T], \tag{7.16}$$

and again

$$\mathcal{S}_\nu^{t,s} : L^p(\Omega) \longrightarrow L^p(\Omega), \quad \text{isometrically isomorphically}, \ \forall\, s,t \in (0,T]. \tag{7.17}$$

Note that

$$\begin{aligned} \mathcal{S}^{t,s}\mathcal{S}^{s,r} &= \mathcal{S}^{t,r}, \quad r,s,t \in [0,T], \\ \mathcal{S}_\nu^{t,s}\mathcal{S}_\nu^{s,r} &= \mathcal{S}_\nu^{t,r}, \quad r,s,t \in (0,T]. \end{aligned} \tag{7.18}$$

We will extend the scope of (7.16) to the case $s = 0$. Then we will show that, given (7.3)–(7.5), $p \in [1,\infty)$, $t \in [0,T]$, $f_0 \in L^p(\Omega)$,

$$\begin{aligned} &u^\nu \to u \ \text{ in } \ L^1([0,T], L^p(\Omega)) \\ &\qquad \Longrightarrow \mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \ \text{ in } \ L^p\text{-norm}. \end{aligned} \tag{7.19}$$

In light of the relationship

$$\mathcal{S}_\nu^{t,0} f_0(x) = f_0(\varphi_\nu^{0,t}(x)), \tag{7.20}$$

which will be established below, this convergence amounts to some sort of convergence

$$\varphi_\nu^{0,t} \longrightarrow \varphi^{0,t} \tag{7.21}$$

for the *backward* flows $\varphi_\nu^{0,t}$.

To construct $\mathcal{S}_\nu^{t,0}$ on $L^p(\Omega)$, we first note that

$$\begin{aligned} &\varepsilon, \delta \in (0, T], \ f_0 \in \operatorname{Lip}(\overline{\Omega}) \\ &\qquad \Longrightarrow \|\mathcal{S}_\nu^{\delta,\varepsilon} f_0 - f_0\|_{L^\infty} \le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} |\varepsilon - \delta|, \end{aligned} \tag{7.22}$$

which in turn implies

$$\begin{aligned} \|\mathcal{S}_\nu^{t,\delta} f_0 - \mathcal{S}_\nu^{t,\varepsilon} f_0\|_{L^\infty} &= \|\mathcal{S}_\nu^{t,\varepsilon}(\mathcal{S}_\nu^{\varepsilon,\delta} f_0 - f_0)\|_{L^\infty} \\ &= \|\mathcal{S}_\nu^{\varepsilon,\delta} f_0 - f_0\|_{L^\infty} \\ &\le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} \cdot |\varepsilon - \delta|. \end{aligned} \tag{7.23}$$

Hence

$$\lim_{\varepsilon \searrow 0} \mathcal{S}_\nu^{t,\varepsilon} f_0 = \mathcal{S}_\nu^{t,0} f_0 \tag{7.24}$$

exists for all $f_0 \in \operatorname{Lip}(\overline{\Omega})$, convergence in (7.24) holding in sup-norm, and a fortiori in L^p-norm. The uniform boundedness from (7.17) then implies that (7.24) holds in L^p-norm for each $f_0 \in L^p(\Omega)$, as long as $p \in [1, \infty)$, so $\operatorname{Lip}(\overline{\Omega})$ is dense in $L^p(\Omega)$. This defines

$$\mathcal{S}_\nu^{t,0} : L^p(\Omega) \longrightarrow L^p(\Omega), \quad 1 \le p < \infty, \ t \in (0, T], \tag{7.25}$$

and we have

$$\|\mathcal{S}_\nu^{t,0} f_0\|_{L^p} = \lim_{\varepsilon \searrow 0} \|\mathcal{S}_\nu^{t,\varepsilon} f_0\|_{L^p} \equiv \|f_0\|_{L^p}, \tag{7.26}$$

so $\mathcal{S}_\nu^{t,0}$ is an isometry on $L^p(\Omega)$ for each $p \in [1, \infty)$.

We note that, parallel to (7.22), for $\varepsilon, \delta \in (0, T], \ x \in \overline{\Omega}$,

$$\operatorname{dist}(\varphi_\nu^{\delta,\varepsilon}(x), x) \le \|u^\nu\|_{L^\infty} \cdot |\varepsilon - \delta|, \tag{7.27}$$

and, parallel to (7.23), if also $t \in (0, T]$,

$$\begin{aligned} \operatorname{dist}(\varphi_\nu^{\varepsilon,t}(x), \varphi_\nu^{\delta,t}(x)) &= \operatorname{dist}(\varphi_\nu^{\varepsilon,\delta}(\varphi_\nu^{\delta,t}(x)), \varphi_\nu^{\delta,t}(x)) \\ &\le \|u^\nu\|_{L^\infty} \cdot |\varepsilon - \delta|. \end{aligned} \tag{7.28}$$

It follows that

$$\varphi_\nu^{0,t}(x) = \lim_{\varepsilon \searrow 0} \varphi_\nu^{\varepsilon,t}(x) \tag{7.29}$$

exists and $\varphi_\nu^{0,t} : \overline{\Omega} \to \overline{\Omega}$ continuously, preserving the volume. Furthermore, we have

$$\mathcal{S}_\nu^{t,0} f_0(x) = f_0(\varphi_\nu^{0,t}(x)), \tag{7.30}$$

first for $f_0 \in \mathrm{Lip}(\overline{\Omega})$, then, by limiting arguments, for all $f_0 \in C(\overline{\Omega})$, and furthermore, for all $f_0 \in L^p(\Omega)$, in which case (7.30) holds, for each $t \in (0,T]$, for a.e. x, i.e., (7.30) is an identity in the Banach space $L^p(\Omega)$.

We derive some more properties of $\mathcal{S}_\nu^{t,0}$. Note from (7.23)–(7.24) that, when $f_0 \in \mathrm{Lip}(\overline{\Omega}),\ \varepsilon \in (0,T]$,

$$\|\mathcal{S}_\nu^{t,0} f_0 - \mathcal{S}_\nu^{t,\varepsilon} f_0\|_{L^\infty} \le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} \cdot \varepsilon, \tag{7.31}$$

and hence, by uniform operator boundedness, for each $f_0 \in L^p(\Omega),\ p \in [1,\infty),\ s,t \in (0,T]$,

$$\begin{aligned} \mathcal{S}_\nu^{t,0} f_0 &= \lim_{\varepsilon \searrow 0} \mathcal{S}_\nu^{t,\varepsilon} f_0 \quad (\text{in } L^p\text{-norm}) \\ &= \lim_{\varepsilon \searrow 0} \mathcal{S}_\nu^{t,s} \mathcal{S}_\nu^{s,\varepsilon} f_0 \quad (\text{by } (7.18)) \\ &= \mathcal{S}_\nu^{t,s} \mathcal{S}_\nu^{s,0} f_0. \end{aligned} \tag{7.32}$$

Hence, $\mathcal{S}_\nu^{s,t} \mathcal{S}_\nu^{t,0} = \mathcal{S}_\nu^{s,0}$ on $L^p(\Omega),\ \forall\, s,t \in (0,T]$, or equivalently,

$$\mathcal{S}_\nu^{\delta,t} \mathcal{S}^{t,0} = \mathcal{S}_\nu^{\delta,0} \quad \text{on } L^p(\Omega). \tag{7.33}$$

We also have from (7.23)–(7.24) that

$$f_0 \in \mathrm{Lip}(\overline{\Omega}) \Longrightarrow \|\mathcal{S}_\nu^{\delta,0} f_0 - f_0\|_{L^\infty} \le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} \cdot \delta, \tag{7.34}$$

and hence, again by uniform operator boundedness and denseness of $\mathrm{Lip}(\overline{\Omega})$,

$$\lim_{\delta \searrow 0} \mathcal{S}_\nu^{\delta,0} f_0 = f_0 \ \text{ in } L^p\text{-norm},\ \forall\, f_0 \in L^p(\Omega). \tag{7.35}$$

We now want to compare $\mathcal{S}^{t,0} f_0$ with $\mathcal{S}_\nu^{t,0} f_0$. To begin, take

$$f_0 \in \mathrm{Lip}(\overline{\Omega}), \quad f(t) = \mathcal{S}^{t,0} f_0. \tag{7.36}$$

Then $f(t)$ satisfies

$$\frac{\partial f}{\partial t} = -\nabla_{u^\nu(t)} f(t) + \nabla_{u^\nu(t) - u(t)} f(t), \quad f(0) = f_0, \tag{7.37}$$

so Duhamel's formula gives

$$f(t) = \mathcal{S}_\nu^{t,0} f_0 + \int_0^t \mathcal{S}_\nu^{t,s} \nabla_{u^\nu(s) - u(s)} f(s)\, ds. \tag{7.38}$$

Now, by hypothesis (7.3) on u, we see that, for $s \in [0, T]$,

$$
\begin{aligned}
f_0 \in \mathrm{Lip}(\overline{\Omega}) &\Longrightarrow \|f(s)\|_{\mathrm{Lip}} \le A\|f_0\|_{\mathrm{Lip}} \\
&\Longrightarrow |\nabla_{u^\nu(s)-u(s)} f(s)| \le A\|f_0\|_{\mathrm{Lip}} |u^\nu(s) - u(s)|.
\end{aligned} \tag{7.39}
$$

Hence

$$
\begin{aligned}
f_0 \in \mathrm{Lip}(\overline{\Omega}) \Longrightarrow \|\mathcal{S}^{t,0} f_0 - \mathcal{S}_\nu^{t,0} f_0\|_{L^p} &\le \int_0^t \|\mathcal{S}_\nu^{t,s} \nabla_{u^\nu(s)-u(s)} f(s)\|_{L^p}\, ds \\
&\le A\|f_0\|_{\mathrm{Lip}} \int_0^t \|u^\nu(s) - u(s)\|_{L^p}\, ds.
\end{aligned} \tag{7.40}
$$

Hence, given $p \in [1, \infty),\ t \in [0, T]$,

$$
\begin{aligned}
&u^\nu \to u \ \text{ in } \ L^1([0, T], L^p(\Omega)) \\
&\Longrightarrow \mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \ \text{ in } \ L^p\text{-norm},
\end{aligned} \tag{7.41}
$$

for all $f_0 \in \mathrm{Lip}(\overline{\Omega})$, and hence, by the uniform operator bounds (7.26) and denseness of $\mathrm{Lip}(\overline{\Omega})$ in $L^p(\Omega)$, we have:

Proposition 7.1. *Under the hypotheses (7.3)–(7.5), given $p \in [1, \infty),\ t \in [0, T]$, convergence in (7.41) holds for all $f_0 \in L^p(\Omega)$.*

In fact, we can improve Proposition 7.1, as follows. (Compare [DL], Theorem II.4.)

Proposition 7.2. *Given $p \in (1, \infty),\ t \in [0, T]$,*

$$
\begin{aligned}
&f_0 \in L^p(\Omega),\ u^\nu \to u \ \textit{ in } \ L^1([0, T], L^1(\Omega)) \\
&\Longrightarrow \mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \ \textit{ in } \ L^p\textit{-norm.}
\end{aligned} \tag{7.42}
$$

Proof. By Proposition 7.1, the hypotheses of (7.42) imply

$$
\mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \tag{7.43}
$$

in L^1-norm. We also know that

$$
\|\mathcal{S}_\nu^{t,0} f_0\|_{L^p} = \|\mathcal{S}^{t,0} f_0\|_{L^p} = \|f_0\|_{L^p}, \tag{7.44}
$$

for each $\nu \in (0, 1],\ t \in [0, T]$. These bounds imply weak* compactness in $L^p(\Omega)$, and we see that convergence in (7.43) holds weak* in $L^p(\Omega)$. Then another use of (7.44), together with the *uniform convexity* of $L^p(\Omega)$ for each $p \in (1, \infty)$ gives convergence in L^p-norm in (7.43).

A. Regularity for the Stokes system on bounded domains

The following result is the basic ingredient in the proof of Proposition 5.2. Assume that $\overline{\Omega}$ is a compact, connected Riemannian manifold, with smooth boundary, that

$$u \in H^1(\Omega, T^*), \quad f \in L^2(\Omega, T^*), \quad p \in L^2(\Omega),$$

and that

$$-\Delta u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0. \tag{A.1}$$

We claim that $u \in H^2(\Omega, T^*)$. More generally, we claim that, for $s \geq 0$,

$$f \in H^s(\Omega, T^*) \Longrightarrow u \in H^{s+2}(\Omega, T^*). \tag{A.2}$$

Indeed, given any $\lambda \in [0, \infty)$, it is an equivalent task to establish the implication (A.2) when we replace (A.1) by

$$(\lambda - \Delta)u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0. \tag{A.3}$$

In this appendix we prove this result. We also treat the following related problem. Assume $v \in H^1(\Omega, T^*)$, $p \in L^2(\Omega)$, and

$$(\lambda - \Delta)v = dp, \quad \delta v = 0, \quad v\big|_{\partial\Omega} = g. \tag{A.4}$$

Then we claim that, for $s \geq 0$,

$$g \in H^{s+3/2}(\partial\Omega, T^*) \Longrightarrow v \in H^{s+2}(\Omega, T^*). \tag{A.5}$$

Here, for any $x \in \overline{\Omega}$ (including $x \in \partial\Omega$), $T_x^* = T_x^*(\overline{\Omega}) = T_x^* M$, where we take M to be a compact Riemannian manifold without boundary, containing Ω as an open subset (with smooth boundary $\partial\Omega$). In fact, take M to be diffeomorphic to the double of $\overline{\Omega}$.

We will represent solutions to (A.4) in terms of layer potentials, in a fashion parallel to constructions in § 11 of Chap. 7. Such an approach is taken in [Sol1]; see also [Lad]. A different sort of proof, appealing to the theory of systems elliptic in the sense of Douglis–Nirenberg, is given in [Tem]. An extension of the boundary-layer approach to Lipschitz domains is given in [FKV]. This work has been applied to the Navier–Stokes equations on Lipschitz domains in [DW]. Here the analysis was restricted to Lipschitz domains with connected boundary. This topological restriction was removed in [MiT]. Subsequently, [Mon] produced strong, short time solutions on 3D domains with arbitrarily rough boundary.

Pick $\lambda \in (0, \infty)$. We now define some operators on $\mathcal{D}'(M)$, so that

$$(\lambda - \Delta)\Phi - dQ = I \text{ on } \mathcal{D}'(M, T^*), \quad \delta\Phi = 0. \tag{A.6}$$

To get these operators, start with the Hodge decomposition on M:

$$d\delta G + \delta dG + P_h = I \quad \text{on } \mathcal{D}'(M, \Lambda^*), \tag{A.7}$$

where P_h is the orthogonal projection onto the space $\mathcal{H}$ of harmonic forms on M, and G is Δ^{-1} on the orthogonal complement of $\mathcal{H}$. Then (A.6) holds if we set

$$\begin{aligned} \Phi &= (\lambda - \Delta)^{-1}(\delta dG + P_h) \in OPS^{-2}(M), \\ Q &= -\delta G \in OPS^{-1}(M). \end{aligned} \tag{A.8}$$

Let $F(x, y)$ and $Q(x, y)$ denote the Schwartz kernels of these operators. Thus

$$(\lambda - \Delta_x)F(x, y) - d_x Q(x, y) = \delta_y(x)I, \quad \delta_x F(x, y) = 0. \tag{A.9}$$

Note that as $\operatorname{dist}(x, y) \to 0$, we have (for $\dim \Omega = n \geq 3$)

$$\begin{aligned} F(x, y) &\sim A_0(x, y) \operatorname{dist}(x, y)^{2-n} + \cdots, \\ Q(x, y) &\sim B_0(x, y) \operatorname{dist}(x, y)^{1-n} + \cdots, \end{aligned} \tag{A.10}$$

where $A_0(\operatorname{Exp}_y v, y)$ and $B_0(\operatorname{Exp}_y v, y)$ are homogeneous of degree zero in $v \in T_y M$.

We now look for solutions to (A.4) in the form of layer potentials:

$$\begin{aligned} v(x) &= \int_{\partial\Omega} F(x, y)\, w(y)\, dS(y) = \mathcal{F}w(x), \\ p(x) &= \int_{\partial\Omega} Q(x, y) \cdot w(y)\, dS(y) = \mathcal{Q}w(x). \end{aligned} \tag{A.11}$$

The first two equations in (A.4) then follow directly from (A.9), and the last equation in (A.4) is equivalent to

$$\Psi w = g, \tag{A.12}$$

where

$$\Psi w(x) = \int_{\partial\Omega} F(x, y) w(y)\, dS(y), \quad x \in \partial\Omega, \tag{A.13}$$

defines

$$\Psi \in OPS^{-1}(\partial\Omega, T^*). \tag{A.14}$$

Note that Ψ is self-adjoint on $L^2(\partial\Omega, T^*)$. The following lemma is incisive:

Lemma A.1. *The operator Ψ is an elliptic operator in $OPS^{-1}(\partial\Omega)$.*

We can analyze the principal symbol of Ψ using the results of § 11 in Chap. 7, particularly the identity (11.12) there. This implies that, for $x \in \partial\Omega$, $\xi \in T_x(\partial\Omega)$, ν the outgoing unit normal to $\partial\Omega$ at x,

$$\sigma_\Psi(x,\xi) = C_n \int_{-\infty}^{\infty} \sigma_\Phi(x, \xi + \tau\nu)\, d\tau. \tag{A.15}$$

From (A.8), we have

$$\sigma_\Phi(x,\zeta)\beta = |\zeta|^{-4}\iota_\zeta \wedge_\zeta \beta, \quad \zeta, \beta \in T_x^* M. \tag{A.16}$$

This is equal to $|\zeta|^{-2}P_\zeta^\perp \beta$, where $P_\zeta^\perp$ is the orthogonal projection of T_x^* onto $(\zeta)^\perp$. Thus

$$\sigma_\Phi(x,\zeta)\beta = A(\zeta)\beta - B(\zeta)\beta, \tag{A.17}$$

with

$$A(\zeta)\beta = |\zeta|^{-2}\beta, \quad B(\zeta)\beta = |\zeta|^{-4}(\beta\cdot\zeta)\zeta.$$

Hence

$$\int_{-\infty}^{\infty} A(\xi + \tau\nu)\, d\tau = \int_{-\infty}^{\infty} (|\xi|^2 + \tau^2)^{-1}\, d\tau = \gamma_1 |\xi|^{-1}, \tag{A.18}$$

with

$$\gamma_1 = \int_{-\infty}^{\infty} \frac{1}{1+\tau^2}\, d\tau.$$

Also

$$\begin{aligned} \int_{-\infty}^{\infty} B(\xi + \tau\nu)\beta\, d\tau &= \int_{-\infty}^{\infty} \big(|\xi|^2 + \tau^2\big)^{-2}\big[(\beta\cdot\xi)\xi + \tau^2(\beta\cdot\nu)\nu\big]\, d\tau \\ &= \gamma_2|\xi|^{-3}(\beta\cdot\xi)\xi + \gamma_3|\xi|^{-1}(\beta\cdot\nu)\nu, \end{aligned} \tag{A.19}$$

with

$$\gamma_2 = \int_{-\infty}^{\infty} \frac{1}{(1+\tau^2)^2}\, d\tau, \quad \gamma_3 = \int_{-\infty}^{\infty} \frac{\tau^2}{(1+\tau^2)^2}\, d\tau. \tag{A.20}$$

We have

$$\sigma_\Psi(x,\xi) = C_n|\xi|^{-1}\big[\gamma_1 I - \gamma_2 P_\xi - \gamma_3 P_\nu\big], \tag{A.21}$$

where P_ξ is the orthogonal projection of T^*_x onto the span of ξ, and P_ν is similarly defined. Note that $\gamma_2+\gamma_3=\gamma_1,\ 0<\gamma_j$. Hence $0<\gamma_2<\gamma_1$ and $0<\gamma_3<\gamma_1$. In fact, use of residue calculus readily gives

$$\gamma_1=\pi,\quad \gamma_2=\frac{\pi}{4},\quad \gamma_3=\frac{3\pi}{4}.$$

Thus the symbol (A.21) is invertible, in fact positive-definite. Lemma A.1 is proved.

We also have, for any $\sigma\in\mathbb{R}$,

$$\Psi : H^\sigma(\partial\Omega,T^*)\longrightarrow H^{\sigma+1}(\partial\Omega,T^*),\ \text{Fredholm, of index zero.} \tag{A.22}$$

We next characterize Ker Ψ, which we claim is a one-dimensional subspace of $C^\infty(\partial\Omega,T^*)$.

The ellipticity of Ψ implies that Ker Ψ is a finite-dimensional subspace of $C^\infty(\partial\Omega,T^*)$. If $w\in$ Ker Ψ, consider $v=\mathcal{F}w,\ p=\mathcal{Q}w$, defined by (A.11), on $\Omega\cup\mathcal{O}$ (where $\mathcal{O}=M\setminus\overline{\Omega}$). We have $(\lambda-\Delta)v=dp$ on $\Omega,\ \delta v=0$ on Ω, and $v\big|_{\partial\Omega}=0$, so, since solutions to (A.3) are unique for any $\lambda>0$, we deduce that $v=0$ on Ω. Similarly, $v=0$ on $\mathcal{O}$. In other words,

$$\Phi(w\sigma)=0 \quad \text{on } \Omega\cup\mathcal{O}, \tag{A.23}$$

where σ is the area element of $\partial\Omega$, so $w\sigma$ is an element of $\mathcal{D}'(M,T^*)$, supported on $\partial\Omega$. Since $\Phi\in OPS^{-2}(M),\ \Phi(w\sigma)\in C(M,T^*)$, so (A.23) implies $\Phi(w\sigma)=0$ on M. Consequently, by (A.6),

$$w\sigma = dQ(w\sigma) \text{ on } M. \tag{A.24}$$

The right side is equal to $d\delta G(w\sigma)=P_d(w\sigma)$. It follows that $d(w\sigma)=0$, which uniquely determines w, up to a constant scalar multiple, on each component of $\partial\Omega$, namely as a constant multiple of ν. It follows that

$$w\in \text{Ker }\Psi \Longleftrightarrow w\sigma = C\ d\chi_\Omega, \tag{A.25}$$

for some constant C, assuming Ω and $\mathcal{O}$ are connected. In our situation, $\mathcal{O}$ is diffeomorphic to Ω, which is assumed to be connected.

Consequently, whenever $g\in H^{s+3/2}(\partial\Omega,T^*)$ satisfies

$$\int_{\partial\Omega}\langle g,\nu\rangle\, dS=0, \tag{A.26}$$

the unique solution to (A.4) is given by (A.11), with

$$w \in H^{s+1/2}(\partial\Omega, T^*). \tag{A.27}$$

Note that if $\delta v = 0$ on Ω and $v\big|_{\partial\Omega} = g$, then the divergence theorem implies that (A.26) holds. Thus this construction applies to all solutions of (A.4).

Next we reduce the analysis of (A.3) to that of (A.4). Thus, let $f \in H^s(\Omega, T^*)$. Extend f to $\widetilde{f} \in H^s(M, T^*)$. Now let $u_1 \in H^{s+2}(M, T^*)$, $p_1 \in H^{s+1}(M)$ solve

$$(\lambda - \Delta)u_1 = \widetilde{f} + dp_1, \quad \delta u_1 = 0 \quad \text{on } M, \tag{A.28}$$

hence $u_1 = \Phi\widetilde{f}$ and $p_1 = Q\widetilde{f}$. If u solves (A.3), take $v = u - u_1\big|_{\Omega}$, which solves (A.4), with p replaced by $p - p_1$, and

$$g = -u_1\big|_{\partial\Omega} \in H^{s+3/2}(\partial\Omega, T^*). \tag{A.29}$$

Furthermore, since $\delta u_1 = 0$ on M, we have (A.26), as remarked above.

We are in a position to establish the results stated at the beginning of this appendix, namely:

Proposition A.2. *Assume $u, v \in H^1(\Omega, T^*)$, $f \in L^2(\Omega, T^*)$, $p \in L^2(\Omega)$, and $\lambda > 0$. If*

$$(\lambda - \Delta)u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0, \tag{A.30}$$

then, for $s \ge 0$,

$$f \in H^s(\Omega, T^*) \Longrightarrow u \in H^{s+2}(\Omega, T^*), \tag{A.31}$$

and if

$$(\lambda - \Delta)v = dp, \quad \delta v = 0, \quad v\big|_{\partial\Omega} = g, \tag{A.32}$$

then, for $s \ge 0$,

$$g \in H^{s+3/2}(\partial\Omega, T^*) \Longrightarrow v \in H^{s+2}(\Omega, T^*). \tag{A.33}$$

Proof. As seen above, it suffices to deduce (A.33) from (A.32), and we can assume g satisfies (A.26), so

$$v(x) = \int_{\partial\Omega} F(x, y)w(y)\, dS(y), \quad x \in \Omega, \tag{A.34}$$

where $F(x, y)$ is the Schwartz kernel of the operator Φ in (A.6)–(A.8), and

$$g \in H^{s+3/2}(\partial\Omega, T^*) \Longrightarrow w \in H^{s+1/2}(\partial\Omega, T^*). \tag{A.35}$$

Now $V = dv$ satisfies

$$(\lambda - \Delta)V = 0,$$

$$V(x) = \lim_{x' \to x, x' \in \Omega} \int_{\partial\Omega} d_x F(x', y) w(y)\, dS(y) = \mathcal{G}w(x), \quad x \in \partial\Omega, \tag{A.36}$$

where, parallel to Proposition 11.3 of Chap. 7, we have

$$\mathcal{G} \in OPS^0(\partial\Omega). \tag{A.37}$$

Hence (A.35) implies $\mathcal{G}w \in H^{s+1/2}(\partial\Omega, \Lambda^2 T^*)$. Now standard estimates for the Dirichlet problem (A.36) yield $V \in H^{s+1}(\Omega)$ if $w \in H^{s+1/2}(\partial\Omega)$; hence, if v satisfies (A.32),

$$\Delta v = \delta V \in H^s(\Omega), \quad v\big|_{\partial\Omega} = g, \tag{A.38}$$

and regularity for the Dirichlet problem yields the desired conclusion (A.33). Thus Proposition A.2 is proved.

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18

Einstein's Equations

Introduction

In this chapter we discuss Einstein's gravitational equations, which state that the presence of matter and energy creates curvature in spacetime, via

$$G_{jk} = 8\pi\kappa T_{jk}, \tag{0.1}$$

where $G_{jk} = \mathrm{Ric}_{jk} - (1/2)Sg_{jk}$ is the Einstein tensor, T_{jk} is the stress-energy tensor due to the presence of matter, and κ is a positive constant. In § 1 we introduce this equation and relate it to previous discussions of stress-energy tensors and their relation to equations of motion. We recall various stationary action principles that give rise to equations of motion and show that (0.1) itself results from adding a term proportional to the scalar curvature of spacetime to standard Lagrangians and considering variations of the metric tensor.

In § 2 we consider spherically symmetric spacetimes and derive the solution to the empty-space Einstein equations due to Schwarzschild. This solution provides a model for the gravitational field of a star. After some general comments on stationary and static spacetimes in § 3, we study in § 4 orbits of free particles in Schwarzschild spacetime. Comparison with orbits for the classical Kepler problem enables us to relate the formula for a Schwarzschild metric to the mass of a star.

In § 5 we consider the coupling of Einstein's equations with Maxwell's equations for an electromagnetic field. In § 6 we consider fluid motion and study a relativistic version of the Euler equations for fluids. We look at some steady solutions, and comparison with the Newtonian analogue leads to identification of the constant κ in (0.1) with the gravitational constant of Newtonian theory. In § 7 we consider some special cases of gravitational collapse, showing that in some cases no amount of fluid pressure can prevent such collapse, a phenomenon very different from that predicted by the classical theory.

In § 8 we consider the initial-value problem for Einstein's equations, first in empty space. We discuss two ways of transforming the equations into hyperbolic form: via the use of "harmonic coordinates" (following [CBr2]), and via a

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modification of the equation due to [DeT]. We then consider Einstein's equations in the presence of an electromagnetic field, and in the presence of matter, with emphasis on the initial-value problem for relativistic fluids.

In §§ 9 and 10, we consider an alternative picture of the initial-value problem for Einstein's equations, regarding the initial data as specifying the first and second fundamental forms of a spacelike hypersurface (subject to constraints arising from the Gauss–Codazzi equations) and discussing the solution in terms of the evolution of such hypersurfaces (as "time slices"). Such a picture has been prominent in investigations by physicists for some time (see [MTW]) and has also played a significant role in recent mathematical work, such as in [CBR, CK, CBY2].

1. The gravitational field equations

According to general relativity, the presence of matter in the universe (a four-dimensional spacetime) influences its Lorentz metric tensor, via the equation

$$G^{jk} = 8\pi\kappa T^{jk}, \tag{1.1}$$

where κ is a positive (experimentally determined) constant, G^{jk} is the Einstein tensor, defined in terms of the Ricci tensor by

$$G^{jk} = \mathrm{Ric}^{jk} - \frac{1}{2} S g^{jk}, \tag{1.2}$$

$S = \mathrm{Ric}^j{}_j$ being the scalar curvature, and T^{jk} is the stress-energy tensor due to the presence of matter.

We will review some facts about the stress-energy tensor, introduced in Chap. 2, and show how the stationary action principle–as used in § 11 of Chap. 2 to produce Maxwell's equations for an electromagnetic field, and the Lorentz force law for the influence of this field on charged matter, from a Lagrangian–can be extended to a variational principle that also leads to (1.1). This cannot be regarded as a derivation of (1.1), from more elementary physical principles, but it does provide a context for the equation. We follow the point of view of [Wey].

In the relativistic set-up, as mentioned in § 18 of Chap. 1, one has a four-dimensional manifold M with a Lorentz metric (g_{jk}), which we take to have signature $(-,+,+,+)$. A particle with positive mass moves on a timelike curve in M, that is, one whose tangent Z satisfies $\langle Z, Z\rangle < 0$. One parameterizes such a path by arc length, or "proper time," so that $\langle Z, Z\rangle = -1$. The stress-energy tensor T due to some matter field on M is a symmetric tensor field of type $(0, 2)$ with the property that an observer on such a path (with basically a Newtonian frame of mind) "observes" an energy density equal to $T(Z, Z)$.

For example, consider a diffuse cloud of matter. We will model this as a continuous substance, whose motion is described by a vector field u, satisfying $\langle u, u\rangle = -1$. Suppose this substance has mass density $\mu\ dV$, measured by an observer whose velocity is u. Suppose this matter does not interact with itself; sometimes this is called a "dust." Then an observer measures the mass-energy of the moving matter. The stress-energy tensor is given by

$$\widehat{T} = \mu u \otimes u, \quad \text{i.e., } T^{jk} = \mu u^j u^k, \tag{1.3}$$

where $\widehat{T}$ is the tensor field of type (2,0) obtained from T via the metric, that is, by raising indices.

For the electromagnetic field $\mathcal{F}$, an antisymmetric tensor field of type (0,2) on M, in (11.34) of Chap. 2 we produced the formula

$$T^{jk} = \frac{1}{4\pi}\Bigl(\mathcal{F}^j{}_\ell \mathcal{F}^{k\ell} - \frac{1}{4} g^{jk} \mathcal{F}^{i\ell}\mathcal{F}_{i\ell}\Bigr). \tag{1.4}$$

Also in § 11 of Chap. 2 we considered the equations governing the interaction of the electromagnetic field on a Lorentz manifold with a charged dust cloud, modeled as a charged continuous substance. We produced the Lagrangian

$$L = -\frac{1}{8\pi}\langle \mathcal{F}, \mathcal{F}\rangle + \langle \mathcal{A}, \mathcal{J}\rangle + \frac{1}{2}\mu\langle u, u\rangle = L_1 + L_2 + L_3. \tag{1.5}$$

Here, $\mathcal{F}$, μ, and u are as above. Part of Maxwell's equations assert

$$d\mathcal{F} = 0, \tag{1.6}$$

so, at least locally, we can write $\mathcal{F} = d\mathcal{A}$ for a 1-form $\mathcal{A}$ on M, called the electromagnetic potential. The vector field $\mathcal{J}$ is the current, which has the form $\mathcal{J} = \sigma u$, where $\sigma\ dV$ is the charge density of the substance, measured by an observer with velocity u. We assumed there is only one type of matter present, so σ is a constant multiple of μ. Also we assumed the law of conservation of mass:

$$\operatorname{div}(\mu u) = 0. \tag{1.7}$$

We then examined the action integral

$$I(\mathcal{A}, u) = \int L\, dV \tag{1.8}$$

and showed that, for a compactly supported 1-form β on M,

$$\frac{d}{d\tau} I(\mathcal{A} + \tau\beta, u)\big|_{\tau=0} = \int \Bigl[-\frac{1}{4\pi}\langle \beta, d^\star \mathcal{F}\rangle + \langle \beta, \mathcal{J}\rangle\Bigr]\, dV. \tag{1.9}$$

Thus the condition that $I(\mathcal{A}, u)$ be stationary with respect to variations of $\mathcal{A}$ implies the remaining Maxwell equation:

$$d^{\star}\mathcal{F} = 4\pi \mathcal{J}^b, \tag{1.10}$$

where $\mathcal{J}^b$ is the 1-form obtained from $\mathcal{J}$ by lowering indices. A popular way to write (1.9) is as

$$\delta \int L\, dV = \int \Big[-\frac{1}{4\pi}\mathcal{F}^{jk}{}_{;k} + \mathcal{J}^j\Big]\beta_j\, dV. \tag{1.11}$$

Furthermore, we showed that when the motion of the charged substance was varied, leading to a variation $u(\tau)$ of u, with $w = \partial_\tau u$, compactly supported, then

$$\frac{d}{d\tau} I\big(\mathcal{A}, u(\tau)\big)\big|_{\tau=0} = -\int \big\langle \mu\nabla_u u - \widetilde{\mathcal{F}}\mathcal{J}, w\big\rangle\, dV, \tag{1.12}$$

or equivalently, for the variation of the motion of the charged matter,

$$\delta \int L\, dV = \int \big[\mu u^k u^j{}_{;k} - \mathcal{F}^j{}_k \mathcal{J}^k\big] w_k\, dV. \tag{1.13}$$

Then the condition that $I(\mathcal{A}, u)$ be stationary with respect to variations of u is that

$$\mu\nabla_u u - \widetilde{\mathcal{F}}\mathcal{J} = 0, \tag{1.14}$$

the Lorentz force law.

Having varied $\mathcal{A}$ and u in the action integral, we next vary the metric. We claim that the variation of an action integral of the form (1.8) with respect to the metric is given by

$$\delta \int L\, dV = \frac{1}{2}\int T^{jk}(\delta g_{jk})\, dV, \tag{1.15}$$

where T^{jk} is the stress-energy tensor associated with the Lagrangian L. We look separately at the three terms in (1.5). First, we consider

$$L_3 = \frac{1}{2}\mu\langle u, u\rangle, \quad T_3^{jk} = \mu u^j u^k. \tag{1.16}$$

To examine the variation in $\int L_3\, dV$, it is necessary to recognize that μ depends on the metric, via the identity $\mu\, dV = m\, dy\, ds$, where m is constant. Thus

$$
\begin{aligned}
\delta \int L_3 \, dV &= \delta \int \frac{1}{2} g_{jk} u^j u^k m \, dy \, ds \\
&= \frac{1}{2} \int u^j u^k (\delta g_{jk}) m \, dy \, ds \\
&= \frac{1}{2} \int \mu u^j u^k (\delta g_{jk}) \, dV,
\end{aligned} \tag{1.17}
$$

yielding (1.15) in this case.

Next, consider

$$
L_1 = -\frac{1}{8\pi} \langle \mathcal{F}, \mathcal{F} \rangle, \quad T_1^{jk} = \frac{1}{4\pi} \Big(\mathcal{F}^j{}_\ell \mathcal{F}^{k\ell} - \frac{1}{4} g^{jk} \mathcal{F}^{i\ell} \mathcal{F}_{i\ell} \Big). \tag{1.18}
$$

Now $\langle \mathcal{F}, \mathcal{F} \rangle = (1/2) \mathcal{F}_{jk} \mathcal{F}_{i\ell} g^{ji} g^{k\ell}$, and

$$
\delta \big(\mathcal{F}_{jk} \mathcal{F}_{i\ell} g^{ji} g^{k\ell} \big) = \mathcal{F}_{jk} \mathcal{F}_{i\ell} \big[(\delta g^{ji}) g^{k\ell} + g^{ji} (\delta g^{k\ell}) \big], \tag{1.19}
$$

while

$$
dV = \sqrt{|g|} \, dx \Longrightarrow \delta(dV) = -\frac{1}{2} g_{jk} \, \delta g^{jk} \, dV. \tag{1.20}
$$

Hence

$$
-8\pi \, \delta \int L_1 \, dV = \frac{1}{2} n \int (\delta g^{jk}) \Big[\mathcal{F}_j{}^\ell \mathcal{F}_{k\ell} + \mathcal{F}^\ell{}_k \mathcal{F}_{\ell j} - \frac{1}{2} g_{jk} \mathcal{F}^{i\ell} \mathcal{F}_{i\ell} \Big] \, dV. \tag{1.21}
$$

Using $\delta g^{jk} = -g^{j\ell} (\delta g_{\ell i}) g^{ik}$ and $\mathcal{F}_{jk} = -\mathcal{F}_{kj}$, which implies $\mathcal{F}_j{}^\ell = -\mathcal{F}^\ell{}_j$, we obtain

$$
-8\pi \, \delta \int L_1 \, dV = -\frac{1}{2} \int (\delta g_{jk}) \Big[2 \mathcal{F}^j{}_\ell \mathcal{F}^{k\ell} - \frac{1}{2} g^{jk} \mathcal{F}^{i\ell} \mathcal{F}_{i\ell} \Big] \, dV, \tag{1.22}
$$

which also yields (1.15).

For the middle term in (1.5), namely, $L_2 = \langle \mathcal{A}, \mathcal{J} \rangle = (e/m) \langle \mathcal{A}, u \rangle \mu$, we have

$$
\delta \int \langle \mathcal{A}, \mathcal{J} \rangle \, dV = \delta \int \frac{e}{m} \langle \mathcal{A}, u \rangle \mu \, dV = 0, \tag{1.23}
$$

consistent with the standard choice of stress-energy tensor for the coupled system:

$$
T^{jk} = \mu u^j u^k + \frac{1}{4\pi} \Big(\mathcal{F}^j{}_\ell \mathcal{F}^{k\ell} - \frac{1}{4} g^{jk} \mathcal{F}^{i\ell} \mathcal{F}_{i\ell} \Big). \tag{1.24}
$$

As noted in Chap. 2, § 11, if the stationary conditions (1.10) and (1.14) are satisfied, and (1.6)–(1.7) hold, then this tensor has zero divergence (i.e., $T^{jk}{}_{;k} = 0$).

Now Einstein hypothesized that "gravity" is a purely geometrical effect. Independently, both Einstein and Hilbert hypothesized that it could be captured by adding a fourth term, L_4, to the Lagrangian (1.5). The term L_4 should depend only on the metric tensor on M, not on the electromagnetic or matter fields (or any other field). It should be "natural." The most natural scalar field to take is one proportional to the scalar curvature:

$$L_4 = \alpha S, \tag{1.25}$$

where α is a real constant.

We are hence led to calculate the variation of the integral of scalar curvature, with respect to the metric:

Theorem 1.1. *If M is a manifold with nondegenerate metric tensor (g_{jk}), associated Einstein tensor $G_{jk} = Ric_{jk} - (1/2)Sg_{jk}$, scalar curvature S, and volume element dV, then, with respect to a compactly supported variation of the metric, we have*

$$\delta \int S\, dV = \int G_{jk}\, \delta g^{jk}\, dV = -\int G^{jk}\, \delta g_{jk}\, dV. \tag{1.26}$$

To establish this, we first obtain formulas for the variation of the Riemann curvature tensor, then of the Ricci tensor and the scalar curvature. Let $\Gamma^i{}_{jk}$ be the connection coefficients. Then $\delta\Gamma^i{}_{jk}$ is a tensor field. The formula (3.54) of Appendix C states that if $\widetilde{R}$ and R are the curvatures of the connections $\widetilde{\nabla}$ and $\nabla = \widetilde{\nabla} + \varepsilon C$, then

$$(R - \widetilde{R})(X, Y)u = \varepsilon(\widetilde{\nabla}_X C)(Y, u) - \varepsilon(\widetilde{\nabla}_Y C)(X, u) + \varepsilon^2[C_X, C_Y]u. \tag{1.27}$$

It follows that

$$\delta R^i{}_{jk\ell} = \delta\Gamma^i{}_{j\ell;k} - \delta\Gamma^i{}_{jk;\ell}. \tag{1.28}$$

Contracting, we obtain

$$\delta\, \mathrm{Ric}_{jk} = \delta\Gamma^i{}_{ji;k} - \delta\Gamma^i{}_{jk;i}. \tag{1.29}$$

Another contraction yields

$$g^{jk}\, \delta\, \mathrm{Ric}_{jk} = \big(g^{jk}\, \delta\Gamma^\ell{}_{j\ell}\big)_{;k} - \big(g^{jk}\, \delta\Gamma^\ell{}_{jk}\big)_{;\ell} \tag{1.30}$$

since the metric tensor has vanishing covariant derivative. The identities (1.28)–(1.30) are called "Palatini identities."

Note that the right side of (1.30) is the divergence of a vector field. This will be significant for our calculation of (1.26). By the divergence theorem, it implies that

$$\int g^{jk}\big(\delta \operatorname{Ric}_{jk}\big)\, dV = 0, \tag{1.31}$$

as long as δg_{jk} (hence $\delta \operatorname{Ric}_{jk}$) is compactly supported.

We now compute the left side of (1.26). Since $S = g^{jk} \operatorname{Ric}_{jk}$, we have

$$\delta S = \operatorname{Ric}_{jk} \delta g^{jk} + g^{jk}\, \delta \operatorname{Ric}_{jk}. \tag{1.32}$$

Thus, since $\delta(dV)$ is given by (1.20), we have

$$\begin{aligned} \delta(S\, dV) &= \operatorname{Ric}_{jk}\, \delta g^{jk}\, dV + g^{jk}(\delta \operatorname{Ric}_{jk})\, dV + S\, \delta(dV) \\ &= \big(\operatorname{Ric}_{jk} - \frac{1}{2} S g_{jk}\big)\delta g^{jk}\, dV + g^{jk}(\delta \operatorname{Ric}_{jk})\, dV. \end{aligned} \tag{1.33}$$

The last term integrates to zero, by (1.31), so we have (1.26).

For some purposes it is useful to consider analogues of (1.26), involving variations of the metric which do not have compact support (see, e.g., [Yo4] and [Yo5]).

Note that verifying (1.26) did not require the computation of $\delta\Gamma^i{}_{jk}$ in terms of δg_{jk}, though this can be done explicitly. Indeed, formula (3.63) of Appendix C implies

$$\delta\Gamma_{\ell jk} = \frac{1}{2}\Big[\delta g_{\ell j;k} - \delta g_{\ell k;j} + \delta g_{jk;\ell}\Big]. \tag{1.34}$$

From Theorem 1.1 together with (1.15), we see that if L is a matter Lagrangian, such as in (1.5), then the stationary condition for

$$\delta \int \Big(\frac{1}{2} S + 8\pi\kappa L\Big)\, dV, \tag{1.35}$$

with respect to variations of the metric tensor, yields the gravitational equation (1.1).

An alternative formulation of (1.1) is the following. Take the trace of both sides of (1.1). We have

$$G^j{}_j = \operatorname{Ric}^j{}_j - \frac{1}{2} S g^j{}_j = \Big(1 - \frac{n}{2}\Big) S \tag{1.36}$$

when $n = \dim M$. Since $n = 4$ here, this implies

$$-S = 8\pi\kappa\tau, \quad \tau = T^j{}_j. \tag{1.37}$$

Then substitution of $-8\pi\kappa\tau$ for S in (1.1) yields

$$\operatorname{Ric}^{jk} = 8\pi\kappa\Big(T^{jk} - \frac{1}{2}\tau g^{jk}\Big). \tag{1.38}$$

We now derive a geometrical interpretation of the Einstein tensor. Let $\xi = e_0$ be any unit timelike vector in T_pM, part of an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of T_pM, where $\langle e_j, e_j\rangle = +1$ for $j \ge 1$, -1 for $j = 0$. From the definiton of the Ricci tensor, we obtain

$$\operatorname{Ric}(\xi,\xi) = \sum_{j=1}^{3} \big\langle R(e_j,\xi)\xi, e_j\big\rangle = -\sum_{j=1}^{3} K(\xi \wedge e_j), \tag{1.39}$$

where $K(\xi \wedge e_j)$ denotes the sectional curvature with respect to the 2-plane in T_pM spanned by ξ and e_j. Compare with Proposition 4.7 of Appendix C, but note the sign change due to the different signature of the metric here. Also, the scalar curvature of M at p is given by

$$S = \sum_{j\neq k} K(e_j \wedge e_k) \qquad (0 \le j,k \le 3). \tag{1.40}$$

Hence, for $G = \operatorname{Ric} - (1/2)Sg$, we have

$$G(\xi,\xi) = \operatorname{Ric}(\xi,\xi) + \frac{1}{2}S = \sum_{1\le j<k} K(e_j \wedge e_k). \tag{1.41}$$

Now let V_ξ be the spacelike hypersurface, formed by the geodesics through p normal to ξ. The second fundamental form of V_ξ vanishes at p (see the proof of Proposition 4.7 in Appendix C, analyzing the sectional curvature). It follows that the scalar curvature of V_ξ at p is given by

$$S(V_\xi) = 2 \sum_{1\le j<k} K(e_j \wedge e_k), \tag{1.42}$$

with $K(e_j \wedge e_k)$ as in (1.41). Hence

$$S(V_\xi) = 2G(\xi,\xi). \tag{1.43}$$

Thus the gravitational equation (1.1) can be written as

$$S(V_\xi) = 16\pi\kappa\rho, \tag{1.44}$$

where

$$\rho = T(\xi,\xi) \tag{1.45}$$

is the energy density, measured by an observer with 4-velocity ξ. Note that if T is given by (1.3), then $T(\xi,\xi) = \mu\langle u,\xi\rangle^2$, which is nonnegative (in fact, positive where $\mu \neq 0$). Also, if T is the stress-energy tensor (1.4) of the electromagnetic field, then, as observed in calculations leading to (11.33) in Chap. 2, $T(\xi, \xi)$ is the observed measurement of $(1/8\pi)\big(|E|^2 + |B|^2\big)$, also nonnegative (and positive where the electromagnetic field does not vanish). Typically, a stress-energy tensor for ordinary matter has the property that

$$T(\xi, \xi) \geq 0, \quad \text{for } \xi \text{ timelike.} \tag{1.46}$$

If $\rho > 0$ at p, we see that $S(V_\xi)$ has the same sign as the constant κ. Below we will argue that κ is *positive*.

Note that the equation (1.14) indicates that uncharged matter should move along geodesics. Let us consider the influence of the geometry on the relative motion of nearby neutral particles, whose motion is along nearby timelike geodesics in M. Say one geodesic $\gamma_0(s)$ has unit timelike tangent vector $\xi = \gamma_0'(s)$; $\langle \xi, \xi \rangle = -1$. If there is a one-parameter family of geodesics $\gamma_\tau(s)$, then $W(s) = \partial_\tau \gamma_\tau(s)\big|_{\tau=0}$ is a vector field along γ_0 that satisfies the Jacobi equation

$$\nabla_\xi \nabla_\xi W = R(\xi, W)\xi. \tag{1.47}$$

See Exercise 10 in § 3 of Appendix C.

Let us vary the geodesic γ_0 in the following specific fashion. Let V_ξ be the hypersurface described above, spanned by geodesics through $p = \gamma_0(0)$ with tangent vector in $(\xi)^\perp \subset T_pM$. We extend ξ over V_ξ by radial parallel transport and, given $q \in V_\xi$, close to p, consider the geodesic γ satisfying $\gamma(0) = q$, $\gamma'(0) = \xi$ (see Fig. 1.1). If γ_τ is a one-parameter family of such geodesics, with $W(s) = \partial_\tau \gamma_\tau(s)\big|_{\tau=0}$, then

$$W(0) \perp \xi \in T_pM \quad \text{and } \nabla_\xi W(0) = 0. \tag{1.48}$$

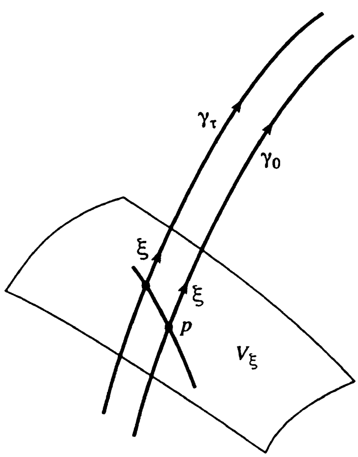

FIGURE 1.1 Nearby Timelike Geodesics

Note that $(d/ds)\langle W, \xi \rangle = \langle \nabla_\xi W, \xi \rangle$ and

$$\frac{d^2}{ds^2}\langle W, \xi \rangle = \langle \nabla_\xi^2 W, \xi \rangle = \big\langle R(\xi, W)\xi, \xi \big\rangle = 0.$$

Hence, if (1.48) holds, then $W(s) \perp \xi(s)$ all along γ_0.

Now we have $(d/ds)\langle W, W \rangle = 2\langle \nabla_\xi W, W \rangle$, and

$$\frac{d^2}{ds^2}\langle W, W \rangle = 2\frac{d}{ds}\langle \nabla_\xi W, W \rangle = 2\langle \nabla_\xi^2 W, W \rangle + 2\langle \nabla_\xi W, \nabla_\xi W \rangle. \tag{1.49}$$

Hence, at p,

$$\frac{d^2}{ds^2}\langle W, W \rangle = -2\langle R(W, \xi)\xi, W \rangle. \tag{1.50}$$

If we let W_j be an orthonormal basis of $T_p(V_\xi)$, we obtain, via (1.39),

$$\frac{d^2}{ds^2}\sum_{j=1}^{3}\langle W_j, W_j \rangle = -2\,\mathrm{Ric}(\xi, \xi) = -16\pi\kappa\Big[T(\xi, \xi) + \frac{1}{2}\tau\Big], \tag{1.51}$$

at p.

Note that if T is given by (1.3), then

$$T(\xi, \xi) + \frac{1}{2}\tau = \mu\langle \xi, u \rangle^2 + \frac{1}{2}\mu\langle u, u \rangle \geq 0, \tag{1.52}$$

when ξ and u are both unit timelike vectors. In particular, $T(u, u) + \tau/2 = \mu/2$, in this case. Generally, a stress-energy tensor T is said to satisfy the "strong energy condition" if $T(\xi, \xi) + \tau/2 \geq 0$ for all unit timelike ξ. For such stress-energy tensors, we have (at p)

$$\begin{aligned} \frac{d^2}{ds^2}\sum_{j=1}^{3}\langle W_j, W_j \rangle &\leq 0 \quad \text{if } \kappa > 0, \\ &= 0 \quad \text{if } \kappa = 0, \\ &\geq 0 \quad \text{if } \kappa < 0. \end{aligned} \tag{1.53}$$

Now, it is clear that an attractive gravitational force would make (1.53) ≤ 0 (and in fact < 0 in a nontrivial matter field), while a repulsive force would make (1.53) ≥ 0. Since we observe gravity to be attractive, we conclude that the constant κ in (1.1) is > 0. Further discussion of the determination of κ will be given in § 6, after (6.73)–(6.74).

Exercises

1. If M is a Riemannian manifold of dimension 2, show that $G_{jk} = 0$. Deduce from Theorem 1.1 that

$$\int_M K\,dA = C(M) \tag{1.54}$$

is independent of the metric on M. Relate this to the Gauss–Bonnet formula established in § 5 of Appendix C, on connections and curvature.

2. As shown in (3.31) of Appendix C, the Einstein tensor satisfies

$$G^{jk}{}_{;k} = 0, \tag{1.55}$$

as a consequence of the Bianchi identity. Hence, Einstein's equations (1.1) imply

$$T^{jk}{}_{;k} = 0. \tag{1.56}$$

Compute this when T^{jk} is given by (1.24). Compare with the calculation (11.54) of Chap. 2.

3. A fluid, with 4-velocity field u (satisfying $\langle u, u\rangle = -1$), density ρ, and pressure p (measured by an observer with velocity u), has stress-energy tensor

$$T_{jk} = (\rho + p)u_j u_k + p g_{jk}. \tag{1.57}$$

Thus a dust, with the stress-energy tensor (1.3), is a zero-pressure fluid. Compute $T^{jk}{}_{;k}$ in this case, and show that the conservation law (1.7) and the geodesic equation $\nabla_u u = 0$ (which is (1.14) in the absence of an electromagnetic field) are modified to

$$\operatorname{div}(\rho u) = -p \operatorname{div} u, \quad (\rho + p)\nabla_u u = -\Pi(u) \operatorname{grad} p, \tag{1.58}$$

where $\Pi(u)$ denotes projection orthogonal to u (with respect to the Lorentz metric), namely, in components, $(\Pi(u) \operatorname{grad} p)^j = p_{;k}(g^{jk} + u^j u^k)$.

4. Recall from (1.37) that when (1.1) holds, the scalar curvature of spacetime is given by $S = -8\pi\kappa T^j{}_j$. Deduce that if T^{jk} is given by (1.24), then

$$S = 8\pi\kappa\mu, \tag{1.59}$$

while if it is given by (1.57), then

$$S = 8\pi\kappa(\rho - 3p). \tag{1.60}$$

5. Show that if T^{jk} is given by (1.3), then

$$\operatorname{Ric}(u, u) = 4\pi\kappa\mu, \tag{1.61}$$

and if it is given by (1.24), then

$$\operatorname{Ric}(u, u) = 4\pi\kappa\mu + \kappa(|E|^2 + |B|^2), \tag{1.62}$$

where E and B are the electric and magnetic fields, measured by an observer with 4-velocity u, while if it is given by (1.57), then

$$\operatorname{Ric}(u,u) = 4\pi\kappa(\rho + 3p). \tag{1.63}$$

2. Spherically symmetric spacetimes and the Schwarzschild solution

We investigate solutions to Einstein's equations (1.1) which are *spherically symmetric*. Generally, a Lorentz 4-manifold (M, g) is said to be spherically symmetric provided there is an effective action of SO(3) as a group of isometries of M. The generic orbit $\mathcal{O}$ will be diffeomorphic to S^2. We will assume that $\mathcal{O}$ is spacelike, that is, the metric induced on $\mathcal{O}$ is positive-definite. Given $p \in \mathcal{O}$, let K_p be the subgroup of SO(3) fixing p; K_p is a circle group. Thus K_p acts as a group of rotations on $T_p\mathcal{O}$, and it also acts on $N_p\mathcal{O} = T_p\mathcal{O}^\perp$. Since $N_p\mathcal{O}$ has a metric of signature $(-1, 1)$, and K_p acts on it as a compact, connected group of isometries, it follows that K_p acts trivially on $N_p\mathcal{O}$.

On a neighborhood of $\mathcal{O} \subset M$, diffeomorphic to $(a, b) \times (c, d) \times S^2$, we can introduce coordinates so that the metric is

$$ds^2 = -C(r,t)\, dt^2 + D(r,t)\, dr^2 + 2E(r,t)\, dr\, dt + F(r,t)\, d\omega^2, \tag{2.1}$$

Where the functions C, D, E, and F are smooth and positive, and $d\omega^2$ is the standard Riemannian metric on the unit sphere, $S^2 \subset \mathbb{R}^3$. Assume that $\partial F/\partial r \neq 0$ on $\mathcal{O}$. Then we can change variables, replacing r by $r' = \sqrt{F(r,t)}$, and get the simpler form

$$ds^2 = -C(r,t)\, dt^2 + D(r,t)\, dr^2 + 2E(r,t)\, dr\, dt + r^2\, d\omega^2, \tag{2.2}$$

with new functions $C(r,t)$, and so on. Next, we can replace t by t', such that

$$dt' = \eta(r,t)\big[C(r,t)\, dt - E(r,t)\, dr\big], \tag{2.3}$$

where $\eta(r,t)$ is an integrating factor, chosen to be positive and to make the right side of (2.3) a closed form. Then the metric on M takes the form

$$ds^2 = -e^{\nu(r,t)}\, dt^2 + e^{\lambda(r,t)}\, dr^2 + r^2\, d\omega^2. \tag{2.4}$$

We take spherical coordinates (φ, θ) on S^2, where $\varphi = 0$ defines the north pole and $\varphi = \pi/2$ defines the equator. (Physics texts often give φ and θ the opposite roles.) Then

$$d\omega^2 = d\varphi^2 + \sin^2\varphi\, d\theta^2.$$

The formula for the Einstein tensor G_{jk} for such a metric is fairly complicated. Rather than just write it down, we will take a leisurely path through the calculation, making some general observations about the Einstein tensor, and other measures of curvature, along the way. Some of these calculations will have further uses in subsequent sections. Among alternative derivations of the formula for the Ricci tensor for a metric of the form (2.4), we mention one using differential forms, on pp. 87–90 of [HT].

The metric (2.4) has the general form

$$g_{jk} = g^U_{jk} + \psi g^S_{jk}, \quad \psi \in C^\infty(U), \tag{2.5}$$

on a product $M = U \times S$, where g^U is the metric tensor of a manifold U, g^S is the metric tensor of S. To be more precise, if

$$(x', x'') = (x_0, \dots, x_{L-1}, x_L, \dots, x_{L+M-1}) \in U \times S,$$

g^U_{jk} is the metric tensor for U if $0 \le j, k \le L-1$, and we fill in g^U_{jk} to be zero for other indices. Similarly, we set $g^S_{jk} = h_{j+L,k+L}$ for $0 \le j, k \le M-1$, where h_{jk} is the metric tensor for S, and we fill in g^S_{jk} to be zero for other indices. In the example (2.4), we have $U \subset \mathbb{R}^2$, $S = S^2$, so $L = M = 2$. With obvious notation,

$$g^{jk} = g^{jk}_U + \psi^{-1} g^{jk}_S. \tag{2.6}$$

We want to express the curvature tensor $R^j{}_{k\ell m}$ of M in terms of the tensors ${}^U R^j{}_{k\ell m}$ and ${}^S R^j{}_{k\ell m}$ and the function ψ and then obtain formulas for the Ricci tensor, scalar curvature, and Einstein tensor of M. Recall that if $\Gamma^j{}_{k\ell}$ are the connection coefficients on M, then

$$R^j{}_{k\ell m} = \partial_\ell \Gamma^j{}_{km} - \partial_m \Gamma^j{}_{k\ell} + \Gamma^j{}_{\nu\ell}\Gamma^\nu{}_{km} - \Gamma^j{}_{\nu m}\Gamma^\nu{}_{k\ell}, \tag{2.7}$$

where we use the summation convention (sum on ν). Meanwhile,

$$\Gamma^\ell{}_{jk} = \frac{1}{2} g^{\ell\mu}\big[\partial_k g_{j\mu} + \partial_j g_{k\mu} - \partial_\mu g_{jk}\big]. \tag{2.8}$$

Using (2.5) and (2.6), we can first express $\Gamma^\ell{}_{jk}$ in terms of the connection coefficients on the factors U and S:

$$\Gamma^\ell{}_{jk} = {}^U\Gamma^\ell{}_{jk} + {}^S\Gamma^\ell{}_{jk} + B^\ell{}_{jk}, \tag{2.9}$$

where

$$B^\ell{}_{jk} = \frac{1}{2} g^{\ell\mu} \big[g^S_{j\mu}\, \partial_k \psi + g^S_{k\mu}\, \partial_j \psi - g^S_{jk}\, \partial_\mu \psi \big] \tag{2.10}$$
$$= \frac{1}{2} \big[\sigma^\ell{}_j\, \partial_k \vartheta + \sigma^\ell{}_k\, \partial_j \vartheta - g^S_{jk}\, \partial^\ell \psi \big].$$

Here we have set

$$\vartheta = \log \psi, \tag{2.11}$$

and, in a product coordinate system such as described above, $\sigma^\ell{}_j = 1$ if $j = \ell$ is an index for S, 0 otherwise. We can write

$$\sigma^\ell{}_j = e(\ell) \delta^\ell{}_j, \tag{2.12}$$

where $e(\ell) = 1$ if $\ell \geq L$, $e(\ell) = 0$ if $\ell < L$. Note that $\sigma^\ell{}_j$ produces a well-defined tensor field of type $(1, 1)$ on $M = U \times S$, namely $T_x M \approx T_{x'} U \oplus T_{x''} S$, and $\sigma(x)$ is the projection of $T_x M$ onto $T_{x''} S$, annihilating $T_{x'} U$.

Now, given $p = (p', p'') \in M = U \times S$, let us use product-exponential coordinates centered at p, namely, the product of an exponential coordinate system on U centered at p' and an exponential coordinate system on S centered at p''. In such a coordinate system, we have

$$R^j{}_{k\ell m} = {}^U R^j{}_{k\ell m} + {}^S R^j{}_{k\ell m} + D^j{}_{k\ell m}, \tag{2.13}$$

with

$$D^j{}_{k\ell m} = \partial_\ell B^j{}_{km} - \partial_m B^j{}_{k\ell} + B^j{}_{\nu\ell} B^\nu{}_{km} - B^j{}_{\nu m} B^\nu{}_{k\ell}, \quad \text{at } p. \tag{2.14}$$

The formula (2.10) for $B^\ell{}_{jk}$ is valid in any coordinate system. In product-exponential coordinates we have

$$\partial_\ell B^j{}_{km} = \frac{1}{2} \big[\sigma^j{}_k\, \partial_\ell \partial_m \vartheta + \sigma^j{}_m\, \partial_\ell \partial_k \vartheta - g^S_{km}\, \partial_\ell \partial^j \psi \big], \tag{2.15}$$

at p. Also,

$$\partial_m B^j{}_{k\ell} = \frac{1}{2} \big[\sigma^j{}_k\, \partial_m \partial_\ell \vartheta + \sigma^j{}_\ell\, \partial_m \partial_k \vartheta - g^S_{k\ell}\, \partial_m \partial^j \psi \big],$$

at p, so

$$\partial_\ell B^j{}_{km} - \partial_m B^j{}_{k\ell} \tag{2.16}$$
$$= \frac{1}{2} \big[\sigma^j{}_m\, \partial_\ell \partial_k \vartheta - \sigma^j{}_\ell\, \partial_m \partial_k \vartheta - g^S_{km}\, \partial_\ell \partial^j \psi + g^S_{k\ell}\, \partial_m \partial^j \psi \big],$$

at p. From (2.10) we have

$$4B^j{}_{\nu\ell}B^\nu{}_{km} = \sigma^j{}_k\,\partial_\ell\vartheta\,\partial_m\vartheta + \sigma^j{}_m\,\partial_\ell\vartheta\,\partial_k\vartheta - g^S_{k\ell}\,\partial^j\psi\,\partial_m\vartheta - g^S_{\ell m}\,\partial^j\psi\,\partial_k\vartheta - \sigma^j{}_\ell\langle d\vartheta, d\psi\rangle g^S_{km}, \tag{2.17}$$

where

$$\langle d\vartheta, d\psi\rangle = \partial_\nu\vartheta\,\partial^\nu\psi. \tag{2.18}$$

Antisymmetrizing (2.17) with respect to ℓ and m, we have

$$\begin{aligned} 4\big(B^j{}_{\nu\ell}B^\nu{}_{km} - B^j{}_{\nu m}B^\nu{}_{k\ell}\big) = {} & \big(\sigma^j{}_m\,\partial_\ell\vartheta - \sigma^j{}_\ell\,\partial_m\vartheta\big)\,\partial_k\vartheta \\ & + \big(g^S_{km}\,\partial_\ell\vartheta - g^S_{k\ell}\,\partial_m\vartheta\big)\,\partial^j\psi \\ & + \big(\sigma^j{}_m g^S_{k\ell} - \sigma^j{}_\ell g^S_{km}\big)\langle d\vartheta, d\psi\rangle. \end{aligned} \tag{2.19}$$

We can produce a formula for $D^j{}_{k\ell m}$ where each term is manifestly a tensor. To get this, first note that the tensor whose components in a product-exponential coordinate system are $\partial_\ell\partial_k\vartheta$ is $\vartheta_{;\ell;k} - (1/2)\langle d\psi, d\vartheta\rangle g^S_{\ell k}$. In fact, in any product coordinate system,

$$\begin{aligned} \vartheta_{;\ell;k} &= \partial_\ell\partial_k\vartheta - \Gamma^\nu{}_{\ell k}\,\partial_\nu\vartheta \\ &= \partial_\ell\partial_k\vartheta - {}^U\Gamma^\nu{}_{\ell k}\,\partial_\nu\vartheta - B^\nu{}_{\ell k}\,\partial_\nu\vartheta \\ &= \partial_\ell\partial_k\vartheta + \frac{1}{2}\langle d\psi, d\vartheta\rangle g^S_{\ell k} - {}^U\Gamma^\nu{}_{\ell k}\,\partial_\nu\vartheta. \end{aligned} \tag{2.20}$$

The last identity follows from (2.10) plus the fact that we are summing only over $\nu < L$. Then, from (2.16) and (2.19) we obtain

$$\begin{aligned} D^j{}_{k\ell m} = {} & \frac{1}{2}\Big[\sigma^j{}_m\vartheta_{;k;\ell} - \sigma^j{}_\ell\vartheta_{;k;m} - g^S_{km}\psi^{;j}{}_{;\ell} + g^S_{k\ell}\psi^{;j}{}_{;m}\Big] \\ & + \frac{1}{4}\Big[\sigma^j{}_m\vartheta_{;k}\vartheta_{;\ell} - \sigma^j{}_\ell\vartheta_{;k}\vartheta_{;m} + g^S_{km}\psi^{;j}\vartheta_{;\ell} - g^S_{k\ell}\psi^{;j}\vartheta_{;m}\Big], \end{aligned} \tag{2.21}$$

in any product coordinate system.

Contracting (2.13), we have

$$\mathrm{Ric}_{km} = \mathrm{Ric}^U_{km} + \mathrm{Ric}^S_{km} + F_{km}, \quad F_{km} = D^j{}_{kjm}. \tag{2.22}$$

Thus, in product-exponential coordinates centered at $p \in M$, we have

$$F_{km} = \partial_j B^j{}_{km} - \partial_m B^j{}_{kj} + B^j{}_{\nu j}B^\nu{}_{km} - B^j{}_{\nu m}B^\nu{}_{kj}, \tag{2.23}$$

at p. We evaluate this more explicitly, using (2.16) and (2.19).

Contracting (2.16) over $j = \ell$, we have

$$
\begin{aligned}
&\partial_j B^j{}_{km} - \partial_m B^j{}_{kj} \\
&= \frac{1}{2}\big[\sigma^j{}_m\, \partial_j \partial_k \vartheta - \sigma^j{}_j\, \partial_m \partial_k \vartheta - g^S_{km}\, \partial_j \partial^j \psi + g^S_{kj}\, \partial_m \partial^j \psi\big] \\
&= -\frac{1}{2} M\, \partial_m \partial_k \vartheta - \frac{1}{2} g^S_{km} \mathcal{L}\psi,
\end{aligned} \tag{2.24}
$$

at p, where $M = \dim S$ and $\mathcal{L}\psi = \partial_j \partial^j \psi$. Since $\psi \in C^\infty(U)$, we have

$$
\mathcal{L}\psi = \sum_{j \le L-1} \partial_j \partial^j \psi = g^{km}_U\, \partial_k \partial_m \psi = \Box_U \psi, \tag{2.25}
$$

at p. Note that $\Box_U$ is the Laplace operator on U. (For our case of primary interest, U has a Lorentz metric, so we use $\Box_U$ rather than Δ_U.) Contracting (2.19) over $j = \ell$, we have

$$
\begin{aligned}
&B^j{}_{\nu j} B^\nu{}_{km} - B^j{}_{\nu m} B^\nu{}_{kj} \\
&\qquad = -\frac{1}{4} M (\partial_m \vartheta)(\partial_k \vartheta) - \frac{1}{4}(M-2)\langle d\vartheta, d\psi\rangle g^S_{km}.
\end{aligned} \tag{2.26}
$$

Thus we obtain, at p,

$$
\begin{aligned}
F_{km} = &-\frac{1}{2} M\, \partial_m \partial_k \vartheta - \frac{1}{2}(\Box_U \psi) g^S_{km} - \frac{1}{4} M (\partial_m \vartheta)(\partial_k \vartheta) \\
&- \frac{1}{4}(M-2)\langle d\vartheta, d\psi\rangle g^S_{km}.
\end{aligned} \tag{2.27}
$$

To write this in tensor form, we recall the computation (2.20). Hence, in any product coordinate system,

$$
F_{km} = -\frac{1}{2} M \big(\vartheta_{;k;m} + \frac{1}{2}\vartheta_{;k}\vartheta_{;m}\big) - \frac{1}{2}\big(\Box_U \psi - \langle d\vartheta, d\psi\rangle\big) g^S_{km}. \tag{2.28}
$$

The scalar curvature of M is $S = g^{km}\, \mathrm{Ric}_{km}$. By (2.22), we have

$$
S = S_U + \psi^{-1} S_S + \beta, \qquad \beta = g^{km} F_{km}. \tag{2.29}
$$

The formula (2.27) yields

$$
\begin{aligned}
\beta = &-\frac{1}{2} M g^{km}_U\, \partial_m \partial_k \vartheta - \frac{1}{2} M \psi^{-1} \Box_U \psi \\
&- \frac{1}{4} M g^{km}_U (\partial_m \vartheta)(\partial_k \vartheta) - \frac{1}{4}(M-2) M \psi^{-1}\langle d\vartheta, d\psi\rangle,
\end{aligned} \tag{2.30}
$$

at p. Note that

$$g_U^{km}\,\partial_m\partial_k\vartheta = g_U^{km}\partial_m\bigl(\psi^{-1}\partial_k\psi\bigr) = \psi^{-1}\Box_U\psi - \psi^{-2}g_U^{km}(\partial_m\psi)(\partial_k\psi), \tag{2.31}$$

at p. Hence

$$\beta = -M\psi^{-1}\Box_U\psi - \frac{1}{4}(M^2-3M)\psi^{-2}\langle d\psi, d\psi\rangle. \tag{2.32}$$

As a check on this calculation, consider the simple case $\dim U = \dim S = 1$, with

$$g^U = dx_0^2, \quad g^S = dx_1^2, \quad g = dx_0^2 + \psi(x_0)\,dx_1^2. \tag{2.33}$$

Of course, $S_U = S_S = 0$ here, and (2.29) and (2.32) yield for the scalar curvature of M,

$$S = -\psi^{-1}(x_0)\psi''(x_0) + \frac{1}{2}\psi^{-2}(x_0)\psi'(x_0)^2. \tag{2.34}$$

Since $S = 2K$, K being the Gauss curvature of the two-dimensional surface M, this formula agrees with the $E = 1$, $G = \psi$ case of formula (3.37) in Appendix C, for the Gauss curvature of a surface with metric $E\,du^2 + G\,dv^2$.

We now look at the Einstein tensor of M, $G_{jk} = \mathrm{Ric}_{jk} - (1/2)Sg_{jk}$. In view of (2.22) and (2.29), we have

$$\begin{aligned} G_{jk} = {}&\mathrm{Ric}^U_{jk} + \mathrm{Ric}^S_{jk} + F_{jk} - \frac{1}{2}S_U(g^U_{jk} + \psi g^S_{jk}) \\ &- \frac{1}{2}\psi^{-1}S_S(g^U_{jk} + \psi g^S_{jk}) - \frac{1}{2}\beta g_{jk}. \end{aligned} \tag{2.35}$$

Rearranging terms, we can write

$$G_{jk} = G^U_{jk} + G^S_{jk} - \frac{1}{2}(S_U\psi g^S_{jk} + S_S\psi^{-1}g^U_{jk}) + F_{jk} - \frac{1}{2}\beta g_{jk}. \tag{2.36}$$

Before considering the case $\dim S = 2$, let us first consider the case $\dim U = \dim S = 1$. Then U and S are flat, and (2.36) becomes $G_{jk} = F_{jk} - (1/2)\beta g_{jk}$. Let us parameterize U and S by arc length. The case $M = 1$ of (2.27) is

$$F_{jk} = -\frac{1}{2}\partial_j\partial_k\vartheta - \frac{1}{2}(\Box_U\psi)g^S_{jk} - \frac{1}{4}(\partial_j\vartheta)(\partial_k\vartheta) + \frac{1}{4}\psi^{-1}\langle d\psi, d\psi\rangle g^S_{jk}.$$

Here, $\Box_U\psi = \psi''(x_0)$, $\partial_j\partial_k\vartheta = \vartheta''(x_0)$ for $j = k = 0$, 0 otherwise, and $\partial_j\vartheta = \vartheta'$ for $j = 0$, 0 otherwise. Also, in this case (2.32) (or (2.34)) implies $\beta = -\psi^{-1}\psi'' + (1/2)\psi^{-2}(\psi')^2$. It readily follows that $F_{jk} = (1/2)\beta g_{jk}$, hence $G_{jk} = 0$. This is part of a more general result.

Lemma 2.1. *If M is a two-dimensional manifold with a nondegenerate metric tensor, then its Einstein tensor always vanishes, $G_{jk} = 0$.*

Proof. Generally, $\mathrm{Ric}^k{}_m$ is produced from $R^{jk}{}_{\ell m}$ via a natural map

$$\kappa : \mathrm{End}(\Lambda^2 T_p M) \longrightarrow \mathrm{End}(T_p M). \tag{2.37}$$

Since $\delta^{jk}{}_{jm} = (n-1)\delta^k{}_m$, we see that $\kappa(I) = (n-1)I$, when $n = \dim M$. Now, if $\dim M = 2$, then $\Lambda^2 T_p M$ is one-dimensional. Hence $\mathrm{Ric}^k{}_m$ must be a scalar multiple of $\delta^k{}_m$, so Ric_{km} must be a scalar multiple of g_{km}. Comparing traces, we see that the multiple must be $S/2$, so

$$\mathrm{Ric}_{jk} = \frac{1}{2} S g_{jk} \quad \text{when } \dim M = 2.$$

This precisely says that $G_{jk} = 0$. Compare the derivation of (3.35) in Appendix C.

Let us now consider the case $\dim S = 2$. From (2.28), we have

$$F_{jk} = -\vartheta_{;j;k} - \frac{1}{2}(\Box_U \psi) g^S_{jk} + \frac{1}{2}\langle d\psi, d\vartheta\rangle g^S_{jk} - \frac{1}{2}\vartheta_{;j}\vartheta_{;k} \tag{2.38}$$

in any product coordinate system. Contracting this, or alternatively taking the $M = 2$ case of (2.32), we have

$$\beta = -2\psi^{-1}\Box_U \psi + \frac{1}{2}\psi^{-2}\langle d\psi, d\psi\rangle. \tag{2.39}$$

When $\dim S = 2$, we have from Lemma 2.1 that $G^S_{jk} = 0$.

If also $\dim U = 2$, then $G^U_{jk} = 0$, so (2.36) yields

$$G_{jk} = -\frac{1}{2}\big(S_U \psi g^S_{jk} + S_S \psi^{-1} g^U_{jk}\big) + F_{jk} - \frac{1}{2}\beta g_{jk}, \tag{2.40}$$

with F_{jk} and β given by (2.38) and (2.39).

Now, whenever a two-dimensional surface U has a metric tensor of the form

$$\gamma_0\, dx_0^2 + \gamma_1\, dx_1^2, \tag{2.41}$$

with $\gamma_j = \gamma_j(x_0, x_1)$, we readily obtain from (2.8) the formulas for the connection coefficients:

$$\left({}^U\Gamma^0{}_{jk}\right) = \frac{1}{2\gamma_0}\begin{pmatrix} \partial_0\gamma_0 & \partial_1\gamma_0 \\ \partial_1\gamma_0 & -\partial_0\gamma_1 \end{pmatrix}, \quad \left({}^U\Gamma^1{}_{jk}\right) = \frac{1}{2\gamma_1}\begin{pmatrix} -\partial_1\gamma_0 & \partial_0\gamma_1 \\ \partial_0\gamma_1 & \partial_1\gamma_1 \end{pmatrix}.$$

In the case (2.4), we have

$$\gamma_0 = -e^{\nu}, \qquad \gamma_1 = e^{\lambda}, \tag{2.42}$$

where ν and λ are functions of (u_0, u_1). This yields

$$\begin{aligned} \left({}^{U}\Gamma^{0}{}_{jk}\right) &= \frac{1}{2}\begin{pmatrix} \partial_0\nu & \partial_1\nu \\ \partial_1\nu & e^{\lambda-\nu}\partial_0\lambda \end{pmatrix}, \\ \left({}^{U}\Gamma^{1}{}_{jk}\right) &= \frac{1}{2}\begin{pmatrix} e^{\nu-\lambda}\partial_1\nu & \partial_0\lambda \\ \partial_0\lambda & \partial_1\lambda \end{pmatrix}. \end{aligned} \tag{2.43}$$

Also, in the case (2.4), $\psi = \psi(r) = r^2$, where $r = x_1$, and $\vartheta = 2\log r$. Consequently, by (2.20),

$$\vartheta_{;j;k} = \partial_j\partial_k\vartheta + \frac{1}{2}\psi^{-1}\langle d\psi, d\psi\rangle g^{S}_{jk} - w_{jk}, \tag{2.44}$$

where $\partial_1\partial_1\vartheta = -2/r^2,\ \partial_j\partial_k\vartheta = 0$ for other indices, and

$$\left(w_{jk}\right) = \frac{1}{r}\begin{pmatrix} e^{\lambda-\nu}\partial_1\nu & \partial_0\lambda \\ \partial_0\lambda & \partial_1\lambda \end{pmatrix}, \tag{2.45}$$

for $0 \le j, k \le 1,\ w_{jk} = 0$ for other indices.

Hence for the metric tensor (2.4), the 4×4 matrix (G_{jk}) splits into two 2×2 blocks:

$$G_{jk} = \widetilde{G}_{jk} + \widehat{G}_{jk}. \tag{2.46}$$

The upper left block is

$$\begin{aligned} \widetilde{G}_{jk} &= -\frac{1}{2}S_S\psi^{-1}g^{U}_{jk} - \frac{1}{2}\beta g^{U}_{jk} - \partial_j\partial_k\vartheta + w_{jk} - \frac{1}{2}(\partial_j\vartheta)(\partial_k\vartheta) \\ &= -\frac{1}{2}\psi^{-1}\Big(S_S - 2\Box_U\psi + \frac{1}{2}\psi^{-1}\langle d\psi, d\psi\rangle\Big)g^{U}_{jk} - w_{jk}, \end{aligned} \tag{2.47}$$

since, for $\vartheta = 2\log r$, we have $\partial_j\partial_k\vartheta + (1/2)(\partial_j\vartheta)(\partial_k\vartheta) = 0$. The lower right block is

$$\begin{aligned} \widehat{G}_{jk} &= -\frac{1}{2}S_U\psi g^{S}_{jk} - \frac{1}{2}\beta\psi g^{S}_{jk} - \frac{1}{2}(\Box_U\psi)g^{S}_{jk} \\ &= -\frac{1}{2}\Big(S_U\psi - \Box_U\psi + \frac{1}{2}\psi^{-1}\langle d\psi, d\psi\rangle\Big)g^{S}_{jk}. \end{aligned} \tag{2.48}$$

Thus, for metrics of the form (2.4), G_{jk} has 6 nonzero components, out of 16 (or, if symmetry is taken into account in counting components, it has 5 nonzero components, out of 10).

When the metric tensor of U has the form $-e^\nu\,dt^2 + e^\lambda\,dr^2$, the calculation of Gauss curvature in (3.37) of Appendix C gives

$$\begin{aligned} S_U &= e^{-(\lambda+\nu)/2}\big[-\partial_r(\nu_r e^{(\nu-\lambda)/2}) + \partial_t(\lambda_t e^{(\lambda-\nu)/2})\big] \\ &= -e^{-\lambda}\nu_{rr} + e^{-\nu}\lambda_{tt} - \frac{1}{2}\nu_r(\nu_r - \lambda_r)e^{-\lambda} + \frac{1}{2}\lambda_t(\lambda_t - \nu_t)e^{-\nu}. \end{aligned} \tag{2.49}$$

Here, $\lambda_r = \partial\lambda$, $\lambda_t = \partial\lambda/\partial t$, and so forth. Of course, the unit sphere S^2 has Gauss curvature 1, so $S_S = 2$. Also, we have, for $\psi(r) = r^2$,

$$\Box_U\psi = 2e^{-\lambda} + r(\nu_r - \lambda_r)e^{-\lambda}, \qquad \psi^{-1}\langle d\psi, d\psi\rangle = 4e^{-\lambda}. \tag{2.50}$$

The formulas (2.45), (2.49), and (2.50) specify all the ingredients in (2.47) and (2.48).

We conclude that, for a metric of the form (2.4), with $(x_0, x_1, x_2, x_3) = (t, r, \varphi, \theta)$, all the nontrivial components of G are specified by the following five formulas:

$$G_{00} = -\frac{e^{\nu-\lambda}}{r^2}\big(1 - r\lambda_r - e^\lambda\big), \tag{2.51}$$

$$G_{01} = G_{10} = \frac{\lambda_t}{r}, \tag{2.52}$$

$$G_{11} = \frac{1}{r^2}\big(1 + r\nu_r - e^\lambda\big), \tag{2.53}$$

$$\begin{aligned} G_{22} = {}& \frac{1}{2}r^2 e^{-\lambda}\Big(\nu_{rr} + \frac{1}{2}\nu_r^2 + \frac{1}{r}(\nu_r - \lambda_r) - \frac{1}{2}\nu_r\lambda_r\Big) \\ &- \frac{1}{2}r^2 e^{-\nu}\Big(\lambda_{tt} + \frac{1}{2}\lambda_t^2 - \frac{1}{2}\lambda_t\nu_t\Big), \end{aligned} \tag{2.54}$$

$$G_{33} = \sin^2\varphi\; G_{22}. \tag{2.55}$$

Having determined the Einstein tensor for a spherically symmetric spacetime that has been put in the form (2.4), we now examine when the empty-space Einstein equation is satisfied, namely, when $G_{jk} = 0$. If we require all components G_{jk} to vanish, then (2.52) implies $\partial\lambda/\partial t = 0$, or $\lambda = \lambda(r)$. Furthermore, (2.51) and (2.53) imply

$$\partial_r(\lambda + \nu) = 0, \tag{2.56}$$

or $\lambda(r) + \nu(r,t) = f(t)$. Now, replacing t by $t' = \varphi(t)$ has the effect of adding an arbitrary function of t to ν in (2.4), so we can arrange that $\nu + \lambda = 0$. Thus the metric (2.4) takes the form

$$ds^2 = -e^{\nu(r)}\,dt^2 + e^{-\nu(r)}\,dr^2 + r^2\,d\omega^2. \tag{2.57}$$

Note that the coefficients are independent of t! We say the metric is *static*. The observation that a spherically symmetric solution to $G = 0$ must be static (under the additional hypotheses made at the beginning of this section) is known as *Birkhoff's theorem*.

For the metric (2.57), the component G_{01}, given by (2.52), certainly vanishes, and $G_{00} = 0 = G_{11}$ if and only if

$$r\nu'(r) = e^{-\nu(r)} - 1. \tag{2.58}$$

If we set $\psi(r) = e^{\nu(r)}$, this ODE becomes $r\psi'(r) = 1 - \psi(r)$, a nonhomogeneous Euler equation with general solution $\psi(r) = 1 - K/r$. Hence

$$e^{\nu(r)} = 1 - \frac{K}{r}. \tag{2.59}$$

It remains to check that G_{22} vanishes for this metric, that is, that

$$\nu''(r) + \nu'(r)^2 + \frac{2}{r}\nu'(r) = 0. \tag{2.60}$$

This is straightforward to check. Rather than substituting $\nu(r)$, given by (2.59), into (2.60), we can differentiate (2.58) to get $r\nu'' + \nu' = -\nu' e^{-\nu}$; adding $r(\nu')^2 + \nu'$ to both sides and again using (2.58), we obtain (2.60).

We have derived the following metric, known as the *Schwarzschild metric*, satisfying the vacuum Einstein equation $G = 0$:

$$ds^2 = -\Big(1 - \frac{K}{r}\Big)\,dt^2 + \Big(1 - \frac{K}{r}\Big)^{-1}\,dr^2 + r^2\,d\omega^2. \tag{2.61}$$

We can readily check that this is not a flat metric in a funny coordinate system, unless $K = 0$. Indeed, by (2.13) we have $R^0{}_{101} = {}^U R^0{}_{101}$, since $D^0{}_{101} = 0$ by (2.21). Now ${}^U R^0{}_{101}$ is a nonzero multiple of S_U, and, by (2.49), we have

$$S_U = \frac{2K}{r^3},$$

in this case.

We have a solution to $G = 0$ upon taking any real K in (2.61), but the metrics most relevant to observed phenomena are those for which $K > 0$. Indeed, as will be seen in § 4, geodesic orbits for the metric (2.61) have the property that, for large r and small "velocity," they approximate orbits for the Newtonian problem

$$\ddot{x} = -\operatorname{grad} V(x), \qquad V(x) = -\frac{1}{2}\frac{K}{|x|}. \tag{2.62}$$

If $K > 0$, these are orbits for the Kepler problem, that is, for the two-body planetary motion problem. If $K < 0$, these are orbits for the Coulomb problem, for the motion of charged particles with like charges, hence for motion under a repulsive force. Repulsive gravitational fields have not been observed.

Note that if we take $K > 0$ in (2.61), then the formula is degenerate at $r = K$. Only on $\{r > K\}$ is $\partial/\partial t$ a timelike vector. It is this region that is properly said to carry the Schwarzschild metric.

It follows from the fact that the sectional curvature of the plane spanned by $\partial/\partial t$ and $\partial/\partial r$ is S_U that the Schwarzschild metric (2.61) is singular at $r = 0$. On the other hand, the apparent singularity in the Schwarzschild metric at $r = K$ actually arises from a coordinate singularity, which can be removed as follows. First, set

$$v = t + \int \Big(1 - \frac{K}{r}\Big)^{-1} dr = t + r + K \log(r - K). \tag{2.63}$$

Using coordinates (v, r, θ, φ), the metric tensor (2.61) takes the form

$$ds^2 = -\Big(1 - \frac{K}{r}\Big)\, dv^2 + 2\, dv\, dr + r^2\, d\omega^2. \tag{2.64}$$

These coordinates are called *Eddington–Finkelstein coordinates*. The region $\{r > K\}$ in (t, r, θ, φ)-coordinates corresponds to the region $\{r > K\}$ in the new coordinate system, but the metric (2.64) is smooth and nondegenerate on the larger region $\{r > 0\}$. Note that if $\varphi,\ \theta$, and v are held constant and $r \searrow K$, then $t \nearrow \infty$.

The shell $\Sigma = \{r = K\}$ is a null surface for the metric (2.64); that is, the restriction of the metric to Σ is everywhere degenerate. Thus, for each $p \in \Sigma$, the light cone formed by null geodesics through p is tangent to Σ at p. Figure 2.1 depicts the extended Schwarzschild metric, in Eddington–Finkelstein coordinates.

The function v arises from considering null geodesics in the Schwarzschild spacetime for which $\omega \in S^2$ is constant or, equivalently, considering null geodesics in the two-dimensional spacetime

$$ds^2 = -\Big(1 - \frac{K}{r}\Big)\, dt^2 + \Big(1 - \frac{K}{r}\Big)^{-1} dr^2. \tag{2.65}$$

On the region $r > K$, there are a family of null geodesics given by $v =$ const. and a family of null geodesics given by $u =$ const., where

$$u = t - r - K \log(r - K). \tag{2.66}$$

The coordinates (u, r, θ, φ) are called *outgoing* Eddington–Finkelstein coordinates (the ones above then being called *incoming*), and in this coordinate system the Schwarzschild metric takes the form

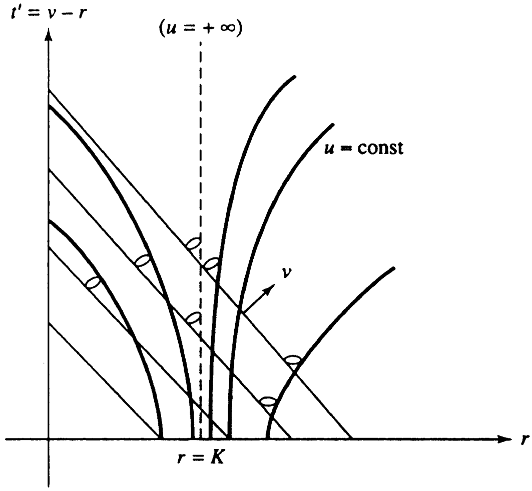

FIGURE 2.1 Extended Schwarzschild Metric

$$ds^2 = -\left(1 - \frac{K}{r}\right) du^2 - 2\, du\, dr + r^2\, d\omega^2. \tag{2.67}$$

As above, the region $\{r > K\}$ in (t, r, θ, φ)-coordinates corresponds to the region $\{r > K\}$ in the new coordinate system.

The incoming and outgoing Eddington–Finkelstein coordinates yield two different extensions of Schwarzschild spacetime. These two extensions were sewn together by M. Kruskal and P. Szekeres. As an intermediate step from (2.64) and (2.67) to "Kruskal coordinates," use the coordinates (u, v, θ, φ). In this coordinate system, the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{K}{r}\right) du\, dv + r^2\, d\omega^2, \tag{2.68}$$

where r is determined by

$$\frac{1}{2}(v - u) = r + K \log(r - K). \tag{2.69}$$

Now make a further coordinate change:

$$\xi = \frac{1}{2}\left(e^{v/2K} - e^{-u/2K}\right), \quad \tau = \frac{1}{2}\left(e^{v/2K} + e^{-u/2K}\right). \tag{2.70}$$

Then, in the Kruskal coordinates $(\xi, \tau, \theta, \varphi)$, the metric becomes

$$ds^2 = F(\tau, \xi)^2(-d\tau^2 + d\xi^2) + r(\xi, \tau)^2\, d\omega^2, \tag{2.71}$$

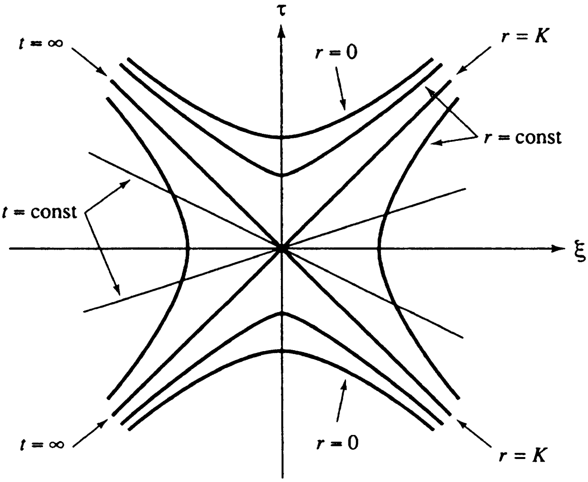

FIGURE 2.2 Kruskal Coordinates

where r is determined by

$$\tau^2 - \xi^2 = -(r - K)e^{r/K}, \tag{2.72}$$

and F is given by

$$F(\tau, \xi)^2 = \frac{4K^2}{r^2} e^{-r/K}. \tag{2.73}$$

Figure 2.2 depicts the extended Schwarzschild spacetime in Kruskal coordinates.

Exercises

1. Use Lemma 2.1 together with Theorem 1.1 to show that whenever M is a compact manifold of dimension 2, endowed with a Riemannian metric, the integrated scalar curvature
$$\int_M S\, dA$$
is independent of the choice of Riemannian metric on M. How does this fit in with proofs of the Gauss–Bonnet theorem, given in § 5 of Appendix C?
2. Suppose M is a manifold with a nondegenerate metric tensor. Show that
$$\dim M = 3, \ \mathrm{Ric}_{jk} = 0 \Longrightarrow R^i{}_{jk\ell} = 0.$$
(*Hint*: Show that the map (2.37) is an isomorphism when $\dim M = 3$.)

3. Suppose in (2.4) you replace S^2, with its standard metric, by *hyperbolic space* $\mathcal{H}^2$, with Gauss curvature -1, obtaining

$$ds^2 = -e^\nu\, dt^2 + e^\lambda\, dr^2 + r^2 g_{\mathcal{H}}. \tag{2.74}$$

Show that in the formulas (2.46)–(2.48) for the Einstein tensor, the only change occurs in (2.47), where $S_S = 2$ is replaced by -2. Show that this has the effect precisely of replacing $-e^\lambda$ by $+e^\lambda$ in the formulas (2.51) and (2.53) for G_{00} and G_{11}. Deduce that a solution to the vacuum Einstein equations arises if $\lambda = -\nu$ and

$$e^\nu = -1 - \frac{K}{r},$$

for some $K \in \mathbb{R}$. Taking $K > 0$, we have a metric of the form

$$ds^2 = \Bigl(1 + \frac{K}{r}\Bigr)\, dt^2 - \Bigl(1 + \frac{K}{r}\Bigr)^{-1} dr^2 + r^2 g_{\mathcal{H}}, \tag{2.75}$$

so r, rather than t, takes the place of "time," and the Killing vector $\partial/\partial t$ is not timelike. Taking $K = -\kappa,\ \kappa > 0$, we have a metric of the form

$$ds^2 = -\Bigl(\frac{\kappa}{r} - 1\Bigr)\, dt^2 + \Bigl(\frac{\kappa}{r} - 1\Bigr)^{-1} dr^2 + r^2 g_{\mathcal{H}} \quad (r \neq \kappa), \tag{2.76}$$

and the Killing vector $\partial/\partial t$ is timelike on $\{r < \kappa\}$.
Show that

$$-dr^2 + r^2 g_{\mathcal{H}} \tag{2.77}$$

can be interpreted as the flat Minkowski metric on the interior of the forward light cone in $\mathbb{R}^3$. What does this mean for (2.75)?

3. Stationary and static spacetimes

Let M be a four-dimensional manifold with a Lorentz metric, of signature $(-1, 1, 1, 1)$. We say M is stationary if there is a timelike Killing field Z on M, generating a one-parameter group of isometries. We then have a fibration $M \to S$, where S is a three-dimensional manifold and the fibers are the integral curves of Z, and S inherits a natural Riemannian metric. We call S the "space" associated to the spacetime M.

Given $x \in M$, let $\mathcal{V}_x$ denote the subspace of T_xM consisting of vectors parallel to $Z(x)$, and let $\mathcal{H}_x$ denote the orthogonal complement of $\mathcal{V}_x$, with respect to the Lorentz metric on M. We then have complementary bundles $\mathcal{V}$ and $\mathcal{H}$. Indeed, $p : M \to S$ has the structure of a principal G-bundle with connection, with $G = \mathbb{R}$. For each $x, \mathcal{H}_x$ is naturally isomorphic to $T_{p(x)}S$. The curvature of this bundle is the $\mathcal{V}$-valued 2-form ω given by

$$\omega(X, Y) = P_0[X, Y] \tag{3.1}$$

whenever X and Y are smooth sections of $\mathcal{H}$. Here, P_0 is the orthogonal projection of T_xM onto $\mathcal{V}$. Since $G = \mathbb{R}$, this gives rise in a natural fashion to an ordinary 2-form on M.

We remark that the integral curves of Z are all geodesics if and only if the length of Z is constant on M. This is a restrictive condition, which we certainly will not assume to hold. Thus such an integral curve $\mathcal{C}$ can have a nonvanishing second fundamental form $II_{\mathcal{C}}(X, Y)$, which for $X, Y \in \mathcal{V}_x$ takes values in $\mathcal{H}_x$. We have the following quantitative statement:

Proposition 3.1. *If Z is a Killing field and U_1 is a smooth section of $\mathcal{H}$, then*

$$\langle II_{\mathcal{C}}(Z, Z), U_1 \rangle = -\frac{1}{2} \mathcal{L}_{U_1} \langle Z, Z \rangle. \tag{3.2}$$

Proof. The left side of (3.2) is equal to

$$\langle \nabla_Z Z, U_1 \rangle = -\langle Z, \nabla_Z U_1 \rangle = -\langle Z, \nabla_{U_1} Z - \mathcal{L}_Z U_1 \rangle. \tag{3.3}$$

Now $\langle Z, \mathcal{L}_Z U_1 \rangle = -(\mathcal{L}_Z g)(Z, U_1) = 0$, so the right side of (3.3) is equal to the right side of (3.2), and the proof is complete.

Let E_0 and E_1 denote the bundles $\mathcal{V}$ and $\mathcal{H}$, respectively, so $TM = E_0 \oplus E_1$. Let $P_j(x)$ denote the orthogonal projection of T_xM onto E_{jx}. Thus P_0 is as in (3.1). If ∇ is the Levi–Civita connection on M, we define another metric connection (with torsion)

$$\widetilde{\nabla} = \nabla^0 \oplus \nabla^1, \tag{3.4}$$

where $\nabla^j_X = P_j \nabla_X$ on sections of E_j. Thus

$$\widetilde{\nabla}_X = P_0 \nabla_X P_0 + P_1 \nabla_X P_1 = \nabla_X - C_X, \tag{3.5}$$

where C_X has the form

$$C_X = \begin{pmatrix} 0 & II^1_X \\ II^0_X & 0 \end{pmatrix} \tag{3.6}$$

as in (4.40) of Appendix C; C is a section of $\mathrm{Hom}(TM \otimes TM, TM)$. Let us set

$$T_X = -C_{P_0 X}, \quad A_X = -C_{P_1 X}. \tag{3.7}$$

The Weingarten formula states that

$$C^t_X = -C_X; \tag{3.8}$$

see (4.41) of Appendix C. Note that if $x \in \mathcal{C}$, an integral curve of Z, then

$$X, Y \in \mathcal{V}_x \Longrightarrow C_X Y = II_{\mathcal{C}}(X, Y). \tag{3.9}$$

The following is a special case of a result of B. O'Neill, [ON]. It says that A in (3.7) measures the extent to which $\mathcal{H}$ is not integrable.

Proposition 3.2. *If X and Y are sections of $\mathcal{H}$, then*

$$C_X Y = -\frac{1}{2} P_0[X, Y]. \tag{3.10}$$

Proof. Since C is clearly a tensor, it suffices to prove this when X and Y are "basic," namely, when they, arise from vector fields on M. Note that

$$P_0[X, Y] = P_0 \nabla_X Y - P_0 \nabla_Y X = A_X Y - A_Y X,$$

so it suffices to show that $A_X X = 0$. If U is a section of $\mathcal{V}$, then

$$\langle U, A_X X \rangle = \langle U, \nabla_X X \rangle = -\langle \nabla_X U, X \rangle,$$

where $\langle \ , \ \rangle$ is the inner product on $T_x\mathcal{X}$. Now, under our hypotheses, $[X, U]$ is *vertical*, so $\langle \nabla_X U, X \rangle = \langle \nabla_U X, X \rangle$, hence

$$\langle U, A_X X \rangle = \frac{1}{2} \mathcal{L}_U \langle X, X \rangle = 0,$$

since $\langle X, X \rangle$ is constant on each integral curve $\mathcal{C}$.

Note that C_X is uniquely determined by (3.8)–(3.10), together with the fact that it interchanges $\mathcal{V}$ and $\mathcal{H}$.

We want to study the behavior of a geodesic on a stationary spacetime M. We begin with the following result:

Proposition 3.3. *Let γ be a constant-speed geodesic on M, with velocity vector T. If Z is a Killing field, then $\langle T, Z \rangle$ is constant on γ.*

Proof. We have

$$\frac{d}{ds} \big\langle T(s), Z(\gamma(s)) \big\rangle = \langle T, \nabla_T Z \rangle, \tag{3.11}$$

if $\nabla_T T = 0$. Now generally the Lie derivative of the metric tensor g is given by $(\mathcal{L}_Z g)(X, Y) = \langle \nabla_X Z, Y \rangle + \langle X, \nabla_Y Z \rangle$, so the right side of (3.11) is equal to $(1/2)(\mathcal{L}_Z g)(T, T)$. Since Z is a Killing field precisely when $\mathcal{L}_Z g = 0$, the proposition is proved.

Thus, if γ is a geodesic on M, satisfying

$$\langle T, T\rangle = C_2, \tag{3.12}$$

we have

$$\langle T, Z\rangle = C_1. \tag{3.13}$$

There is the following relation. Set

$$T = T_0 + T_1 = \alpha Z + T_1, \tag{3.14}$$

where T_0 is a section of $\mathcal{V}$ and T_1 a section of $\mathcal{H}$. Then, by orthogonality, $C_2 = \alpha^2\langle Z, Z\rangle + \langle T_1, T_1\rangle$, while $\langle T, Z\rangle = \alpha\langle Z, Z\rangle = C_1$, so

$$C_2 = \frac{C_1^2}{\langle Z, Z\rangle} + \langle T_1, T_1\rangle, \qquad \alpha = \frac{C_1}{\langle Z, Z\rangle}. \tag{3.15}$$

In Einstein's theory, a constant-speed, timelike geodesic in M represents the path of a freely falling observor. Let us consider the corresponding path in "space," namely, the path $\sigma(s) = p \circ \gamma(s)$, where $p : M \to S$ is the natural projection. We want a formula for the acceleration of σ.

Note that if $\gamma'(s) = T = T_0 + T_1 = \alpha Z + T_1$, as in (3.14), then $\sigma'(s) = V(s)$ is the vector in $T_{\sigma(s)}S$ whose horizontal lift is $T_1(s)$. By slight abuse of notation, we simply say $V(s) = T_1(s)$. Similarly,

$$\nabla^S_V V = P_1 \nabla_T T_1, \tag{3.16}$$

where P_1 is the orthogonal projection of T_xM on $\mathcal{H}_x,\ x = \gamma(s)$. We can restate this, using a modification of the Levi–Civita connection ∇ on M to $\widetilde{\nabla}$, given by (3.5). Then, via the identification used in (3.16), we have

$$\nabla^S_V V = \widetilde{\nabla}_T T_1 = -C_T T_0, \tag{3.17}$$

using $\nabla_T T = 0$. In fact, this plus (3.5) yields

$$\nabla^S_V V = -\widetilde{\nabla}_T T_0 - C_T T_1 - C_T T_0,$$

where the first two terms on the right are sections of $\mathcal{V}$ and the last term is a section of $\mathcal{H}$. Thus we get (3.17), plus the identity $\widetilde{\nabla}_T T_0 = -C_T T_1$.

Consequently, if U_1 is a vector field on M, identified with a section of $\mathcal{H}$ on $\mathcal{X}$, we have

$$\begin{aligned} \langle \nabla^S_V V, U_1\rangle &= -\big\langle C_{T_0} T_0, U_1\big\rangle + \big\langle T_0, C_{T_1} U_1\big\rangle \\ &= -\big\langle II_C(T_0, T_0), U_1\big\rangle - \frac{1}{2}\big\langle T_0, \omega(T_1, U_1)\big\rangle. \end{aligned} \tag{3.18}$$

Here, $II_{\mathcal{C}}$ is the second fundamental form of the integral curve $\mathcal{C}$ of Z, and ω is the "bundle curvature" of $M \to S$, as in (3.1). The first identity in (3.18) makes use of (3.8), while the last identity follows from (3.9) to (3.10).

Consequently, if we define $\omega_{T_1} : \mathcal{H} \to \mathcal{V}$ by $\omega_{T_1} U_1 = \omega(T_1, U_1)$, with adjoint $\omega^t_{T_1} : \mathcal{V} \to \mathcal{H}$, we have

$$\nabla^S_V V = -II_{\mathcal{C}}(T_0, T_0) - \frac{1}{2}\omega^t(T_1, T_0), \tag{3.19}$$

where $\omega^t(T_1, T_0) = \omega^t_{T_1} T_0$. Note that the formula (3.2) for $II_{\mathcal{C}}$ can be rewritten as

$$II_{\mathcal{C}}(Z, Z) = \frac{1}{2} \operatorname{grad} \Phi, \qquad \Phi = -\langle Z, Z\rangle, \tag{3.20}$$

where $\langle Z, Z\rangle$ is a smooth function on M, constant on each integral curve $\mathcal{C}$, hence effectively a function on S. Thus

$$II_{\mathcal{C}}(T_0, T_0) = \alpha^2 II_{\mathcal{C}}(Z, Z) = \frac{C_1^2}{2} \frac{1}{\Phi^2} \operatorname{grad} \Phi, \tag{3.21}$$

where C_1 is the constant $C_1 = \langle T, Z\rangle$ of Proposition 3.3.

We can rewrite $\omega^t(T_1, T_0)$ as follows. Let $\beta : \mathcal{H} \to \mathcal{H}$ be the skew-adjoint map satisfying

$$\omega(T_1, U_1) = \langle \beta(T_1), U_1\rangle Z. \tag{3.22}$$

We then have

$$\omega^t(T_1, T_0) = C_1\, \beta(T_1), \tag{3.23}$$

using the identity $\alpha\langle Z, Z\rangle = C_1$ from (3.15). Note that effectively β is a section of End TS, that is, a tensor field of type (1,1) on S.

In summary, recalling the identification of V and T_1, we have the following:

Proposition 3.4. *If γ is a constant-speed, timelike geodesic on a stationary spacetime M, then the curve $\sigma = p \circ \gamma$ on S, with velocity $V(s) = \sigma'(s)$, has acceleration satisfying*

$$\nabla^S_V V = \frac{1}{2} C_1^2 \, grad\, \Phi^{-1} - \frac{1}{2} C_1 \beta(V). \tag{3.24}$$

Note a formal similarity between the "force" term containing $\beta(V)$ here and the Lorentz force due to an electromagnetic field, on a Lorentz 4-manifold. Given initial data for $\gamma(s)$, namely,

$$\gamma(0) = x_0 \in M, \qquad \gamma'(0) = T(0) = T_0(0) + T_1(0), \tag{3.25}$$

we have $C_1 = \langle T_0(0), Z(x_0)\rangle$. The initial condition for σ is

$$\sigma(0) = p(x_0), \quad \sigma'(0) = T_0(0). \tag{3.26}$$

Conversely, once we obtain the path $\sigma(s)$ on S, by solving (3.24) subject to the initial data (3.26), we can reconstruct $\gamma(s)$ as follows. We define T on the surface $\Sigma = p^{-1}(\sigma)$ so that

$$p(x) = \sigma(s) \Longrightarrow T(x) = \alpha(s)Z + V(s), \tag{3.27}$$

with $\alpha(s)$ specified by the identity (3.15), namely,

$$\alpha(s) = -C_1\Phi\big(\sigma(s)\big)^{-1}. \tag{3.28}$$

Then T is tangent to Σ and γ is the integral curve of T through x_0.

The Lorentz manifold M is said to be a static spacetime if the subbundle $\mathcal{H}$ is integrable, that is, the bundle curvature ω of (3.1) vanishes. Note that if ζ is the 1-form on M obtained from Z by lowering indices, then

$$(d\zeta)(X, Y) = X\langle Z, Y\rangle - Y\langle Z, X\rangle - \langle Z, [X, Y]\rangle. \tag{3.29}$$

If X and Y are sections of $\mathcal{H}$, this gives

$$\langle Z, [X, Y]\rangle = -(d\zeta)(X, Y), \tag{3.30}$$

so vanishing of $d\zeta$ on $\mathcal{H} \times \mathcal{H}$ is a necessary and sufficient condition for integrability of $\mathcal{H}$. As a complement to (3.30), we remark that, since Z is a Killing field,

$$\langle X, d\Phi\rangle = (d\zeta)(X, Z), \tag{3.31}$$

for any vector field X on M, where, as in (3.20), $\Phi = -\langle Z, Z\rangle$. This follows from the identities

$$(\mathcal{L}_Z\zeta)(X) = (\mathcal{L}_Z g)(Z, X), \qquad \mathcal{L}_Z\zeta = d(\zeta\rfloor Z) - (d\zeta)\rfloor Z.$$

If M is static, a calculation using (3.30)–(3.31) implies

$$d\left(\Phi^{-1}\zeta\right) = 0. \tag{3.32}$$

Hence there is a function $t \in C^\infty(M)$ such that

$$\zeta = -\Phi\, dt. \tag{3.33}$$

It follows that the tangent space to any three-dimensional surface $\{t = c\} = S_c$ is given by $T_pS_c = \mathcal{H}_p$, for $p \in S_c$, and furthermore, the flow $\mathcal{F}_Z^t$ generated by Z (which preserves $\mathcal{H}$) takes S_c to S_{c+t}. Each S_c is naturally isometric to the Riemannian manifold S, and the metric tensor on M has the form

$$ds^2 = -\Phi(x)\,dt^2 + g_S(dx, dx), \tag{3.34}$$

where Φ is given by (3.20) and g_S is the metric tensor on S.

So, when M is static, we obtain a diffeomorphism $\Psi : S \times \mathbb{R} \to M$ by identifying S with $S_0 = \{t = 0\}$ and then setting $\Psi(x, t) = \mathcal{F}_Z^t x$. The geodesic γ on M yields a path on $S \times \mathbb{R}$:

$$\Psi^{-1}(\gamma(s)) = \big(\sigma(s), t(s)\big), \tag{3.35}$$

where $\sigma(s)$ is the path in S studied above. The function $t(s)$ is defined by (3.35). Note that

$$\frac{dt}{ds} = \alpha(s), \tag{3.36}$$

where $\alpha(s)$ is given by (3.27)–(3.28). Thus we can reparameterize γ by t, obtaining $\tilde{\gamma}(t)$ such that $\tilde{\gamma}(t(s)) = \gamma(s)$. We see that

$$\tilde{\gamma}(t) = \big(x(t), t\big), \quad x(t) = \sigma(s). \tag{3.37}$$

The quantities $v(t) = x'(t)$ and $a(t) = \nabla_v^S v(t)$ are the velocity and acceleration vectors of the path $x(t)$. We have

$$v(t) = x'(t) = \frac{1}{\alpha(s)} V(s) = -\frac{1}{C_1}\Phi(x)V(s). \tag{3.38}$$

Furthermore,

$$\nabla_v^S v = \frac{\Phi^2}{C_1^2}\nabla_V^S V - \frac{1}{C_1}\Big(\frac{d}{dt}\Phi\Big)V. \tag{3.39}$$

Note that $d\Phi/dt = \langle v, \text{grad}\,\Phi\rangle$. If we use (3.24), recalling that $\beta = 0$ in this case, we obtain the following result:

Proposition 3.5. *A static spacetime M can be written as a product $S \times \mathbb{R}$, with Lorentz metric of the form (3.34). A timelike geodesic on such a static spacetime can be reparameterized to have the form (3.37), with velocity $v(t) = x'(t)$, and with acceleration given by*

$$\nabla_v^S v = -\frac{1}{2}\,\mathit{grad}\,\Phi + \frac{1}{\Phi}\langle v, \mathit{grad}\,\Phi\rangle v. \tag{3.40}$$

By (3.15) we have $\langle V, V\rangle = C_2 + C_1^2/\Phi$, hence

$$\langle v, v\rangle = \Phi + \frac{C_2}{C_1^2}\Phi^2. \tag{3.41}$$

In particular, if $\gamma(s)$ is *lightlike*, so $C_2 = 0$, we have

$$\langle v, v\rangle = \Phi. \tag{3.42}$$

This identity suggests rescaling the metric on S, that is, looking at $g^\# = \Phi^{-1} g_S$. We will pursue this next.

Note that the *null* geodesics on a Lorentz manifold M (i.e., the "light rays"), coincide with those of any conformally equivalent metric, though they may be parameterized differently. This is particularly easy to see via identifying the geodesic flow with the Hamiltonian flow on T^*M, using the Lorentz metric to define the total "energy." If M is static, we can multiply the metric (3.34) by Φ^{-1}, obtaining the new metric

$$ds^2 = -dt^2 + g^\#(dx, dx), \qquad g^\# = \Phi^{-1} g_S. \tag{3.43}$$

If γ is a geodesic for this new metric on M, the equation (3.40) for the projected path $x(t)$ on S becomes

$$\nabla^\#_v v = 0, \tag{3.44}$$

as the $\Phi = 1$ case of (3.40). Consequently, null geodesics in a static spacetime project to geodesics on the space S, with the rescaled metric $g^\# = \Phi^{-1} g_S$.

Let us see what happens to geodesics that need not be lightlike. For the moment, we take M to be stationary, and define Φ by (3.20). In order to clarify the role of the exponent of Φ, we consider on S a conformally rescaled metric of the form $g^\# = \Phi^a\, g_S$. Farther along, we will again take $a = -1$, and then we will specialize to the case of M static.

The connection coefficients for the two metrics g_S and $g^\#$ are related by

$${}^\#\Gamma^j{}_{k\ell} = {}^S\Gamma^j{}_{k\ell} + \frac{a}{2\Phi}\Big((\partial_\ell\Phi)\,\delta^j{}_k + (\partial_k\Phi)\,\delta^j{}_\ell - g^{j\mu}(\partial_\mu\Phi)g_{k\ell}\Big). \tag{3.45}$$

Equivalently, the connections ∇^S and $\nabla^\#$ are related by

$$\begin{aligned}\nabla^\#_V W = \nabla^S_V W + \frac{a}{2\Phi}\Big(&\langle V, \operatorname{grad}\Phi\rangle W + \langle W, \operatorname{grad}\Phi\rangle V \\ &- \langle V, W\rangle \operatorname{grad}\Phi\Big).\end{aligned} \tag{3.46}$$

In particular,

$$\nabla^{\#}_V V = \nabla^S_V V + \frac{a}{\Phi}\langle V, \text{grad }\Phi\rangle V - \frac{a}{2\Phi}\langle V, V\rangle \text{ grad }\Phi. \tag{3.47}$$

Here, $\langle\ ,\ \rangle$ is the inner product for g_S, and grad Φ is obtained from $d\Phi$ via the metric g_S.

If $\gamma(s)$ is a geodesic (not necessarily lightlike) on a stationary spacetime M, then the construction of the projected path $\sigma(s)$ on S given by $\sigma = p \circ \gamma$ shows that $V = \sigma'(s)$ satisfies

$$\langle V, V\rangle = C_2 + \frac{C_1^2}{\Phi}, \tag{3.48}$$

as noted after Proposition 3.5. Hence, in the lightlike case, $g^{\#}(V, V) = \Phi^{a-1}C_1^2$. If we want to reparameterize σ to have constant speed (in the lightlike case), we set

$$\tilde{\sigma}(r) = \sigma(s), \qquad \frac{dr}{ds} = \Phi^{(1-a)/2}, \tag{3.49}$$

so $g^{\#}(\tilde{\sigma}', \tilde{\sigma}') = C_1^2$ if $\gamma(s)$ is lightlike. Let

$$w = \tilde{\sigma}'(r) = \Phi^{(1-a)/2} V(s). \tag{3.50}$$

Then (regardless of whether γ is lightlike)

$$\nabla^{\#}_w w = \Phi^{1-a}\ \nabla^{\#}_V V + \frac{1-a}{2}\Phi^{-a}\langle \text{grad }\Phi, V\rangle V. \tag{3.51}$$

If we use (3.46), this becomes

$$\begin{aligned}\Phi^{1-a}\Big(\nabla^S_V V + \frac{a}{\Phi}\langle V, \text{grad }\Phi\rangle V - \frac{a}{2\Phi}\langle V, V\rangle \text{ grad }\Phi\Big)\\ + \frac{1-a}{2}\Phi^{-a}\langle \text{grad }\Phi, V\rangle V.\end{aligned} \tag{3.52}$$

If we use (3.48) for $\langle V, V\rangle$ and (3.24) for $\nabla^S_V V$, we see that (3.51) is equal to

$$\begin{aligned}-\frac{C_1^2}{2}\Phi^{-1-a} \text{ grad }\Phi - \frac{a}{2}\Phi^{-a}\Big(C_2 + \frac{C_1^2}{\Phi}\Big) \text{ grad }\Phi\\ + \frac{1+a}{2}\Phi^{-a}\langle V, \text{grad }\Phi\rangle V - \Phi^{1-a}\frac{C_1}{2}\beta(V).\end{aligned}$$

From here on, we take $a = -1$. We have

$$\nabla^{\#}_w w = \frac{C_2}{2} \Phi \text{ grad } \Phi - \frac{C_1}{2} \Phi \beta(w). \tag{3.53}$$

This is a generalization of (3.44). If M is static, $\beta(V) = 0$, and if γ is lightlike, $C_2 = 0$, so $\nabla^{\#}_w w = 0$ in that case.

Note that if M is static, then w is a constant multiple of v, defined by (3.38), and (3.53) is then equivalent to

$$\nabla^{\#}_v v = \frac{C_2}{2C_1^2} \Phi \text{ grad } \Phi. \tag{3.54}$$

In (3.54), grad Φ is obtained from $d\Phi$ via g_S. If we use $g^{\#}$, call the vector field so produced $\text{grad}^{\#}\, \Phi$. We have

$$\text{grad}^{\#}\, f = \Phi \text{ grad } f.$$

We have established the following:

Proposition 3.6. *Let $M = S \times \mathbb{R}$ be a static spacetime, with metric of the form (3.34). Let γ be a timelike geodesic on M, reparameterized to have the form (3.37), yielding a curve $x(t)$ on S, with velocity $v(t) = x'(t)$. With respect to the rescaled metric $g^{\#} = \Phi^{-1} g_S$ on S, this curve satisfies the equation*

$$\nabla^{\#}_v v = \frac{C_2}{2C_1^2}\, \mathit{grad}^{\#}\, \Phi. \tag{3.55}$$

Recall that C_1 and C_2 are given by (3.12)–(3.13). We mention that, while the factor C_2/C_1^2 is constant on each orbit, it varies from orbit to orbit. For example, if at time t_0 a particle moving along γ is "at rest," so $\gamma'(s_0)$ is parallel to Z, where $t(s_0) = t_0$, then $C_2/C_1^2 = -\Phi(x_0)^{-1}$, where $x_0 = x(t_0)$, as follows from (3.15).

Exercises

Exercises 1–5 deal with the Kerr metric, which, in (t, r, θ, φ)-coordinates, is

$$ds^2 = -\frac{A}{\rho^2}\big[dt - a \sin^2\varphi \, d\theta\big]^2 + \frac{\rho^2}{A}\, dr^2 + \rho^2\, d\varphi^2 + \frac{\sin^2\varphi}{\rho^2}\big[(r^2 + a^2)\, d\theta - a\, dt\big]^2, \tag{3.56}$$

where

$$A = r^2 - Kr + a^2, \qquad \rho^2 = r^2 + a^2 \cos^2\varphi.$$

Here, $K > 0$ and a are constants. Note that the case $a = 0$ gives the Schwarzschild metric (2.61).

1. Show that the Kerr metric provides a solution to the vacuum Einstein equation

$$G_{jk} = 0.$$

2. Show that $\partial/\partial t$ is a Killing field and hence that the Kerr metric is stationary.
3. Show that the Kerr metric is not static (if $a \neq 0$). Compute the 2-form ω of (3.1).
4. Contrast the metric induced by (3.56) on surfaces $t = \text{const.}$ with the three-dimensional Riemannian metric constructed at the beginning of this section.
5. Try to provide a "simple" derivation of the metric (3.56).

4. Orbits in Schwarzschild spacetime

We want to describe timelike geodesics in the Schwarzschild metric, which (for $r > K$) is a static, Ricci-flat metric of the form

$$ds^2 = -\Phi(r)\, dt^2 + \Phi^{-1}(r)\, dr^2 + r^2\, d\omega^2, \quad \Phi(r) = 1 - \frac{K}{r}, \tag{4.1}$$

where $d\omega^2$ is the standard metric on the unit sphere S^2. This is of the form (3.34), with Φ depending only on r, and $g_S(dx, dx) = \Phi(r)^{-1}\, dr^2 + r^2\, d\omega^2$. Thus, by Proposition 3.6, a timelike geodesic (reparameterized) within the region $r > K$ has the form $\tilde{\gamma}(t) = \big(x(t), t\big)$, where $v(t) = x'(t)$ satisfies the equation (3.55), namely,

$$\nabla^{\#}_v v = -\sigma \text{ grad}^{\#} \Phi, \quad \sigma = -\frac{C_2}{2C_1^2}. \tag{4.2}$$

This involves the rescaled metric $g^{\#} = \Phi^{-1} g_S$, which has the form

$$g^{\#}(dx, dx) = \Phi(r)^{-2}\, dr^2 + \Phi(r)^{-1} r^2\, d\omega^2. \tag{4.3}$$

Symmetry implies that $x(t)$ is confined to a plane, so we can restrict attention to the associated planar problem, with

$$\begin{aligned} g^{\#}(dx, dx) &= \Phi(r)^{-2}\, dr^2 + \Phi(r)^{-1} r^2\, d\theta^2 \\ &= \alpha(r)^{-1}\, dr^2 + \beta(r)^{-1}\, d\theta^2. \end{aligned} \tag{4.4}$$

We use the method discussed in § 17 of Chap. 1 to treat this problem. We have a Hamiltonian system of the form

$$\dot{y}_j = \frac{\partial F}{\partial \eta_j}, \quad \dot{\eta}_1 = -\frac{\partial F}{\partial y_1}, \quad \dot{\eta}_2 = 0, \tag{4.5}$$

with

$$F(y_1, \eta_1, \eta_2) = \frac{1}{2}\alpha(y_1)\eta_1^2 + \frac{1}{2}\beta(y_1)\eta_2^2 + \sigma\Phi(y_1), \tag{4.6}$$

where $y_1 = r$, $y_2 = \theta$. The first set of equations in (4.5) yields

$$\dot{r} = \Phi(r)^2 \eta_1, \qquad \dot{\theta} = L\frac{\Phi(r)}{r^2}, \tag{4.7}$$

where L is the constant value of η_2 along the integral curve of (4.5). Now $F(y_1, \eta_1, L) = E$ is constant along any such integral curve. Solving for η_1 and substituting into the first equation in (4.7), we have

$$\dot{r} = \pm\Phi(r)\Big(2E - 2\sigma\Phi(r) - L^2\frac{\Phi(r)}{r^2}\Big)^{1/2}. \tag{4.8}$$

We can rewrite this as

$$\dot{r} = \pm\Phi(r)\Big(2\widetilde{E} + \frac{2K\sigma}{r} - \frac{L^2}{r^2} + \frac{KL^2}{r^3}\Big)^{1/2}, \qquad \widetilde{E} = E - \sigma. \tag{4.9}$$

Compare this with (17.16) of Chap. 1:

$$\dot{r} = \pm\Big(2E + \frac{2K}{r} - \frac{L^2}{r^2}\Big)^{1/2}, \tag{4.10}$$

for the Kepler problem, with potential $v(r) = -K/r$. We have a shift in E, a correspondence $K\sigma \mapsto K$, and an extra term KL^2/r^3, within large parentheses in (4.9).

Next, setting $u = 1/r$, we have

$$\frac{dr}{dt} = -r^2\frac{du}{d\theta}\frac{d\theta}{dt}, \tag{4.11}$$

parallel to (17.21) of Chap. 1, and using $\dot{\theta} = L\Phi(r)/r^2$ we have

$$\frac{dr}{dt} = -L\Phi(r)\frac{du}{d\theta}; \tag{4.12}$$

hence, via (4.9),

$$\frac{du}{d\theta} = \mp\frac{1}{L}\Big(2\widetilde{E} + 2K\sigma u - L^2u^2 + KL^2u^3\Big)^{1/2}. \tag{4.13}$$

Compare the Kepler problem, where $dr/dt = -L\,du/d\theta$, and hence, from (4.10),

$$\frac{du}{d\theta} = \mp\frac{1}{L}\Big(2E + 2Ku - L^2u^2\Big)^{1/2}. \tag{4.14}$$

It is useful also to consider a second-order ODE for $u = u(\theta)$. Squaring (4.13) and taking the θ-derivative, we obtain (either $u'(\theta) = 0$ or)

$$\frac{d^2u}{d\theta^2} + u = \frac{K\sigma}{L^2} + \frac{3}{2}Ku^2. \tag{4.15}$$

Following [ABS], p. 207, we write this as

$$\frac{d^2u}{d\theta^2} + u = A + \varepsilon u^2. \tag{4.16}$$

The $\varepsilon = 0$ case arises from the study of the Kepler problem; cf. (17.24) of Chap. 1.

A phase-plane analysis of (4.16) is useful. If $v = du/d\theta$, we have the "Hamiltonian system"

$$\frac{du}{d\theta} = v = \partial_v F(u, v), \qquad \frac{dv}{d\theta} = A - u + \varepsilon u^2 = -\partial_u F(u, v), \tag{4.17}$$

where

$$F(u, v) = \frac{1}{2}v^2 + \frac{1}{2}u^2 - Au - \frac{\varepsilon}{3}u^3. \tag{4.18}$$

Of course, orbits for (4.18) lie on level curves $F(u, v) = E_1$, bringing us back to (4.13). See Figs. 4.1–4.4. In these four figures we have, respectively

$$\varepsilon = 0, \quad 0 < A\varepsilon < \frac{3}{16}, \quad \frac{3}{16} < A\varepsilon < \frac{1}{4}, \quad A\varepsilon = \frac{1}{4}.$$

Also, in Figs. 4.2–4.3, we have

$$\alpha = \frac{1 - \sqrt{1 - 4A\varepsilon}}{2\varepsilon} = \frac{2A}{1 + \sqrt{1 - 4A\varepsilon}},$$
$$\beta = \frac{1 + \sqrt{1 - 4A\varepsilon}}{2\varepsilon}.$$

We perceive the periodicity of u as a function of θ, on those level curves diffeomorphic to the circle. The period is not 2π, generally, if $\varepsilon \neq 0$, so we have precession of the perihelion.

Not all the closed orbits for (4.33) depicted in Figs. 4.2–4.4 correspond to bound orbits for the solution $x(t)$ to (4.2). One mechanism behind this arises already in the $\varepsilon = 0$ case. Take a level curve in Fig. 4.1 which crosses the vertical axis $\{u = 0\}$. Now $u = 0$ means $r = \infty$, so as $u \to 0$ along such an orbit, $x(t)$ tends to infinity; this endures for an infinite span of time. Such situations also arise for positive values of ε.

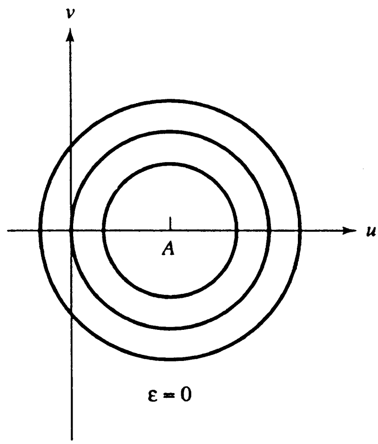

FIGURE 4.1 Orbits for (4.16), $\varepsilon = 0$

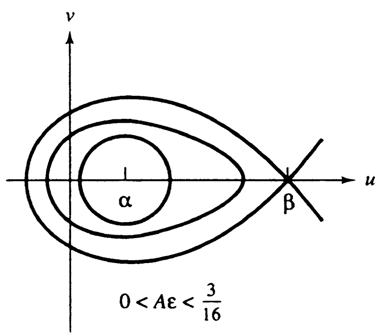

FIGURE 4.2 Orbits for (4.16), $0 < A\varepsilon < \frac{3}{16}$

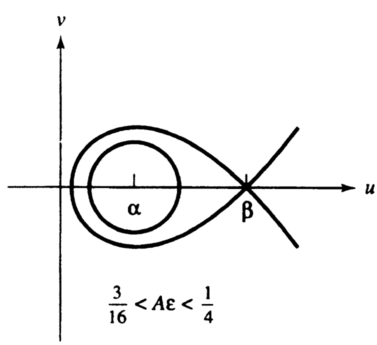

FIGURE 4.3 Orbits for (4.16), $\frac{3}{16} < A\varepsilon < \frac{1}{4}$

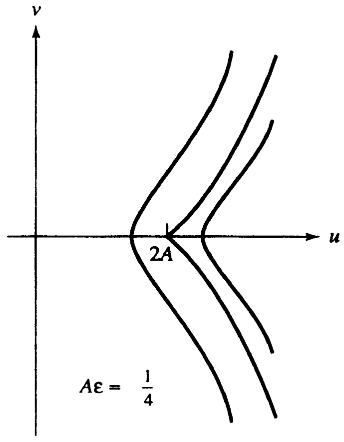

FIGURE 4.4 Orbits for (4.16), A$\varepsilon = \frac{1}{4}$

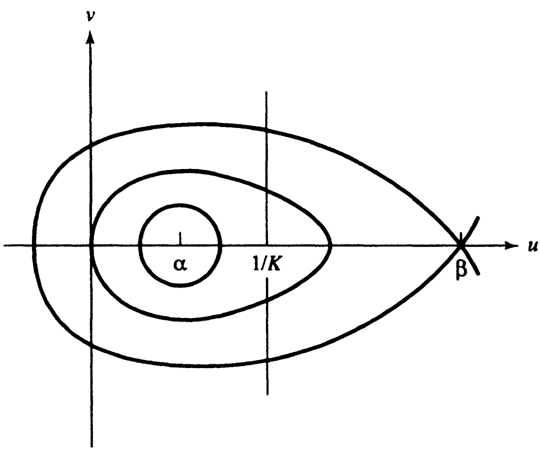

FIGURE 4.5 Crossing the Threshold

In addition, in the case under consideration in this section, there is another mechanism at work. Namely, consider a level curve that crosses the vertical line $\{u = 1/K\}$, as in Fig. 4.5. From (4.9) we see that as $u \nearrow 1/K$ (so $r \searrow K$), $t \nearrow \infty$. Now, in this case, it does not take the body "forever" to cross the threshold. When one switches to Eddington–Finkelstein coordinates (or to Kruskal coordinates), one can see the planet entering the zone $r < K$ in finite "proper time." The analysis of the geodesic within this region is not radically different from that done above, though there are some differences, since in this region the Killing field $\partial/\partial t$ is not timelike. We leave it to the reader to work out the nature of the orbit in this region, but note that indeed, a body crossing the threshold will not be able to exit.

Let us look at the problem of determining the period $p(\varepsilon)$ of a solution $w = w(\varepsilon, \theta)$ to an ODE of the form

$$\frac{d^2 w}{d\theta^2} + w = \varepsilon \psi(w). \tag{4.19}$$

This is related to (4.16) by $w = u - A,\ \psi(w) = (w + A)^2$. To specify uniquely a solution, let us take

$$w(\varepsilon, 0) = a, \qquad \frac{d}{d\theta}\, w(\varepsilon, 0) = 0. \tag{4.20}$$

If $v = dw/d\theta$ (and we denote this by $\dot{w}$), then we have the system

$$\dot{w} = v, \qquad \dot{v} = -w + \varepsilon \psi(w), \tag{4.21}$$

as in (4.17), and orbits lie on level curves of the function F_ε, given by

$$F_\varepsilon(w, v) = \frac{1}{2} v^2 + \frac{1}{2} w^2 - \varepsilon \Psi(w), \quad \Psi(w) = \int \psi(w)\, dw. \tag{4.22}$$

It is clear that, for ε small, we have smooth, real-valued functions $\rho(\varepsilon, \theta)$ and $\varphi(\varepsilon, \theta)$, uniquely specified by

$$w + iv = \rho(\varepsilon, \theta) e^{-i\varphi(\varepsilon, \theta)}. \tag{4.23}$$

By (4.20), we have

$$\rho(\varepsilon, 0) = a, \qquad \varphi(\varepsilon, 0) = 0. \tag{4.24}$$

Note also that

$$\rho(0, \theta) = a, \qquad \varphi(0, \theta) = \theta. \tag{4.25}$$

Then $p(\varepsilon)$ is a smooth function of ε, satisfying

$$p(0) = 2\pi, \qquad \varphi\big(\varepsilon, p(\varepsilon)\big) = 2\pi. \tag{4.26}$$

We can derive a system of ODE for ρ, φ, as functions of θ. The system (4.21) implies

$$\dot{w} + i\dot{v} = v - iw + i\varepsilon\psi(w) = -i(w + iv) + i\varepsilon\psi(w).$$

Writing the left side as $(d/d\theta)(\rho e^{-i\varphi}) = (\dot{\rho} - i\dot{\varphi}\rho)e^{-i\varphi}$ and the right side as $-i\rho e^{-i\varphi} + i\varepsilon\psi(\rho\cos\varphi)$, we obtain

$$\begin{aligned}\dot\rho &= -\varepsilon\psi(\rho\cos\varphi)\sin\varphi,\\ \dot\varphi &= 1-\varepsilon\rho^{-1}\psi(\rho\cos\varphi)\cos\varphi.\end{aligned} \tag{4.27}$$

A particularly significant quantity we can compute is $p'(0)$. By (4.26), we have

$$p'(0)\,\frac{\partial\varphi}{\partial\theta}(0,2\pi)+\frac{\partial\varphi}{\partial\varepsilon}(0,2\pi)=0,$$

so

$$p'(0)=-\frac{\partial\varphi}{\partial\varepsilon}(0,2\pi). \tag{4.28}$$

To compute the right side of (4.28), write

$$\rho=a+\varepsilon\rho_1(\theta)+\cdots,\qquad \varphi=\theta+\varepsilon\varphi_1(\theta)+\cdots. \tag{4.29}$$

Substituting into (4.27), we obtain

$$\dot\rho_1=-\psi(a\cos\theta)\sin\theta,\quad \dot\varphi_1=-\frac{1}{a}\psi(a\cos\theta)\cos\theta. \tag{4.30}$$

Also, from (4.24) we have $\varphi_1(0)=0$, so

$$p'(0)=-\varphi_1(2\pi)=\frac{1}{a}\int_0^{2\pi}\psi(a\cos\theta)\cos\theta\,d\theta. \tag{4.31}$$

In the case $\psi(w)=(w+A)^2$, we can evaluate the integral, to get

$$p'(0)=2\pi A. \tag{4.32}$$

The integral that arises in (4.31) is often interpreted as an average, and the calculation above is sometimes said to involve the "method of averaging." For more on this topic, one can see [SV].

The accuracy achieved when these formulas were applied to the calculation of the precession of the perihelion of the planet Mercury–or rather that part of it not attributable to perturbations produced by the other planets–provided early positive evidence in favor of Einstein's theory of gravity.

We end this section with a remark on the value of K in (4.1), in terms of Newtonian concepts. Newtonian theory should be accurate for a planetary orbit on which r is large and the velocity of the planet is small. Let us evaluate the quantity σ, defined in (4.2). We have

$$\sigma=-\frac{C_2}{2C_1^2}=-\frac{1}{2}\frac{\langle T,T\rangle}{\langle T,Z\rangle^2}, \tag{4.33}$$

by (3.12)–(3.13). Now, "small velocity" means T is essentially parallel to Z; hence $\langle T, T\rangle \approx -\langle T, Z\rangle^2$, and so

$$\sigma \approx \frac{1}{2}. \tag{4.34}$$

If also r is large, then (4.9) becomes

$$\begin{aligned} \dot{r} &\approx \pm\Big(1 - \frac{K}{r}\Big)^{-1}\Big(2\widetilde{E} + \frac{K}{r} - \frac{L^2}{r^2}\Big)^{1/2} \\ &\approx \pm\Big[2\widetilde{E} + (1 - 4\widetilde{E})\frac{K}{r} - \frac{L^2 - 2\widetilde{E}K^2 + 2K^2}{r^2}\Big]^{1/2}. \end{aligned} \tag{4.35}$$

For the Newtonian approximation to be valid, we need $\widetilde{E} << 1$. Also $A\varepsilon << 1$, with $A\varepsilon = 3\sigma K^2/(2L^2)$; in other words, $K^2 << L^2$. Thus,

$$\dot{r} \approx \pm\Big(2\widetilde{E} + \frac{K}{r} - \frac{L^2}{r^2}\Big)^{1/2}. \tag{4.36}$$

If we compare this with the formula (4.10) arising from the Kepler problem, we see one difference; (4.10) has $2K$ instead of K. Since, in appropriate units, the K in (4.10) is the gravitational mass of the attracting body (e.g., the Sun), we conclude that in (4.36), K should be twice the gravitational mass. Thus, it is common to write the Schwarzschild metric as

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)\,dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1} dr^2 + r^2\,d\omega^2 \tag{4.37}$$

and identify M as the "mass" of the solution, as seen at infinity.

Exercises

1. In Fig. 4.3, consider the orbit with $(\beta, 0)$ as forward and backward limit points. Assume $\beta < 1/K$. Interpret the behavior of the corresponding solution $x(t)$ of (4.2).
2. Study timelike geodesics for the metric (4.1) inside $\{r < K\}$.
3. Study the behavior of timelike geodesics on a spacetime with a Kerr metric, given by (3.56). One might consult the treatment in [Chan2].

5. Coupled Maxwell–Einstein equations

The coupled Maxwell–Einstein equations are

$$G_{jk} = 8\pi\kappa T_{jk}, \quad d\mathcal{F} = 0, \quad d^{\star}\mathcal{F} = 0, \tag{5.1}$$

in a spacetime in which there is an electromagnetic field $\mathcal{F}$, but no matter. The last two equations in (5.1) are the Maxwell equations, discussed in § 11 of Chap. 2. The stress-energy tensor of $\mathcal{F}$ is given by (1.4), that is,

$$T_{jk} = \frac{1}{4\pi}\Big(\mathcal{F}_j{}^{\ell}\mathcal{F}_{k\ell} - \frac{1}{4}\mathcal{F}_{i\ell}\mathcal{F}^{i\ell}g_{jk}\Big). \tag{5.2}$$

We look for spherically symmetric solutions to (5.1). Thus, as in § 2, we first take the metric to have the form (2.4), so that G_{jk} is given by (2.51)–(2.55). The hypothesis that $\mathcal{F}$ is spherically symmetric restricts its form severely. In fact, we can write

$$\mathcal{F} = d\mathcal{A} + c(t,r)\sigma,$$

where $\mathcal{A}$ is a 1-form and σ is the standard area form on S^2. The equation $d\mathcal{F} = 0$ implies $c(t,r) = c$, constant. If we assume the electromagnetic field decays to zero as $r \to \infty$, then $c = 0$. We will make this hypothesis. By averaging with respect to the SO(3) action, we can arrange that $\mathcal{A}$ be invariant under this action. This implies that, for each orbit $\mathcal{O}$ of SO(3), the pull-back $j_{\mathcal{O}}^*\mathcal{A} \in \Lambda^1(\mathcal{O})$ vanishes. Indeed, $\Lambda^1(\mathcal{O})$ has no SO(3)-invariant elements other than zero; equivalently, the sphere S^2 has no SO(3)-invariant vector fields, other than zero. Hence, $\mathcal{A}$ has the form

$$\mathcal{A} = a(t,r)\,dt + b(t,r)\,dr, \tag{5.3}$$

so

$$\mathcal{F} = \Big(\frac{\partial b}{\partial t} - \frac{\partial a}{\partial r}\Big)\,dt\wedge dr = E(t,r)\,dt\wedge dr. \tag{5.4}$$

Thus, the only nonzero components of $\mathcal{F}_{jk}$ are $\mathcal{F}_{01} = -\mathcal{F}_{10}$.

We deduce that all off-diagonal components of T_{jk} vanish and that

$$\begin{gathered} 4\pi T_{00} = \frac{1}{2}e^{-\lambda}E^2, \qquad 4\pi T_{11} = -\frac{1}{2}e^{-\nu}E^2, \\ 4\pi T_{jj} = \frac{1}{2}e^{-(\lambda+\nu)}E^2 g_{jj} \quad \text{if } j = 2,3. \end{gathered} \tag{5.5}$$

In particular, since $T_{10} = 0$, it follows that $G_{10} = 0$, so (2.52) implies $\partial\lambda/\partial t = 0$, that is, $\lambda = \lambda(r)$. If we exploit $G_{00} = 8\pi\kappa T_{00}$ and $G_{11} = 8\pi\kappa T_{11}$, using (2.51) and (2.53), we get

$$\kappa E(t,r)^2 = -\frac{e^\nu}{r^2}\big(1 - r\lambda_r - e^\lambda\big) = -\frac{e^\nu}{r^2}\big(1 + r\nu_r - e^\lambda\big). \tag{5.6}$$

In particular, $\partial_r(\lambda + \nu) = 0$. Thus, as in the argument following (2.56), we can fix the t-coordinate so that $\nu + \lambda = 0$, and hence the metric is again in the static form (2.57):

$$ds^2 = -e^{\nu(r)}\,dt^2 + e^{-\nu(r)}\,dr^2 + r^2\,d\omega^2. \tag{5.7}$$

Now the right side of (5.6) is a function of r alone, so $E = E(r)$; that is, the electromagnetic field $\mathcal{F}$ has the form

$$\mathcal{F} = E(r)\,dt \wedge dr. \tag{5.8}$$

Then the equation $d^\star\mathcal{F} = 0$ is equivalent to $\partial_r\big(\sqrt{-g}\mathcal{F}^{01}\big) = 0$, where $g = -e^\nu e^{-\nu} r^2 r^2 = -r^4$, so we have

$$E(r) = \frac{q}{r^2}, \tag{5.9}$$

for some constant q. If we substitute this formula for E in (5.6), we obtain the ODE

$$r\nu'(r) = \Big(1 - \frac{\kappa q^2}{r^2}\Big)e^{-\nu} - 1. \tag{5.10}$$

If we set $\psi(r) = e^{\nu(r)}$, this becomes

$$\Big(r\frac{d}{dr} + 1\Big)\psi(r) = 1 - \frac{\kappa q^2}{r^2}, \tag{5.11}$$

a nonhomogeneous Euler equation with general solution $\psi(r) = 1 - K/r + \kappa q^2/r^2$, so

$$e^{\nu(r)} = 1 - \frac{K}{r} + \frac{Q^2}{r^2}, \tag{5.12}$$

where we have set $Q^2 = \kappa q^2$. Hence we obtain the metric

$$ds^2 = -\Big(1 - \frac{K}{r} + \frac{Q^2}{r^2}\Big)\,dt^2 + \Big(1 - \frac{K}{r} + \frac{Q^2}{r^2}\Big)^{-1}\,dr^2 + r^2\,d\omega^2. \tag{5.13}$$

This is known as the *Reissner–Nordström* solution. It becomes the Schwarzschild solution when $Q = 0$.

It remains to check that $G_{22} = 8\pi\kappa T_{22}$, which by (5.5) is equal to $\kappa E(r)^2 r^2 = \kappa q^2/r^2$. In this case, (2.54) yields

$$G_{22} = \frac{1}{2}r^2 e^\nu\big[\nu''(r) + \nu'(r) + \frac{2}{r}\nu'(r)\big],$$

so we need

$$\nu''(r) + \nu'(r)^2 + \frac{2}{r}\nu'(r) = \frac{2\kappa q^2}{r^4} e^{-\nu(r)}. \tag{5.14}$$

This can be obtained from (5.10) in a fashion similar to the deduction of (2.60) from (2.58), so we can conclude that (5.8), (5.9), (5.13) is our desired spherically symmetric solution to the Maxwell–Einstein equations (5.1).

Exercises

1. Using the method of §§ 3 and 4, study the timelike geodesics (i.e., possible paths of an uncharged particle in free fall) for a spacetime with the Reissner–Nordström metric (5.13).

Exercises 2–4 deal with the Kerr–Newman metric, given in (t, r, θ, φ)-coordinates by

$$ds^2 = -\frac{A}{\rho^2}\big[dt - a\sin^2\varphi\, d\theta\big]^2 + \frac{\rho^2}{A}\, dr^2 + \rho^2\, d\varphi^2 + \frac{\sin^2\varphi}{\rho^2}\big[(r^2+a^2)\, d\theta - a\, dt\big]^2, \tag{5.15}$$

where

$$A = r^2 - Kr + a^2 + Q^2, \quad \rho^2 = r^2 + a^2\cos^2\varphi.$$

Here, $K > 0$, a, and Q are constants. Note that the case $a = 0$ gives the Reissner–Nordström metric (5.13) while the case $Q = 0$ gives the Kerr metric (3.56).

2. Show that (5.15), together with

$$\begin{aligned} \mathcal{F} = {} & \frac{Q}{\rho^4}(r^2 - a^2\cos^2\varphi)\, dr \wedge (dt - a\sin^2\varphi\, d\theta) \\ & + \frac{2Q}{\rho^4} ar(\cos\varphi)(\sin\varphi)\, d\varphi \wedge \big[(r^2+a^2)\, d\theta - a\, dt\big], \end{aligned} \tag{5.16}$$

provides a solution to (5.1).

3. Show that (5.15) is stationary, but not static, if $a \neq 0$.
4. Study the timelike geodesics on a spacetime with the metric (5.15).

6. Relativistic fluids

In general relativity, the motion of an ideal fluid is governed by Einstein's equation

$$G_{jk} = 8\pi\kappa T_{jk}, \tag{6.1}$$

where T_{jk} has the form

$$T_{jk} = (\rho + p)u_j u_k + p g_{jk}. \tag{6.2}$$

This is the stress-energy tensor of a fluid, with 4-velocity u, satisfying $\langle u, u\rangle = -1$. The pressure is p, and the density is ρ, both of these quantities being measured by an observer traveling with velocity u.

The condition that $\operatorname{div} G = 0$ leads to fluid equations. In fact, a computation gives

$$\operatorname{div} T = (\rho + p)\nabla_u u + \mathcal{L}_u(\rho + p)\, u + (\rho + p)(\operatorname{div} u)u + \operatorname{grad} p. \tag{6.3}$$

Note that, since $\langle u, u\rangle = -1,\ u \perp \nabla_u u$. Thus we can separate (6.3) into components orthogonal to and parallel to u and conclude that $\operatorname{div} T = 0$ if and only if

$$\begin{aligned} (\rho + p)\nabla_u u &= -\Pi(u) \operatorname{grad} p, \\ \operatorname{div}(\rho u) &= -p \operatorname{div} u, \end{aligned} \tag{6.4}$$

where $\Pi(u)$ denotes projection orthogonal to u with respect to the Lorentz metric. The case $p = 0$ is that of a *dust*; then (6.4) reduces to

$$\nabla_u u = 0, \quad \operatorname{div}(\rho u) = 0. \tag{6.5}$$

For an isentropic fluid, the pressure p is a function of ρ, so there is an equation of state

$$p = p(\rho). \tag{6.6}$$

One can compare (6.4) with the nonrelativistic fluid equations, (5.12)–(5.13), of Chap. 16, which are, (with $X''(\rho) = \nabla p/\rho$),

$$\begin{aligned} \rho\Big(\frac{\partial v}{\partial t} + \nabla_v v\Big) &= -\nabla p, \\ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) &= 0. \end{aligned} \tag{6.7}$$

This is an approximation to (6.4) when $g_{jk} \approx \eta_{jk}$ (the Minkowski metric), $u \approx (1, v),\ |v| << 1$, and $|p| << \rho$.

As in the study of nonrelativistic fluids in Chaps. 16 and 17, it is of interest to consider the vorticity of a fluid flow. First, if $\tilde{u}$ is the 1-form corresponding to u via the Lorentz metric, we consider the 2-form

$$\tilde{\xi} = d\tilde{u}. \tag{6.8}$$

We can express this in terms of the linear map

$$\Xi_u : T_pM \to T_pM, \quad \Xi_u X = -\nabla_X u. \tag{6.9}$$

In fact,

$$\tilde{\xi}(X,Y) = \langle \nabla_X u, Y\rangle - \langle \nabla_Y u, X\rangle = -\langle (\Xi_u - \Xi_u^*)X, Y\rangle. \tag{6.10}$$

In particular, $\tilde{\xi} = 0$ if and only if $\Xi_u = \Xi_u^*$. Note that since $\langle u,u\rangle = -1$, we have $0 = X\langle u,u\rangle = 2\langle \nabla_X u, u\rangle$, so $\langle \Xi_u X, u\rangle = 0$, that is,

$$\Xi_u : T_pM \to \Sigma_p, \quad \Sigma_p = (u_p)^\perp. \tag{6.11}$$

We also define

$$A_u : \Sigma_p \to \Sigma_p, \quad A_u = \Xi_u\big|_{\Sigma_p}. \tag{6.12}$$

Part of the significance of A_u is in determining whether the subbundle Σ of TM is integrable, as shown by the following:

Lemma 6.1. *The bundle Σ is integrable if and only if $A_u = A_u^*$.*

Proof. If X and Y are sections of Σ, then

$$\tilde{u}\big([X,Y]\big) = -d\tilde{u}(X,Y) = \langle \Xi_u X, Y\rangle - \langle X, \Xi_u Y\rangle, \tag{6.13}$$

the last identity by (6.10). By (6.11)–(6.12), we obtain

$$\tilde{u}\big([X,Y]\big) = \big\langle (A_u - A_u^*)X, Y\big\rangle, \tag{6.14}$$

whenever X and Y are sections of Σ. The lemma follows, by Frobenius's theorem.

It is useful to remark that the following formula holds:

$$\operatorname{div} u = -\operatorname{Tr} A_u, \tag{6.15}$$

whenever $\langle u,u\rangle = -1$. To see this, pick $\{e_j : 1 \le j \le 3\}$ to give a local orthonormal frame field for Σ. We have

$$\operatorname{div} u = \sum_{j=1}^{3} \langle \nabla_{e_j} u, e_j\rangle - \langle \nabla_u u, u\rangle = \sum_{j=1}^{3} \langle \nabla_{e_j} u, e_j\rangle, \tag{6.16}$$

the last identity holding since $2\langle \nabla_u u, u\rangle = u\langle u,u\rangle = 0$. This gives (6.15).

The vorticity is the vector field W, defined as follows. If ω is the volume form on M (a 4-form), W is uniquely specified by the identity

$$\iota_W \omega = \tilde{u} \wedge d\tilde{u}. \tag{6.17}$$

Note that if we wedge both sides of (6.17) with $\tilde{u}$ and use the anticommutator identity $\wedge_{\tilde{u}}\iota_W + \iota_W \wedge_{\tilde{u}} = \langle W, \tilde{u}\rangle I$, we obtain

$$(6.18) \qquad \langle W, \tilde{u}\rangle = 0, \quad \text{i.e., } W_p \in \Sigma_p.$$

We can restate Lemma 6.1 in terms of the behavior of W:

Lemma 6.2. *The bundle Σ is integrable if and only if $W = 0$.*

Proof. By (6.13), we see that Σ is integrable if and only if

$$(6.19) \qquad d\tilde{u}(X, Y) = 0, \qquad \forall\ X, Y \in \Sigma_p,$$

for all $p \in M$. If we pick the basis $\{f_0 = \tilde{u}, f_1, f_2, f_3\}$ of T_p^*M to be the dual basis to $\{u, e_1, e_2, e_3\}$, and write $d\tilde{u}(p)$ as a linear combination of $f_j \wedge f_k$, we see that (6.19) holds if and only if $d\tilde{u}(p) = \tilde{u} \wedge \alpha$, for some $\alpha \in T_p^*$; in turn this holds if and only if $\tilde{u} \wedge d\tilde{u} = 0$, which holds if and only if $W = 0$.

We can derive a "vorticity equation," via calculations parallel to those used in (5.21)–(5.26) of Chap. 16. First, (6.3)–(6.4) imply

$$(6.20) \qquad (\rho + p)\nabla_u \tilde{u} + (\mathcal{L}_u p)\tilde{u} + dp = 0.$$

Next, we have $\mathcal{L}_u \tilde{u} - \nabla_u \tilde{u} = (1/2)d\langle u, u\rangle = 0$, so

$$(6.21) \qquad \mathcal{L}_u \tilde{u} + B\tilde{u} + dq = 0,$$

where, assuming $p = p(\rho)$,

$$(6.22) \qquad dq = \frac{dp}{\rho + p}, \quad B = \frac{\mathcal{L}_u p}{\rho + p} = \mathcal{L}_u q.$$

Applying the exterior derivative to (6.21) yields

$$(6.23) \qquad \mathcal{L}_u \tilde{\xi} = -d(B\tilde{u}).$$

We next produce an equation for $\mathcal{L}_u(\tilde{u} \wedge \tilde{\xi})$. If we start with

$$\mathcal{L}_u(\tilde{u} \wedge \tilde{\xi}) = \tilde{u} \wedge \mathcal{L}_u \tilde{\xi} + (\mathcal{L}_u \tilde{u}) \wedge \tilde{\xi},$$

and use (6.21)–(6.23), we obtain

$$(6.24) \qquad \mathcal{L}_u(\tilde{u} \wedge \tilde{\xi}) = -d(q\tilde{\xi}).$$

We can also produce an equation involving $\mathcal{L}_u W$. Note the following characterization of W, equivalent to (6.17):

$$(\tilde{u} \wedge \tilde{\xi}) \wedge \alpha = \langle W, \alpha \rangle \omega, \tag{6.25}$$

for every 1-form α. Beginning with

$$\mathcal{L}_u(\tilde{u} \wedge \tilde{\xi}) \wedge \alpha = \mathcal{L}_u(\tilde{u} \wedge \tilde{\xi} \wedge \alpha) - \tilde{u} \wedge \tilde{\xi} \wedge \mathcal{L}_u \alpha,$$

and making a computation parallel to that of (5.25)–(5.26) of Chap. 16, one obtains

$$\mathcal{L}_u W + (\operatorname{div} u) W = -\vartheta(q\tilde{\xi}), \tag{6.26}$$

where the vector field $\vartheta(q\tilde{\xi})$ is uniquely defined by

$$\langle \vartheta(q\tilde{\xi}), \alpha \rangle \omega = d(q\tilde{\xi}) \wedge \alpha, \tag{6.27}$$

for all 1-forms α. The equation (6.26) can be compared with (5.26) of Chap. 16; the primary difference is that the right side of (5.26) is zero.

Note that $-d(q\tilde{\xi}) = d(\tilde{u} \wedge dq)$ and $-\vartheta(q\tilde{\xi}) = \vartheta(\tilde{u} \wedge dq)$, so another way to write (6.24) is as

$$\mathcal{L}_u(\tilde{u} \wedge \tilde{\xi}) = d\Big(\frac{\tilde{u} \wedge dp}{\rho + p}\Big), \tag{6.28}$$

and another way to write (6.26) is as

$$\mathcal{L}_u W + (\operatorname{div} u) W = \vartheta\Big(\frac{\tilde{u} \wedge dp}{\rho + p}\Big). \tag{6.29}$$

We can produce a vorticity equation of A. Lichnerowicz as follows. Note that

$$(\mathcal{L}_u + B)\tilde{u} = e^{-q} \mathcal{L}_u(e^q \tilde{u}), \tag{6.30}$$

so (6.21) yields

$$\mathcal{L}_u(e^q \tilde{u}) + d(e^q) = 0, \tag{6.31}$$

and applying the exterior derivative gives

$$\mathcal{L}_u \Omega = 0, \quad \Omega = d\tilde{w}, \quad \tilde{w} = e^q \tilde{u}. \tag{6.32}$$

Note that $\Omega = e^q(\tilde{\xi} + dq \wedge \tilde{u})$, and hence

$$\tilde{u} \wedge \Omega = e^q \tilde{u} \wedge \tilde{\xi}. \tag{6.33}$$

We can rewrite the form (6.31) of the first part of the Euler equations (6.4), using $\mathcal{L}_u(e^q\tilde{u}) = \iota_u d(e^q\tilde{u}) - d(e^q)$. Thus (6.31) is equivalent to

$$\iota_u \Omega = 0. \tag{6.34}$$

The second part of the Euler equations (6.4) can be rewritten as

$$\operatorname{div} u = -\frac{\mathcal{L}_u \rho}{\rho + p}, \tag{6.35}$$

which is also equivalent to

$$\operatorname{div} w = \mathcal{L}_w \Psi(\rho), \tag{6.36}$$

with

$$w = e^q u, \quad \Psi(\rho) = \log e^{-2q}(\rho + p). \tag{6.37}$$

This in turn is equivalent to

$$d^\star \tilde{w} = \mathcal{L}_w \Psi. \tag{6.38}$$

Note that if we multiply (6.34) by e^q and then apply the exterior derivative, we obtain the following variant of (6.32):

$$\mathcal{L}_w \Omega = 0. \tag{6.39}$$

One relation between $\tilde{u} \wedge \tilde{\xi}$ and Ω is given in (6.33). If the Euler equation, in the form (6.34), holds, we can deduce another relation, via the anticommutator relation $\wedge_{\tilde{u}}\iota_u + \iota_u\wedge_{\tilde{u}} = -I$. Applying this to Ω and using (6.33), we obtain

$$\iota_u\Omega = 0 \Longrightarrow \Omega = -e^q \ \iota_u(\tilde{u} \wedge \tilde{\xi}). \tag{6.40}$$

Putting (6.33) and (6.40) together, we get

$$\Omega = 0 \Longleftrightarrow \tilde{u} \wedge \tilde{\xi} = 0, \tag{6.41}$$

whenever Ω satisfies (6.34). This enables us to prove the following:

Proposition 6.3. *Assume u solves the relativistic Euler equation. Let S be a spacelike hypersurface, and assume the vorticity W vanishes on $\mathcal{O} \subset S$. Then W vanishes on the union $\mathcal{U}$ of the integral curves of u through points in $\mathcal{O}$.*

Proof. That Ω vanishes on $\mathcal{U}$ follows from (6.41) and (6.39); applying (6.41) again, we have $\tilde{u} \wedge \tilde{\xi} = 0$ on $\mathcal{U}$, hence $W = 0$ on $\mathcal{U}$.

Note incidentally, that, when $\iota_u \Omega = 0$,

$$\Omega = e^q \iota_W \iota_u \omega. \tag{6.42}$$

Next we will derive a second-order PDE for $\tilde{w}$. To do this, we use (6.32) and (6.38) to compute $\Box \tilde{w}$, where $\Box = -dd^\star - d^\star d$. We have

$$\Box \tilde{w} = -d(\mathcal{L}_w \Psi) - d^\star \Omega. \tag{6.43}$$

It is convenient to write

$$\Psi = \Phi(e^{2q}), \qquad e^{2q} = -\langle w, w \rangle, \tag{6.44}$$

so

$$\mathcal{L}_w \Psi = -2\Phi'(-\langle w, w \rangle)\ \langle \nabla_w w, w \rangle. \tag{6.45}$$

Since $\big\langle X, d\langle w, w \rangle \big\rangle = X\langle w, w \rangle = 2\langle \nabla_X w, w \rangle$, we have

$$d\langle w, w \rangle = 2\iota_w \nabla \tilde{w}. \tag{6.46}$$

Similarly,

$$d\langle \nabla_w w, w \rangle = \iota_w \nabla(\nabla_w \tilde{w}) + \iota_{(\nabla_w w)} \nabla \tilde{w}, \tag{6.47}$$

so

$$d(\mathcal{L}_w \Psi) = -2\Phi'\ \iota_w \nabla(\nabla_w \tilde{w}) + \mathcal{A}(w, \nabla w), \tag{6.48}$$

where

$$\mathcal{A}(w, \nabla w) = -2\Phi'\ \iota_{(\nabla_w w)} \nabla \tilde{w} + 4\Phi''\ \langle \nabla_w w, w \rangle\ \iota_w \nabla \tilde{w}. \tag{6.49}$$

Hence we have the coupled system

$$\Box \tilde{w} - 2\Phi'\ \iota_w \nabla(\nabla_w \tilde{w}) + \mathcal{A}(w, \nabla w) = -d^\star \Omega, \qquad \mathcal{L}_w \Omega = 0. \tag{6.50}$$

A computation using (6.44), (6.37), and (6.22) gives

$$\Phi' = \frac{\rho'(p) - 1}{2e^{2q}}. \tag{6.51}$$

In component notation, $Q = \iota_w \nabla(\nabla_w \tilde{w})$ is given by

$$Q_\ell = (w^j w^k w_{j;k})_{;\ell} = w^j w^k w_{j;k;\ell} + R_\ell, \tag{6.52}$$

where R_ℓ involves only first-order derivatives. Now

$$\iota_w \Omega = 0 \Longrightarrow w^j w_{j;\ell} = w^j w_{\ell;j}. \tag{6.53}$$

Using this, one sees that

$$Q_\ell = w^j w^k w_{\ell;j;k} + \widetilde{R}_\ell, \tag{6.54}$$

where $\widetilde{R}_\ell$ involves only first-order derivatives. Thus, we replace the system (6.50) by

$$\Box \tilde{w} - 2\Phi' \nabla^2_{w,w} \tilde{w} + \widetilde{\mathcal{A}}(w, \nabla w) = -d^\star \Omega, \qquad \mathcal{L}_w \Omega = 0. \tag{6.55}$$

The left side of the first equation in (6.55) is a second-order, quasi-linear operator acting on $\tilde{w}$; its principal symbol is scalar, and provided $\rho'(p) \geq 1$ (i.e., provided $p'(\rho) \leq 1$), it is hyperbolic, and every hypersurface that is spacelike for the Lorentz metric g_{jk} is also spacelike for this operator. Of course, we have $-d^\star \Omega$ on the right, and a second equation involving w and Ω. Since $\mathcal{L}_w \Omega$ involves first-order derivatives of w as well as of Ω, the question of well-posedness of the initial-value problem for (6.55) requires further investigation. Following [CBr3], we clarify this by applying ∇_w to both sides of the first equation.

Since the operator ∇_w has scalar principal symbol,

$$\nabla_w d^\star \Omega = d^\star \nabla_w \Omega + \mathcal{B}_0(w, \nabla w, \nabla \Omega). \tag{6.56}$$

Meanwhile,

$$(\nabla_w \Omega)(X, Y) = (\mathcal{L}_w \Omega)(X, Y) - \Omega(\nabla_X w, Y) - \Omega(X, \nabla_Y w), \tag{6.57}$$

so

$$\mathcal{L}_w \Omega = 0 \Longrightarrow \nabla_w d^\star \Omega = \mathcal{B}(D^2 w, \nabla \Omega). \tag{6.58}$$

Thus we replace the system (6.55) by

$$\nabla_w (\Box - 2\Phi' \nabla^2_{w,w}) \tilde{w} - \widetilde{\mathcal{B}}(D^2 w, \nabla \Omega) = 0, \quad \mathcal{L}_w \Omega = 0, \tag{6.59}$$

which is analytically more tractable. Note that the first equation here contains no higher derivatives on Ω than the first equation of (6.55).

Now the fluid velocity is also coupled to the gravitational field, via (6.1)–(6.2), so all of these equations have to be treated simultaneously. We will discuss this further in § 8. In preparation for that, let us mention that the first equation in (6.59), when written in local coordinates, involves the metric tensor and derivatives of the metric tensor, up to *third order*.

We next construct some examples of static, spherically symmetric solutions to (6.1)–(6.2), which provide models for stable stars. We look for a solution involving a metric of the form

$$ds^2 = -e^{\nu(r)}\,dt^2 + e^{\lambda(r)}\,dr^2 + r^2\,d\omega^2, \tag{6.60}$$

and use $x_0 = t,\ x_1 = r$, as in § 2. For the fluid to be static, we need

$$u^0 = e^{-\nu/2}, \quad u^1 = u^2 = u^3 = 0, \tag{6.61}$$

so

$$T_{jk} = (\rho + p)e^{\nu}\,\delta_{j0}\delta_{k0} + pg_{jk}. \tag{6.62}$$

Using (2.51)–(2.55), we see that (6.1) is equivalent to the following set of equations, recording $G_{jj} = 8\pi\kappa T_{jj}$ for $j = 0, 1$, and 2, respectively:

$$\begin{aligned} 1 - r\lambda_r - e^{\lambda} &= -8\pi\kappa\rho r^2 e^{\lambda}, \\ 1 + r\nu_r - e^{\lambda} &= 8\pi\kappa p r^2 e^{\lambda}, \\ \nu_{rr} + \frac{1}{2}\nu_r^2 + \frac{1}{r}(\nu_r - \lambda_r) - \frac{1}{2}\nu_r\lambda_r &= 16\pi\kappa p\, e^{\lambda}. \end{aligned} \tag{6.63}$$

If we assume p and ρ are related by an equation of state, $p = p(\rho)$, as in (6.6), then (6.63) is a system of three equations, in three unknowns: $\nu(r), \lambda(r)$, and $p(r)$. The system can be simplified a bit.

If we apply $e^{\lambda}(d/dr)e^{-\lambda}$ to the middle equation in (6.63) and subtract r times the last equation, we get

$$\nu'(\nu' + \lambda') = -16\pi\kappa r e^{\lambda} p'. \tag{6.64}$$

Meanwhile, taking the difference of the first two equations in (6.63) gives

$$\nu'(\nu' + \lambda') = 8\pi\kappa r e^{\lambda}(p + \rho)\nu'. \tag{6.65}$$

Comparing these two equations, we have

$$p'(r) = -\frac{1}{2}(p + \rho)\nu'(r). \tag{6.66}$$

It is instructive to rederive this last equation, as a consequence of the vanishing of $T^{jk}{}_{;k}$. In fact, if u is given by (6.61), then div $u = 0$, so by (6.3) the vanishing of div T is equivalent to

$$(\rho + p)\nabla_u u + \mathcal{L}_u(\rho + p)\, u + \text{ grad } p = 0. \tag{6.67}$$

Now $u^j{}_{;k} = \partial_k u^j + \Gamma^j{}_{\ell k} u^\ell$, where $\Gamma^j{}_{\ell k}$ is given by (2.9). When u has the form (6.61), then $u^k u^j{}_{;k} = u^0 u^j{}_{;0} = (\partial_0 u^j + \Gamma^j{}_{00} u^0) u^0$. Also, by (2.9), $\Gamma^j{}_{00} = {}^U\Gamma^j{}_{00} + B^j{}_{00}$, and by (2.10), $B^j{}_{00} = 0$, while (2.43) implies

$$(6.68) \qquad {}^U\Gamma^0{}_{00} = 0, \quad {}^U\Gamma^1{}_{00} = \frac{1}{2}\nu_r e^{\nu-\lambda}.$$

Thus, when u has the form (6.61),

$$(6.69) \qquad u^k u^j{}_{;k} = \frac{1}{2}\nu_r e^{-\lambda}\,\delta_{j1}.$$

Now, in the static, spherically symmetric case, $\rho = \rho(r)$ and $p = p(r)$, so clearly $\mathcal{L}_u(\rho + p) = u^0 \partial_0(\rho + p) = 0$, and the only nontrivial component of the left side of (6.67) is

$$(6.70) \qquad T^{1k}{}_{;k} = \frac{1}{2}(\rho + p)\nu'(r)e^{-\lambda} + p'(r)e^{-\lambda}.$$

Thus we again derive (6.66).

We can use (6.66) to eliminate ν_r from the second equation in (6.63). The result, together with the first equation in (6.63), gives a 2×2 system for $\lambda(r)$ and $p(r)$:

$$(6.71) \qquad \begin{aligned} \lambda'(r) &= \frac{1 - e^\lambda}{r} - 8\pi\kappa r e^\lambda \rho, \\ \frac{2}{p + \rho} p'(r) &= \frac{1 - e^\lambda}{r} - 8\pi\kappa r e^\lambda p, \end{aligned}$$

under the hypothesis that $\rho = \rho(p)$. It is common to replace $\lambda(r)$ by a function giving the metric (6.60) a form more resembling (2.61). We define $M(r)$ by

$$(6.72) \qquad e^{\lambda(r)} = \Bigl(1 - \frac{2M(r)}{r}\Bigr)^{-1},$$

so that $M(r) = r(1 - e^{-\lambda})/2$. The system (6.71) takes the form

$$(6.73) \qquad M'(r) = 4\pi\kappa r^2 \rho, \quad p'(r) = -\frac{(p + \rho)(M + 4\pi\kappa r^3 p)}{r(r - 2M)},$$

known as the *Oppenheimer–Volkov equation*.

In the Newtonian limit, $p << \rho$, $4\pi\kappa r^3 p << M$, and $2M << r$, and these equations become

$$(6.74) \qquad M'(r) = 4\pi\kappa r^2 \rho, \quad p'(r) = -\frac{\rho M}{r^2}.$$

In fact, this is precisely the equation for a static fluid in Newtonian mechanics, in which the force of gravity exactly balances the force due to the pressure gradient. In such a case, $M(r)$ is the gravitational mass of the matter enclosed in the ball $\{|x| \le r\}$ in $\mathbb{R}^3$. The relation between density and gravitational mass, given by the first equation of (6.74) (in the limit when Newtonian mechanics applies) serves to identify the constant κ in Einstein's equation (1.1), with the gravitational constant of Newtonian theory.

The Oppenheimer–Volkov system (6.73) has consequences significantly different from the Newtonian approximation (6.74), for very dense objects. For example, it leads to theoretical upper bounds on the mass of a stable neutron star which are stronger than those obtainable from (6.74). Discussions of this can be found in [Str, Wa, Wein].

In treating (6.73), it is natural to set $M(0) = 0$ and let $p(0) = p_0$ run over a range of values. We assume that $p'(\rho) > 0$ in the equation of state, so $\rho = \rho(p)$ in (6.73), with $\rho'(p) > 0$. Despite the vanishing of the denominator in the second equation of (6.73) at $r = 0$, there is no real singularity there. Indeed, one easily verifies that

$$\begin{aligned} M(r) &= \frac{4\pi\kappa}{3}\rho_0 r^3 + O(r^5), \\ p(r) &= p_0 - \frac{2\pi\kappa}{3}(p_0 + \rho_0)(3p_0 + \rho_0)r^2 + O(r^4), \end{aligned} \tag{6.75}$$

with $\rho_0 = \rho(p_0)$. For a numerical treatment of (6.73), it is convenient to use (6.75) for r very small, and then use a difference scheme, to produce an approximate solution for larger r.

Exercises

1. Assume $u(p) \neq 0$ and $W(p) \neq 0$. Using (6.42), show that the linear span $\mathcal{L}_p$ of $u(p)$ and $W(p)$ is given by
$$\mathcal{L}_p = \{v \in T_pM : \iota_v\Omega = 0\}.$$
Using (6.32), show that the resulting subbundle $\mathcal{L}$ of TM is invariant under the flow generated by u (in regions where u and W are both nonvanishing). In light of this, derive analogues of the Kelvin and Helmholtz theorems, established for nonrelativistic fluids in § 5 of Chap. 16 and § 1 of Chap. 17.
2. Consider a static, spherically symmetric, *charged* fluid and associated electromagnetic field. Discuss the equations of motion.
3. Compute the second terms in the power-series expansions of $M(r)$ and of $p(r)$ about $r = 0$ in (6.75), namely, the coefficients of r^5 and of r^4, respectively.
4. Write some computer programs to solve numerically the Oppenheimer–Volkov system (6.73), with initial data $M(0) = 0,\ p(0) = p_0$. Try various equations of state, such as

$$p(\rho) = k\rho^{4/3}, \tag{6.76}$$

with k = const., used in models of white dwarf stars. For another example, fix $\rho_0 \in (0, \infty)$, and use

$$\begin{aligned} p(\rho) &= \tfrac{1}{3}\rho, \qquad \text{for } \rho \geq \rho_0, \\ p(\rho) &= k\rho^{4/3}, \qquad \text{for } \rho \leq \rho_0, \end{aligned} \tag{6.77}$$

with k picked so $\rho_0/3 = k\rho_0^{4/3}$ (i.e., $k = \rho_0^{-1/3}/3$).

See [Str] and [Wein] for discussions of variants of (6.77) used in models of neutron stars.

5. Suppose the equation of state were

$$p(\rho) = \frac{\rho}{3} \tag{6.78}$$

for all $\rho \in \mathbb{R}^+$. Produce a solution to (6.73) of the form

$$M(r) = Ar, \quad p(r) = Br^{-1}, \tag{6.79}$$

for certain constants A, B. Relate this to the assertion that (6.78) cannot be a realistic equation of state at low density.

7. Gravitational collapse

In many cases, solutions to Einstein's equations, particularly coupled to matter, develop singularities in finite time, sometimes as part of the phenomenon of gravitational collapse. We begin this section with some simple examples in which gravitational collapse occurs.

Let us consider a homogeneous, isotropic universe, containing a fluid with uniform density and pressure. We write the metric as

$$ds^2 = -dt^2 + A(t)g^S, \tag{7.1}$$

where g^S is a constant-curvature metric on a 3-manifold S. The stress-energy tensor has the form (6.2), with

$$\rho = \rho(t), \quad p = p(\rho), \quad u = (1, 0, 0, 0). \tag{7.2}$$

We can compute the Einstein tensor of this metric using formulas from § 2. We have $M = U \times S$, where $\dim U = 1$ and $\dim S = 3$. From (2.22) we have

$$\mathrm{Ric}_{jk} = \mathrm{Ric}^S_{jk} + F_{jk}, \tag{7.3}$$

and F_{jk} is given by (2.28), with $\vartheta = \log A(t)$. Hence (keeping in mind (2.20)) we have

$$F_{00} = -\frac{3}{2}A^{-2}\Big\{AA'' - \frac{1}{2}(A')^2\Big\}, \tag{7.4}$$

and, for $1 \le j, k \le 3$,

(7.5)
$$F_{jk} = -\frac{1}{2} A^{-1} \Big\{ -AA'' - \frac{1}{2} (A')^2 \Big\} g^S_{jk}.$$

By (2.29), the scalar curvature of M is

(7.6)
$$S = A^{-1} S_S + \beta,$$

where

(7.7)
$$\beta = g^{jk} F_{jk} = 3A^{-1} A''.$$

Then, by (2.36), the Einstein tensor of M is given by

(7.8)
$$G_{jk} = G^S_{jk} + \frac{1}{2} A^{-1} S_S\, \delta_{j0} \delta_{k0} + F_{jk} - \frac{1}{2} \beta g_{jk}.$$

In particular,

(7.9)
$$G_{00} = \frac{1}{2} A^{-1} S_S + F_{00} + \frac{1}{2} \beta = \frac{1}{2} A^{-1} S_S + \frac{3}{4} (A^{-1} A')^2,$$

and, for $1 \le j, k \le 3$,

(7.10)
$$G_{jk} = G^S_{jk} + F_{jk} - \frac{3}{2} A'' g^S_{jk} = G^S_{jk} - \Big\{ A'' - \frac{1}{4} A^{-1} (A')^2 \Big\} g^S_{jk}.$$

Now, if S has constant sectional curvature (and hence constant scalar curvature S_S), then ${}^S\mathrm{Ric}^j{}_k$ must be a scalar multiple of $\delta^j{}_k$, and the multiple must be $S_S/3$, so

(7.11)
$$G^S_{jk} = -\frac{1}{6} S_S g^S_{jk}.$$

If T_{jk} is given by (7.2), then Einstein's equations yield the following pair of equations for $A(t)$ and $\rho(t)$:

(7.12)
$$\begin{aligned} \frac{3}{4} \Big(\frac{A'}{A} \Big)^2 + \frac{1}{2} \frac{S_S}{A} &= 8\pi\kappa\rho, \\ A'' - \frac{1}{4} \frac{(A')^2}{A} + \frac{S_S}{6} &= -8\pi\kappa A p(\rho). \end{aligned}$$

To put this in a slightly different form, we set $A(t) = R(t)^2$ and note that if S has constant sectional curvature K, then $S_S = 6K$. The we can rewrite (7.12) as

$$(7.13)\qquad \begin{aligned} (R')^2 + K &= \frac{8}{3}\pi\kappa\rho R^2, \\ 2RR'' + (R')^2 + K &= -8\pi\kappa p(\rho)R^2. \end{aligned}$$

It is useful to perform some elementary operations on these equations. Note that taking the difference yields

$$(7.14)\qquad 3\frac{R''}{R} = -4\pi\kappa(\rho + 3p),$$

while multiplying the first equation in (7.13) by 3 and taking the difference yields

$$(7.15)\qquad \frac{d}{dt}\Big(\frac{R'}{R}\Big) = \frac{K}{R^2} - 4\pi\kappa(\rho + p).$$

On the other hand, applying d/dt to the first part of (7.13) gives

$$(7.16)\qquad \frac{d\rho}{dt} = -6K\frac{R}{R^3} + 6\frac{R'}{R}\frac{d}{dt}\Big(\frac{R'}{R}\Big),$$

and substituting (7.15) into (7.16) then gives

$$(7.17)\qquad \frac{d}{dt}(\rho R^3) = -p\frac{d}{dt}(R^3).$$

One can also deduce (7.17) from the identity $T^{jk}{}_{;k} = 0$ (with $j = 0$), in a fashion analogous to the derivation of (6.66) via (6.67)–(6.70). In turn, (7.17) implies the relation

$$(7.18)\qquad \frac{dR}{R} = -\frac{1}{3}\frac{d\rho}{\rho + p(\rho)},$$

which gives R as a function of ρ, or ρ as a function of R.

Let us fix $R_0 = R(t_0)$ and $\rho_0 = \rho(t_0)$. We can now regard (7.14) as a dynamical equation for R:

$$(7.19)\qquad R'' = -\frac{4}{3}\pi\kappa\varphi(R), \quad \varphi(R) = (\rho + 3p)R,$$

given ρ and $p(\rho)$ as functions of R, and then the first part of (7.13) can be regarded as the conservation law:

$$(7.20)\qquad \frac{1}{2}(R')^2 - \frac{4}{3}\pi\kappa\psi(R) = -K, \quad \psi(R) = \rho R^2.$$

In other words, if we write (7.19) as a first-order system:

$$R' = V, \quad V' = -\frac{4}{3}\pi\kappa\varphi(R), \tag{7.21}$$

then the orbits lie on level curves

$$F(V,R) = -K, \quad F(V,R) = \frac{1}{2}V^2 - \frac{4}{3}\pi\kappa\psi(R). \tag{7.22}$$

Thus we will examine these level curves.

To do this, we look at (7.18), which gives

$$R = R_0 e^{-\lambda(\rho)/3}, \quad \lambda(\rho) = \int_{\rho_0}^{\rho} \frac{d\zeta}{\zeta + p(\zeta)}. \tag{7.23}$$

If we assume that the equation of state satisfies

$$p(0) = 0, \quad p'(\rho) \le 1, \tag{7.24}$$

then (if say $\rho_0 = 1$) for $\rho \ge 1$, we have $(1/2)\log\rho \le \lambda(\rho) \le \log\rho$, so $R_0\rho^{-1/3} \le R \le R_0\rho^{-1/6}$, with reversed inequalities for $\rho \le 1$. Hence $(R_0/R)^3 \le \rho \le (R_0/R)^6$ for $\rho \ge 1$, so $\psi(R)$ has the property

$$R \le R_0 \Longrightarrow \frac{R_0^3}{R} \le \psi(R) \le \frac{R_0^6}{R^4}. \tag{7.25}$$

Similarly,

$$R \ge R_0 \Longrightarrow \frac{R_0^6}{R^4} \le \psi(R) \le \frac{R_0^3}{R}. \tag{7.26}$$

In Fig. 7.1 we depict the level curves of $F(V,R)$ and the resulting phase-plane portrait of the system (7.21). Note that all the orbits $(R(t), V(t))$ in the region $V < 0$ have the property $R(t) \searrow 0$, $V(t) \searrow -\infty$, as t increases. In particular, if $V(t_0) < 0$, then $R'(t)$ is bounded away from zero for $t \ge t_0$, so $R(t)$ must reach zero at a finite time $t_1 > t_0$! Similarly, if $V(t_0) > 0$, then $R(t)$ must vanish for some finite $t < t_0$. Of course, at $R = 0$, $\rho = +\infty$, and the metric is singular. If $K > 0$, one must have a singularity both at a finite time before t_0 and at a finite time after t_0. If $K \ge 0$, there must be such a singularity either at some finite $t < t_0$ or at some finite $t > t_0$.

That such complete collapse must occur is not surprising in the case of a dust, where $p = 0$. However, it is striking that, given any realistic equation of state, the pressure cannot prevent the collapse to infinite density, even in the case $K > 0$ and the total amount of matter in the universe is finite.

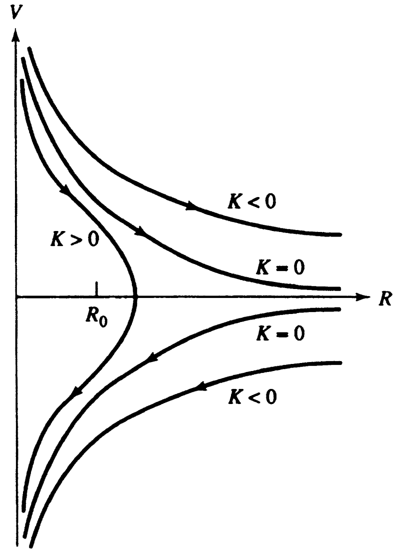

FIGURE 7.1 Orbits for (7.21)

One can cut and paste a portion of some of the spacetimes described above with a portion of Schwarzschild spacetime to give a model of collapse of a star, with spherical symmetry. The collapse of a rotating star is much more complicated. For further discussion, see [MS] and references given therein. It is worth mentioning the widely held belief that such a collapse, generally accompanied by gravitational radiation, should rapidly approximate a Kerr solution.

There are a number of general results on the inevitability of gravitational collapse, accompanied by singularity formation. A detailed treatment is given in [HE], and we mention only one relatively simple case here. We show that under certain mild conditions, an irrotational dust must give rise to a singularity in spacetime. We begin with a pair of geometrical lemmas.

Lemma 7.1. *If u is a vector field satisfying*

$$\langle u, u\rangle = -1, \quad \nabla_u u = 0, \quad A_u = A_u^*, \tag{7.27}$$

then

$$\mathcal{L}_u(div\ u) = -Ric(u,u) - Tr\ A_u^2. \tag{7.28}$$

Proof. Let $\{e_j : 1 \le j \le 3\}$ be a local orthonormal frame field for Σ, the bundle of 3-planes orthogonal to u. Then (6.16) yields

$$\mathcal{L}_u(\text{div}\ u) = \langle \nabla_u \nabla_{e_j} u, e_j\rangle + \langle \nabla_{e_j} u, \nabla_u e_j\rangle. \tag{7.29}$$

Here and below, we use the summation convention. Now write the first term on the right side of (7.29) as

$$
\begin{aligned}
\langle \nabla_u \nabla_{e_j} u, e_j \rangle &= \langle \nabla_{e_j} \nabla_u u, e_j \rangle + \langle \nabla_{[u,e_j]} u, e_j \rangle + \langle R(u, e_j) u, e_j \rangle \\
&= -\langle A_u [u, e_j], e_j \rangle - \operatorname{Ric}(u, u),
\end{aligned}
\tag{7.30}
$$

using $\nabla_u u = 0$. If $A_u = A_u^*$, we have

$$
\begin{aligned}
\langle \nabla_{e_j} u, \nabla_u e_j \rangle - \langle A_u [u, e_j], e_j \rangle &= -\langle A_u e_j, \nabla_u e_j + [u, e_j] \rangle \\
&= -\langle A_u e_j, A_u e_j \rangle - 2 \langle A_u e_j, \nabla_u e_j \rangle,
\end{aligned}
\tag{7.31}
$$

since $[u, e_j] = \nabla_u e_j - \nabla_{e_j} u$.

The expression $-\langle A_u e_j, A_u e_j \rangle$ (summed over j) is equal to $-\operatorname{Tr} A_u^2$ if $A_u^* = A_u$. Furthermore,

$$
\langle A_u e_j, \nabla_u e_j \rangle = A_{jk} \langle e_k, \nabla_u e_j \rangle,
$$

where (A_{jk}) is the matrix of A_u, with respect to the basis $\{e_j\}$, which is symmetric. Since $\langle e_k, \nabla_u e_j \rangle$ is antisymmetric in (j, k), we deduce that $\langle A_u e_j, \nabla_u e_j \rangle = 0$ (summed over j). This proves (7.28).

Recall from Lemma 6.1 that $A_u = A_u^*$ is an integrability condition, and, by Lemma 6.2, it is equivalent to vanishing vorticity. Note that if $A_u = A_u^*$, then

$$
\operatorname{Tr} A_u^2 \geq \frac{1}{3} (\operatorname{Tr} A_u)^2,
$$

as can be seen by putting A_u in diagonal form. Using (6.15), we can deduce the following:

Lemma 7.2. *Under the hypotheses of Lemma 7.1,*

$$
\mathcal{L}_u(\mathit{div}\, u) \leq -\frac{1}{3} (\mathit{div}\, u)^2 - \mathit{Ric}(u, u).
\tag{7.32}
$$

Proposition 7.3. *Suppose M is a spacetime containing a dust, so the Einstein equations hold, with T_{jk} given by*

$$
T_{jk} = \rho u_j u_k,
\tag{7.33}
$$

with $\rho \geq 0$ and $\langle u, u \rangle = -1$. Assume that the motion of the dust is irrotational. Finally, assume that, for some $p \in M$,

$$
\mathit{div}\, u(p) = -b < 0.
\tag{7.34}
$$

Let γ be the orbit of u (a unit-speed geodesic) such that $\gamma(0) = p$. If $\gamma(\tau)$ is defined for $\tau \in [0, a)$, then $a \le 3/b$. Furthermore, if $a = 3/b$, then

$$\rho\big(\gamma(\tau)\big) \to +\infty \quad \textit{and} \quad S\big(\gamma(\tau)\big) \to +\infty, \quad \textit{as } \tau \nearrow a. \tag{7.35}$$

Proof. Under our hypotheses, Corollary 7.2 applies, so we have (7.32). Also, by (1.61),

$$\operatorname{Ric}(u, u) = 4\pi\kappa\rho, \tag{7.36}$$

which is ≥ 0. Hence,

$$f(\tau) = \operatorname{div} u(\gamma(\tau)) \Longrightarrow f'(\tau) \le -\frac{1}{3} f(\tau)^2, \tag{7.37}$$

so the hypothesis (7.34) implies

$$f(\tau) \le -\frac{3b}{3 - b\tau}, \tag{7.38}$$

for $0 \le \tau < a$, provided $f(\tau)$ is smooth on $[0, a)$. This shows that $a \le 3/b$. Also, if $a = 3/b$, then $f(\tau) \to -\infty$ as $\tau \nearrow a$, in such a fashion that

$$\int_0^a f(\tau)\, d\tau = -\infty. \tag{7.39}$$

To conclude the proof of (7.35), note that, given a dust, by (6.5) we have $\operatorname{div}(\rho u) = 0$, hence $\mathcal{L}_u \rho + \rho \operatorname{div} u = 0$, or

$$\mathcal{L}_u(\log \rho) = -\operatorname{div} u, \quad \text{i.e.,} \quad \frac{d}{d\tau} \log \rho(\gamma(\tau)) = -f(\tau).$$

Hence (7.39) implies the first part of (7.35). By (1.59) we have

$$S = 8\pi\kappa\rho,$$

so the second part of (7.35) also holds.

Exercises

1. Obtain more explicit solutions to (7.13) for a dust (i.e., when $p = 0$). (*Hint*: Use (7.17).)
2. Assume $p = 0$. Let M have a metric of the form (7.1), by solving (7.12). Let B be a ball in the 3-manifold S, used in (7.1), and let Ω be the subset of M formed by timelike geodesics through ∂B, orthogonal to S. Glue Ω to an appropriate piece of Schwarzschild spacetime (whose boundary is also swept out by timelike geodesics) so as to model the collapse of a dust ball. In what sense do (6.1) and (6.2) hold on a neighborhood of the interface?

3. Consider the behavior of a homogeneous, isotropic universe filled with a uniform fluid (i.e., consider solutions to (7.12)), when one does not require that the equation of state satisfy $p'(\rho) \le 1$.

8. The initial-value problem

To begin, we look at the initial-value problem for the empty-space Einstein equations, which can be written as

$$\mathrm{Ric}_{jk} = 0. \tag{8.1}$$

In our first formulation of the initial-value problem, we try to prescribe, on a 3-dimensional surface $S = \{x_0 = 0\} \subset M$, the initial data

$$g_{jk}\big|_S = \overset{\circ}{g}_{jk}, \quad \partial_0 g_{jk}\big|_S = \overset{\circ}{k}_{jk}. \tag{8.2}$$

Here, $\partial_0 = \partial/\partial x_0$. Take M open in $\mathbb{R}^4$, $S = M \cap \{x_0 = 0\}$. We will see shortly that compatibility conditions will be required on these data. In local $(x_0, \dots, x_3)$-coordinates we have

$$\begin{aligned} \mathrm{Ric}_{jk} &= \frac{1}{2} g^{\ell m}\big[-\partial_\ell \partial_m g_{jk} - \partial_j \partial_k g_{\ell m} \\ &\qquad + \partial_k \partial_m g_{\ell j} + \partial_\ell \partial_j g_{km}\big] + M_{jk}(g, \nabla g) \\ &= \mathcal{L}_{jk}(g, D) g + M_{jk}(g, \nabla g). \end{aligned} \tag{8.3}$$

Now it is easy to see that $S = \{x_0 = 0\}$ is characteristic for $\mathcal{L}$. This results from the coordinate independence of the condition (8.1). We can get around this problem by choosing coordinate systems of a special nature.

One way to do this is the following, used by [CBr2], following [Lan]. Rewrite (8.3) as

$$\mathrm{Ric}_{jk} = -\frac{1}{2} g^{\ell m}\, \partial_\ell \partial_m g_{jk} + \frac{1}{2} g_{j\ell}\, \partial_k \lambda^\ell + \frac{1}{2} g_{k\ell}\, \partial_j \lambda^\ell + H_{jk}(g, \nabla g), \tag{8.4}$$

where

$$\lambda^\ell = g^{jk} \Gamma^\ell{}_{jk} = \frac{1}{2} g^{jk} g^{\ell m} \big(\partial_k g_{jm} + \partial_j g_{km} - \partial_m g_{jk}\big), \tag{8.5}$$

and the $\Gamma^\ell{}_{jk}$ are the Christoffel symbols for the metric tensor (g_{jk}), in the coordinates $(x_0, \dots, x_3)$. If $\Box$ is the Laplace–Beltrami operator defined by the Lorentz metric (g_{jk}), then

$$\Box u = g^{jk} u_{;j;k} = g^{jk}\, \partial_j \partial_k u - \lambda^\ell\, \partial_\ell u. \tag{8.6}$$

Hence

$$-\Box x_\ell = \lambda^\ell. \tag{8.7}$$

In other words, if the coordinates x_j satisfy $\Box x_j = 0$ (we call them "harmonic coordinates"), then $\lambda^\ell = 0$. Thus, in a harmonic coordinate system, Ric_{jk} is equal to

$$\widetilde{R}_{jk} = -\frac{1}{2} g^{\ell m}\, \partial_\ell \partial_m g_{jk} + H_{jk}(g, \nabla g). \tag{8.8}$$

At this point we can solve the initial-value problem

$$\begin{aligned} g^{\ell m}\, \partial_\ell \partial_m g_{jk} - 2H_{jk}(g, \nabla g) &= 0, \\ g_{jk}\big|_S = \overset{\circ}{g}_{jk}, \quad \partial_0 g_{jk}\big|_S &= \overset{\circ}{k}_{jk}, \end{aligned} \tag{8.9}$$

as long as $\overset{\circ}{g}_{jk}$ defines a Lorentz inner product on $T_p M$, for which $T_p S$ is space-like, for each $p \in S$. In such a case, this is a quasi-linear hyperbolic system, to which the results of Chap. 16, § 3, apply. We will also have a solution to (8.1), if we can show that $\lambda^j = 0$. To that end, we establish the following:

Lemma 8.1. *If $\widetilde{R}_{jk} = 0$, then λ^j satisfies a system of PDE of the form*

$$\partial_k \partial^k \lambda^j + B^{j\ell}_m(g, \nabla g)\, \partial_\ell \lambda^m = 0. \tag{8.10}$$

Proof. By (8.4) and (8.8), if $\widetilde{R}_{jk} = 0$, then

$$\mathrm{Ric}_{jk} = \frac{1}{2} g_{j\ell}\, \partial_k \lambda^\ell + \frac{1}{2} g_{k\ell}\, \partial_j \lambda^\ell,$$

and hence

$$G^{jk} = \frac{1}{2}\big(\partial^k \lambda^j + \partial^j \lambda^k - g^{jk} \partial_\ell\, \lambda^\ell\big). \tag{8.11}$$

Thus

$$2G^{jk}{}_{;k} = \big(\partial^k \lambda^j + \partial^j \lambda^k\big)_{;k} - g^{jk}(\partial_\ell \lambda^\ell)_{;k},$$

and a straightforward calculation gives

$$2G^{jk}{}_{;k} = \partial_k \partial^k \lambda^j + B^{j\ell}_m(g, \nabla g)\, \partial_\ell \lambda^m. \tag{8.12}$$

Since $G^{jk}{}_{;k} = 0$, we have (8.10).

We note for future reference that, without the assumption that $\widetilde{R}_{jk} = 0$, we have

$$(8.13)\quad \partial_k \partial^k \lambda^j + B^{j\ell}_m(g, \nabla g)\, \partial_\ell \lambda^m = -2\widetilde{T}^{jk}{}_{;k}, \quad \widetilde{T}_{jk} = \widetilde{R}_{jk} - \frac{1}{2}\widetilde{R}^\ell{}_\ell g_{jk}.$$

Now, (8.10) is a linear hyperbolic system for λ^j. We can deduce that $\lambda^j = 0$ on a neighborhood of S if we have

$$(8.14)\qquad \lambda^j\big|_S = 0, \quad \partial_0 \lambda^j\big|_S = 0.$$

Arranging this requires placing appropriate compatibility conditions on the Cauchy data (8.2). We now turn to this.

From (8.5) we have

$$(8.15)\qquad \lambda^\ell\big|_S = \overset{\circ}{k}{}^{\ell 0} - \frac{1}{2}\overset{\circ}{g}{}^{\ell 0}\,\overset{\circ}{k}{}^j{}_j + F^\ell(\overset{\circ}{g})\nabla_S \overset{\circ}{g},$$

where the last term is linear in $\nabla_S \overset{\circ}{g} = (\partial_1 \overset{\circ}{g}, \partial_2 \overset{\circ}{g}, \partial_3 \overset{\circ}{g})$. Also, from (8.11) we see that

$$(8.16)\qquad 2G^0{}_k = \partial_k \lambda^0 + g_{k\ell}\, \partial_0 \lambda^\ell - \delta^0{}_k\, \partial_\ell \lambda^\ell,$$

provided $\widetilde{R}_{jk} = 0$. Consequently,

$$(8.17)\qquad \widetilde{R}_{jk} = 0,\ \lambda^\ell\big|_S = 0 \Longrightarrow 2G^0{}_k\big|_S = \overset{\circ}{g}_{k\ell}\partial_0 \lambda^\ell\big|_S.$$

At this point it is convenient to record the following observation:

Lemma 8.2. *The restriction $G^0{}_k\big|_S$ is given in terms of $\overset{\circ}{g}_{jk}$, $\overset{\circ}{k}_{jk}$, and their tangential derivatives; it does not involve $\partial_0^2 g_{jk}$.*

Proof. From (8.3) we have

$$(8.18)\qquad \begin{aligned} G^0{}_k = {} & \frac{1}{2} g^{0j} g^{\ell m}(\partial_k \partial_m g_{\ell j} + \partial_\ell \partial_j g_{km} - \partial_\ell \partial_m g_{jk} - \partial_j \partial_k g_{\ell m}) \\ & - \tfrac{1}{2}\delta^0{}_k g^{ji} g^{\ell m}(\partial_i \partial_m g_{\ell j} - \partial_\ell \partial_m g_{ji}) + \widetilde{H}_k(g, \nabla g). \end{aligned}$$

The contribution of the terms involving ∂_0^2 is $1/2$ times

$$\begin{gathered} g^{0j} g^{\ell 0}\, \delta^0{}_k \partial_0^2 g_{\ell j} + g^{00} g^{0m}\, \partial_0^2 g_{km} - g^{0j} g^{00}\, \partial_0^2 g_{jk} - \delta^0{}_k g^{00} g^{\ell m}\, \partial_0^2 g_{\ell m} \\ -(\delta^0{}_k g^{j0} g^{\ell 0}\, \partial_0^2 g_{\ell j} - \delta^0{}_k g^{ji} g^{00}\, \partial_0^2 g_{ji}), \end{gathered}$$

which clearly vanishes.

Let the resulting formula for $G^0{}_k\big|_S$ be

$$G^0{}_k\big|_S = \mathcal{G}^0{}_k\big(\overset{\circ}{g}_{jk}, D_S^2\overset{\circ}{g}_{jk}, \overset{\circ}{k}_{jk}, \nabla_S\overset{\circ}{k}_{jk}\big). \tag{8.19}$$

We now state a local existence result:

Proposition 8.3. *Suppose the initial data (8.2) are C^∞ on S and satisfy the consistency condition*

$$\mathcal{G}^0{}_k\big(\overset{\circ}{g}_{jk}, D_S^2\overset{\circ}{g}_{jk}, \overset{\circ}{k}_{jk}, \nabla_S\overset{\circ}{k}_{jk}\big) = 0. \tag{8.20}$$

Assume S is spacelike for $\overset{\circ}{g}_{jk}$. Then the initial-value problem (8.1)–(8.2) has a C^∞-solution on a neighborhood of S.

Proof. Start with the tensor field

$$\widetilde{g}_{jk}(x) = \overset{\circ}{g}_{jk}(x') + x_0\overset{\circ}{k}_{jk}(x'), \quad x' = (x_1, x_2, x_3), \tag{8.21}$$

which defines a Lorentz metric on a neighborhood of S. Then the Einstein tensor $\widetilde{G}^{jk}$ of this metric satisfies (8.19) and hence $\widetilde{G}^0{}_k\big|_S = 0$.

Now define smooth "harmonic" coordinates $y_0, \dots, y_3$, by solving

$$\widetilde{\Box} y_j = 0, \tag{8.22}$$

with Cauchy data

$$y_j\big|_S = x_j\big|_S, \qquad dy_j\big|_S = dx_j\big|_S, \tag{8.23}$$

where $\widetilde{\Box}u$ is the analogue of (8.6) for the metric $\widetilde{g}_{jk}$. Rewrite the initial-value problem (8.1)–(8.2), in this new coordinate system.

Then (8.2) takes the form

$$g_{jk}(0, y') = \overset{\circ}{g}_{jk}(y'), \quad \frac{\partial}{\partial y_0} g_{jk}(0, y') = \tilde{k}_{jk}(y'); \tag{8.24}$$

the functions $\overset{\circ}{g}_{jk}$ are not changed, but the $\overset{\circ}{k}_{jk}$ do undergo a change. Due to the tensor character of $\widetilde{G}^{jk}$, we have

$$\mathcal{G}^0{}_k(\overset{\circ}{g}_{jk}, D_S^2\overset{\circ}{g}_{jk}, \tilde{k}_{jk}, \nabla_S\tilde{k}_{jk}) = 0. \tag{8.25}$$

Now solve the system (8.9) for g_{jk}, in the $(y_0, \dots, y_3)$-coordinates, using the initial data (8.24). We claim that $(y_0, \dots, y_3)$ are harmonic coordinates for the Lorentz metric g_{jk}. In fact, by Lemma 8.1 it suffices to show that if λ^ℓ are given by (8.5), then $\lambda^\ell\big|_S = 0$ and $(\partial/\partial y_0)\lambda^\ell\big|_S = 0$. Note that (8.22), together with (8.15), implies that $\lambda^\ell\big|_S = 0$ when λ^ℓ is determined by any metric g_{jk} satisfying

$$g_{jk} - \widetilde{g}_{jk} = O(y_0^2). \tag{8.26}$$

Thus we have $\lambda^\ell\big|_S = 0$ in our case. Next, by (8.25) and Lemma 8.2, $G^0{}_k\big|_S = 0$, so (8.17) implies $(\partial/\partial y_0)\lambda^\ell\big|_S = 0$.

Since $(y_0, \dots, y_3)$ are harmonic coordinates for the metric g_{jk}, we have (8.1) as a consequence of (8.9). Converting back to $(x_0, \dots, x_3)$-coordinates, we also have (8.2), as a consequence of (8.26). This completes the proof.

For simplicity, we have not specified which Sobolev spaces are needed for the initial data. Due to the special structure of Einstein's equations, one can obtain solutions with less regularity than is needed for general second-order, quasi-linear hyperbolic systems. Results on this can be found in [HKM] and in Chap. 5 of [Tay].

We have the following local uniqueness result:

Proposition 8.4. *Suppose g_{jk} and g'_{jk} are two smooth solutions to (8.1)–(8.2), on a neighborhood of S. Then there exists a C^∞-diffeomorphism φ on a neighborhood of S, such that $\varphi\big|_S = id.$ and $\varphi^* g = g'$.*

Proof. Without loss of generality, one can assume that the coordinates $(x_0, \dots, x_3)$ are harmonic for the metric g_{jk}. Parallel to (8.22)–(8.23), solve $\Box' y_j = 0,\ y_j\big|_S = x_j,\ dy_j\big|_S = dx_j$, where $\Box'$ is the Laplace–Beltrami operator for the metric g'. Then the diffeomorphism $\varphi(y) = x$ does the trick, since the system (8.9) for g (in the x-coordinates) is precisely the same as the system for g' (in the y-coordinates), and solutions to this quasi-linear hyperbolic system are locally unique.

We have seen that one way to "hyperbolicize" the equation (8.1) is to use harmonic coordinates. We now discuss an alternative method, due to D. DeTurck [DeT]. In this method, (8.1) is modified to

$$\operatorname{Ric}(g) - \operatorname{div}^*\big(W^{-1} \operatorname{div} \mathfrak{G}(W)\big) = 0, \tag{8.27}$$

where W is a convenient second-order symmetric tensor field, which we will specify below, and $\mathfrak{G}$ acts linearly on S^2T^*, by the rule

$$\mathfrak{G}(W)_{jk} = W_{jk} - \frac{1}{2}(\operatorname{Tr} W) g_{jk}. \tag{8.28}$$

In fact, if the initial data for g_{jk} are given by (8.2), we set

$$W_{jk}(x) = \widetilde{g}_{jk}(x) = \mathring{g}_{jk}(x') + x_0 \mathring{k}_{jk}(x'), \tag{8.29}$$

as in (8.21). Note that, upon lowering an index, W defines an invertible endomorphism on the tangent bundle T; we denote the inverse by W^{-1}.

For any given invertible W, $\mathcal{B} = \operatorname{div}^*(W^{-1} \operatorname{div} \mathfrak{G}(W))$ depends on the metric tensor g_{jk}; a calculation shows that it is given by

$$\mathcal{B}_{jk} = \frac{1}{2} g^{\ell m}\big[-\partial_j \partial_k g_{\ell m} + \partial_m \partial_k g_{\ell j} + \partial_\ell \partial_j g_{mk}\big] + C_{jk}(g, \nabla g). \tag{8.30}$$

Comparison with (8.3) gives

$$\operatorname{Ric}_{jk} - \mathcal{B}_{jk} = -\frac{1}{2} g^{\ell m}\, \partial_\ell \partial_m g_{jk} + M_{jk}(g, \nabla g) - C_{jk}(g, \nabla g). \tag{8.31}$$

Thus the equation (8.27) is strictly hyperbolic. Again, the results of Chap. 16, § 3, apply. We now want to show that if the initial data (8.2) satisfy the compatibility conditions of Proposition 8.3, then $\mathcal{B}_{jk} = 0$, so again we get a solution to (8.1). More precisely, we establish the vanishing of

$$u = W^{-1} \operatorname{div} \mathfrak{G}(W). \tag{8.32}$$

In fact, applying $\operatorname{div} \circ \mathfrak{G}$ to both sides of (8.27) and using $G_{jk}{}^{;k} = 0$, we have

$$\operatorname{div} \mathfrak{G} \operatorname{div}^* u = 0, \tag{8.33}$$

when g satisfies (8.27). Note that, for any covariant vector field v,

$$\begin{aligned} (\operatorname{div} \mathfrak{G} \operatorname{div}^* v)_j &= \frac{1}{2} g^{\ell m}(v_{\ell;m;j} - v_{j;\ell;m} - v_{\ell;j;m}) \\ &= \frac{1}{2}(-g^{\ell m} v_{j;\ell;m} - \operatorname{Ric}^\ell{}_j v_\ell), \end{aligned} \tag{8.34}$$

so (8.33) is a strictly hyperbolic equation for u. Now, the construction (8.29) of W guarantees that $\operatorname{div} \mathfrak{G}(W) = 0$ on $S = \{x_0 = 0\}$, so

$$u\big|_S = 0. \tag{8.35}$$

Thus, the vanishing of u would follow from $\partial_0 u\big|_S = 0$. To get this, we use a lemma:

Lemma 8.5. *If v is a covariant vector field on a Lorentz manifold (M, g), then*

$$v\big|_S = 0,\ \mathfrak{G}(div^* v)^0{}_j\big|_S = 0 \Longrightarrow \partial_0 v\big|_S = 0. \tag{8.36}$$

Proof. We have

$$\mathfrak{G}(\operatorname{div}^* v)^0{}_j = \frac{1}{2}(v_j{}^{;0} + v^0{}_{;j} - \delta^0{}_j g^{\ell m} v_{\ell;m}).$$

Hence

$$v\big|_S = 0 \Longrightarrow \mathfrak{G}(\operatorname{div}^* v)^0{}_j = \frac{1}{2} g^{00} \partial_0 v_j \ \text{ on } \ S, \ \text{ for } j = 1,2,3.$$

In particular, the hypotheses in (8.33) yield that $\partial_0 v_j\big|_S = 0$, for $j = 1,2,3$. Granted that, we see that, on S, $g^{\ell m} v_{\ell;m} = g^{00} v_{0;0}$, so

$$\mathfrak{G}(\operatorname{div}^* v)^0{}_0\big|_S = \frac{1}{2} g^{00} \partial_0 v_0 \quad \text{on } S,$$

and this yields the complete implication (8.36).

Now, the compatibility conditions (8.20) on the initial data imply that $\operatorname{Ric}^0{}_j\big|_S = 0$, so if (8.27) holds, then $u = W^{-1} \operatorname{div} \mathfrak{G}(W)$ satisfies all the hypotheses of (8.36). Thus $\partial_0 u\big|_S = 0$, so we have the following result:

Proposition 8.6. *If the initial data satisfy the compatibility conditions (8.20), then the solution to the hyperbolic system (8.27) is also a solution to (8.1).*

Having examined the empty-space case, we next consider Einstein's equations coupled with Maxwell's equations for an electromagnetic field, (5.1). In parallel with the approach to the empty-space case in (8.9), we will consider the system

$$\begin{aligned} -\frac{1}{2} n g^{\ell m}\, \partial_\ell \partial_m g_{jk} + H_{jk}(g, \nabla g) &= 8\pi\kappa \Big(T_{jk} - \frac{1}{2} \tau g_{jk} \Big), \\ \square \mathcal{F} &= 0, \end{aligned} \tag{8.37}$$

where, in the first equation, $\tau = T^j{}_j$ and

$$T_{jk} = \frac{1}{4\pi} \Big(\mathcal{F}_j{}^\ell \mathcal{F}_{k\ell} - \frac{1}{4} \mathcal{F}_{i\ell} \mathcal{F}^{i\ell} g_{jk} \Big) \tag{8.38}$$

is the stress-energy tensor for the electromagnetic field, as in (5.2). We have obtained the second equation in (8.37) from $d\mathcal{F} = 0 = d^\star \mathcal{F}$, by using $\square = -dd^\star - d^\star d$. In local coordinates, this operator depends on g_{jk} and derivatives up to second order; in fact,

$$(\square \mathcal{F})_{jk} = g^{\ell m}\, \partial_\ell \partial_m \mathcal{F}_{jk} + E_{jk}(D^2 g, \nabla \mathcal{F}). \tag{8.39}$$

Let us pose initial data on a compact hypersurface S, including the data (8.2), with

$$\mathring{g}_{jk} \in H^{s+1}(S), \quad \mathring{k}_{jk} \in H^s(S). \tag{8.40}$$

We assume that S is spacelike for these data and that $s > 7/2$. We also specify

$$\mathcal{F}_{jk}\big|_S \in H^s(S), \quad \partial_0 \mathcal{F}_{jk}\big|_S \in H^{s-1}(S). \tag{8.41}$$

We will postpone placing compatibility conditions on these data. If we identify a neighborhood of S with $I \times S$, $I = (-a, a)$, then, for a sufficiently small, we will obtain a solution to (8.37)–(8.41) satisfying

$$\begin{aligned} g &\in C\big(I, H^{s+1}(S)\big) \cap C^1\big(I, H^s(S)\big), \\ \mathcal{F} &\in C\big(I, H^s(S)\big) \cap C^1(I, H^{s-1}(S)\big). \end{aligned} \tag{8.42}$$

The local existence of solutions to (8.37) is established by a slight variant of the method developed in §§ 1–3 of Chap. 16 to treat hyperbolic systems. One obtains first-order systems for (φ, ψ), with $\varphi = (\Lambda g_{jk}, \partial_0 g_{jk})$ and $\psi = (\Lambda \mathcal{F}_{jk}, \partial_0 \mathcal{F}_{jk})$. From there one solves approximating systems for $(\varphi_\varepsilon, \psi_\varepsilon)$ and uses energy estimates plus Gronwall's inequality to establish

$$\|\varphi_\varepsilon(t)\|^2_{H^s} + \|\psi_\varepsilon(t)\|^2_{H^{s-1}} \le C, \tag{8.43}$$

for $|t| < a$. We require that $s > 7/2$, so $H^s(S) \subset C^{2+r}(S)$ for some $r > 0$. From (8.43) it follows that a limit point exists, yielding a solution to (8.37)–(8.41). As in Chap. 16, one establishes uniqueness, and the continuity described in (8.42). The reason one uses different Sobolev estimates for $\varphi(t)$ and for $\psi(t)$ is the occurence of second-order derivatives of g_{jk} in (8.39), compensated by the fact that no derivatives of $\mathcal{F}_{jk}$ are involved in the first equation of (8.37).

Now, assume that $\mathcal{F}\big|_S$ and $\partial_0 \mathcal{F}\big|_S$ satisfy the compatibility conditions

$$d\mathcal{F} = 0 = d^\star \mathcal{F} \quad \text{on } S. \tag{8.44}$$

Since $\Box d = d\Box$ and $\Box d^\star = d^\star \Box$, we have

$$\Box(d\mathcal{F}) = 0, \quad \Box(d^\star \mathcal{F}) = 0. \tag{8.45}$$

We deduce that $d\mathcal{F} = 0 = d^\star \mathcal{F}$ on $I \times S$. As discussed in § 1, this implies that T_{jk}, given by (8.38), satisfies

$$T^{jk}{}_{;k} = 0. \tag{8.46}$$

We next want to show that if $g_{jk}\big|_S$ and $\partial_0 g_{jk}\big|_S$ satisfy appropriate compatibility conditions, then $\lambda^\ell = -\Box x_\ell = 0$, so in fact the Einstein equations $G_{jk} = 8\pi\kappa T_{jk}$ follow from (8.37).

Lemma 8.7. *Assume that we have a solution to (8.37)–(8.41) on $I \times S$ and that (8.46) holds. Assume that $\lambda^\ell\big|_S = 0$, for $0 \le \ell \le 3$, and that, for $0 \le k \le 3$,*

$$\mathcal{G}^0{}_k(\overset{\circ}{g}_{jk}, D_S^2\overset{\circ}{g}_{jk}, \overset{\circ}{k}_{jk}, \nabla_S\overset{\circ}{k}_{jk}) = 8\pi\kappa T^0{}_k. \tag{8.47}$$

Then $\lambda^\ell = 0$ on $I \times S$.

Proof. If (8.39) holds, then (8.13) holds, with $\widetilde{T}_{jk} = 8\pi\kappa T_{jk}$. In fact, (8.39) implies that $\widetilde{R}^\ell{}_\ell = -8\pi\kappa\tau$, so

$$\widetilde{T}_{jk} = \widetilde{R}_{jk} + 4\pi\kappa\tau g_{jk} = 8\pi\kappa\Big(T_{jk} - \frac{1}{2}\tau g_{jk} + \frac{1}{2}\tau g_{jk}\Big) = 8\pi\kappa T_{jk}.$$

Now, a computation parallel to that yielding (8.17) shows that in the present case,

$$\lambda^\ell\big|_S = 0 \Longrightarrow 2G_k^0\big|_S = \overset{\circ}{g}_{k\ell}\partial_0\lambda^\ell\big|_S + 2\widetilde{T}^0{}_k. \tag{8.48}$$

Hence, if (8.47) holds, the hypothesis $\lambda^\ell = 0$ on S yields

$$\lambda^\ell\big|_S = 0, \quad \partial_0\lambda^\ell\big|_S = 0. \tag{8.49}$$

Also, by (8.46), we have $\widetilde{T}^{jk}{}_{;k} = 0$, so (8.13) gives

$$\partial_k\partial^k\lambda^j + B_m^{j\ell}(g, \nabla g)\partial_\ell\lambda^m = 0, \tag{8.50}$$

and the initial-value problem (8.49)–(8.50) has only $\lambda^\ell = 0$ as a solution, so the lemma is proved.

From here, one obtains the following parallel to Proposition 8.3:

Proposition 8.8. *Suppose the initial data in (8.40)–(8.41) satisfy the consistency conditions (8.44) and (8.47). Then there is a solution to*

$$G_{jk} = 8\pi\kappa T_{jk}$$

satisfying these initial conditions, where T_{jk} is the electromagnetic stress-energy tensor, given by (8.38).

We next consider Einstein's equations coupled with the equations of fluid motion. We use the form (6.59) of these equations, namely,

$$\nabla_w(\Box - 2\Phi'\nabla^2_{w,w})\tilde{w} - \widetilde{\mathcal{B}}(D^3g, D^2w, \nabla\Omega) = 0 \tag{8.51}$$

and

$$\mathcal{L}_w\Omega = 0. \tag{8.52}$$

As in (8.37), we write Einstein's equations as

$$-\frac{1}{2}g^{\ell m}\,\partial_\ell\partial_m g_{jk} + H_{jk}(g,\nabla g) = 8\pi\kappa\Big(T_{jk} - \frac{1}{2}\tau g_{jk}\Big), \tag{8.53}$$

where $\tau = T^j{}_j$ and this time

$$T_{jk} = (\rho + p)u_j u_k + p g_{jk} \tag{8.54}$$

is the stress-energy tensor for a fluid with 4-velocity u, density ρ, and pressure p. As in (6.37),

$$w = e^q u, \quad dq = \frac{dp}{\rho + p}, \tag{8.55}$$

and $\tilde{w}$ is the 1-form associated with w. Since we want (8.51)–(8.53) to be a system of equations for (w, Ω, g), let us rewrite (8.54) as

$$T_{jk} = -\frac{\rho + p}{\langle w, w\rangle} w_j w_k + p g_{jk}, \quad \rho = \rho\big(-\langle w, w\rangle\big), \quad p = p(\rho). \tag{8.56}$$

The formula for ρ as a function of $\langle w, w\rangle$ follows implicitly from (8.55), since $e^{2q} = -\langle w, w\rangle$.

We have made a slight notational change from (6.59) to (8.51), recording the dependence of $\widetilde{\mathcal{B}}$ on $D^3 g$. Clearly, the coefficients of the operator $\nabla_w(\Box - 2\Phi'\nabla^2_{w,w})$ also depend on $D^3 g$. Recall that

$$\Phi' = \frac{\rho'(p) - 1}{2e^{2q}}. \tag{8.57}$$

Also, as long as the equation of state satisfies $\rho'(p) \geq 1$, (8.51) is strictly hyperbolic, and any hypersurface that is spacelike for the metric g_{jk} is also spacelike for (8.51).

Let us pose initial data on a compact hypersurface S, including the data (8.2), with

$$\overset{\circ}{g}_{jk} \in H^{\ell+2}(S), \quad \overset{\circ}{k}_{jk} \in H^{\ell+1}(S). \tag{8.58}$$

We assume that S is spacelike for these data and that $\ell > 7/2$. We also specify

$$w_j\big|_S \in H^{\ell+1}(S), \quad \partial_0 w_j\big|_S \in H^{\ell}(S), \quad \partial_0^2 w_j\big|_S \in H^{\ell-1}(S), \tag{8.59}$$

and

$$\Omega_{jk}\big|_S \in H^{\ell}(S). \tag{8.60}$$

Of course, there will be compatibility conditions that we will ultimately want to place on such data, as discussed below, but these are not needed for the solvability of (8.51)–(8.53). We will assume that $w\big|_S$ is timelike:

$$\langle w, w\rangle\big|_S \le -C_0 < 0. \tag{8.61}$$

We identify a neighborhood of S with $I \times S$, $I = (-\varepsilon, \varepsilon)$. Using the methods of Chap. 16, in a fashion parallel to our discussion of (8.37)–(8.41), we obtain a solution to (8.51)–(8.53) satisfying (8.58)–(8.60) and

$$\begin{aligned} g &\in C\big(I, H^{\ell+2}(S)\big) \cap C^1\big(I, H^{\ell+1}(S)\big), \\ w &\in C\big(I, H^{\ell+1}(S)\big) \cap C^2\big(I, H^{\ell-1}(S)\big), \\ \Omega &\in C\big(I, H^{\ell}(S)\big). \end{aligned} \tag{8.62}$$

We leave to the reader the demonstration that appropriate consistency conditions on the initial data then imply that the Euler equations (6.4) are satisfied. This in turn implies $T^{jk}{}_{;k} = 0$, and hence Lemma 8.7 is applicable. From there one proceeds as before to show that when the initial data satisfy the consistency conditions, one has a solution to (6.1)–(6.2).

As in the case of nonrelativistic (compressible) fluids, one also considers the phenomenon of shock waves in relativistic fluids. For some work on this, see [Lich3, Lich4, ST, ST2, Tau2].

Exercises

1. Write out the principal symbol $\mathcal{L}(x,\xi)$ of the operator $\mathcal{L}$ in (8.3), and verify that
$$\det \mathcal{L}(x,\xi) = 0,$$
for all $\xi \in \mathbb{R}^4 \setminus 0$.
2. Show that under appropriate consistency conditions on the initial data, solutions to (8.51)–(8.53) also satisfy the Euler equations (6.4). (*Hint*: For one approach, see [CBr3].)
3. Discuss the appropriate initial-value problem for the relativistic motion of a charged fluid, coupled both to the metric of spacetime and to an electromagnetic field. The resulting equations are the equations of relativistic magnetohydrodynamics.
Material on this can be found in [Lich3].
4. Make use of finite propagation speed to eliminate the hypothesis that the initial surface S be compact, in the results of this section.

9. Geometry of initial surfaces

It is of interest to consider further when the initial data (8.2) satisfy the consistency condition (8.20). When $\overset{\circ}{g}_{jk}$ is restricted to TS, we get a Riemannian metric on S; $h_{jk} = \overset{\circ}{g}_{jk}$, for $1 \le j \le 3$. Let us assume for now that the coordinate system $(x_0, \dots, x_3)$ has not only the property that $S = \{x_0 = 0\}$ but also the property that the vector field $\partial/\partial x_0$ is *orthogonal* to S. Thus $\overset{\circ}{g}_{0k} = 0$ for $1 \le k \le 3$. Suppose also that $\partial/\partial x_0 = N$ is a unit vector on S, $\langle N, N\rangle = -1$ (i.e., $\overset{\circ}{g}_{00} = -1$). Using the Gauss–Codazzi equations, we can express $G^0{}_k\big|_S$ in terms of the metric tensor h_{jk} of S and the second fundamental form of $S \subset M$, which we denote as K_{jk}. Denote the associated Weingarten map by $A : T_pS \to T_pS$.

The Gauss equation, (4.14) of Appendix C, implies that, for X tangent to S,

$$\begin{aligned} &\mathrm{Ric}_M(X,X) - \mathrm{Ric}_S(X,X) \\ &\qquad = \langle R^M(N,X)X, N\rangle + \sum_{j=1}^{3} \big\langle (\Lambda^2 A)(E_j \wedge X), X \wedge E_j\big\rangle, \end{aligned} \tag{9.1}$$

where $\{E_1, E_2, E_3\}$ is an orthonormal basis of T_pS. From this, we have

$$S_M - S_S = -2\,\mathrm{Tr}\,\Lambda^2 A - 2\,\mathrm{Ric}_M(N,N), \tag{9.2}$$

where S_M is the scalar curvature of M, and S_S the scalar curvature of S. Compare with (4.72) of Appendix C. There is a sign difference, since here $\langle N, N\rangle = -1$. Since $G_{00} = \mathrm{Ric}_{00} - (1/2)S_M g_{00}$, we have

$$S_M - S_S = -2\,\mathrm{Tr}\,\Lambda^2 A - 2G_{00} - S_M g_{00},$$

or equivalently,

$$G^0{}_0\big|_S = -\frac{1}{2} S_S + \mathrm{Tr}\,\Lambda^2 A. \tag{9.3}$$

Meanwhile, the Codazzi equation, (4.16) of Appendix C, implies

$$K_{jk;\ell} = K_{\ell k;j} - R_{k0j\ell}. \tag{9.4}$$

We define the mean curvature H of $S \subset M$ to be $H = (1/3)\mathrm{Tr}\,A = (1/3)K^j{}_j$. The identity (9.4) implies $K_j{}^k{}_{;\ell} = K_\ell{}^k{}_{;j} - R^k{}_{0j\ell}$, and hence

$$K_j{}^k{}_{;k} = 3H_{;j} + \mathrm{Ric}_{0j}. \tag{9.5}$$

Since $G^0{}_j = \mathrm{Ric}^0{}_j - (1/2)S_M g^0{}_j = \mathrm{Ric}^0{}_j$, for $1 \le j \le 3$, we have

$$G^0{}_j\big|_S = K_j{}^k{}_{;k} - 3H_{;j}, \quad 1 \le j \le 3. \tag{9.6}$$

We have the following result:

Proposition 9.1. *If S is a Riemannian 3-manifold with metric tensor h_{jk}, and K_{jk} is a smooth section of $S^2T^*(S)$, then there exists a Lorentz 4-manifold M that is Ricci flat (i.e., satisfies (8.1)), for which S is a spacelike hypersurface, with induced metric tensor h_{jk} and second fundamental form K_{jk}, if and only if*

$$S_S - K_{jk}K^{jk} + K^j{}_j K^k{}_k = 0 \tag{9.7}$$

and

$$K_j{}^k{}_{;k} - 3H_{;j} = 0. \tag{9.8}$$

Proof. We have just seen the necessity of (9.7) and (9.8). For the converse, if h_{jk} and K_{jk} are given, satisfying (9.7) and (9.8), set

$$\overset{\circ}{g}_{jk} = h_{jk} \text{ if } 1 \le j,k \le 3, \quad \overset{\circ}{g}_{00} = -1, \quad \overset{\circ}{g}_{jk} = 0, \text{ otherwise.} \tag{9.9}$$

Also, set

$$\overset{\circ}{k}_{jk} = -2K_{jk} \quad \text{if } 1 \le j,k \le 3, \quad \overset{\circ}{k}_{jk} = 0, \text{ otherwise.} \tag{9.10}$$

Note that, for *any* metric g_{jk} satisfying (8.2) with these initial data, h_{jk} and K_{jk} do specify the first and second fundamental forms of $S = \{x_0 = 0\}$. See (4.69) of Appendix C. The fact that this prescription yields $\overset{\circ}{g}_{jk}$ and $\overset{\circ}{h}_{jk}$, which satisfy the compatibility conditions of Proposition 8.3, thus follows from (9.4) and (9.6).

The index raising and covariant differentiation performed in (9.7) and (9.8) are operations defined by the metric tensor h_{jk} on S. Note that (9.7) follows from (9.3), via the identity

$$\mathrm{Tr}\,\Lambda^2 A = \frac{1}{2}(\mathrm{Tr}\,A)^2 - \frac{1}{2}\,\mathrm{Tr}\,A^2,$$

which is readily verified by using a basis that diagonalizes A. In the physics literature, (9.7) is called the Hamiltonian constraint and (9.8) is called the momentum constraint. Together, (9.7) and (9.8) are called the *constraint equations*.

Note that special hypotheses about the coordinates used on M disappear in the formulation of Proposition 9.1, which is convenient.

We can define the trace-free part of K_{jk}:

$$Q_{jk} = K_{jk} - Hh_{jk}. \tag{9.11}$$

Then the system (9.7)–(9.8) becomes

$$S_S - Q_{jk}Q^{jk} + 6H^2 = 0, \tag{9.12}$$

$$Q_j{}^k{}_{;k} - 2H_{;j} = 0. \tag{9.13}$$

This system has been studied in [Lich1, Yo1, OY1, CBr5, CBr6, CBY]. Following their work, we will investigate this system more closely in the special case when H is taken to be *constant* on S, which we now assume to be *compact*. Then (9.13) specifies that Q_{jk} is a divergence-free (trace-free, order 2, symmetric) tensor field. We show how to construct all such fields on a compact Riemannian manifold (S, h).

Let us define $\mathcal{D}_{TF}$ on vector fields by

$$\mathcal{D}_{TF}X = \text{Def } X - \frac{1}{n}(\text{div } X)h, \quad \mathcal{D}_{TF} : C^\infty(S,T) \to C^\infty(S, S_0^2T^*), \tag{9.14}$$

where Def is the deformation operator, $(\text{Def } X)_{jk} = (X_{j;k} + X_{k;j})/2$, and where $S_0^2T^*$ denotes the bundle of second-order, trace-free, covariant tensors. A calculation yields

$$\mathcal{D}_{TF}^* = -\text{div}\big|_{S_0^2T^*}. \tag{9.15}$$

Hence

$$\mathcal{D}_{TF}^*\mathcal{D}_{TF}X = -\text{div Def } X + \frac{1}{n} \text{ grad div } X. \tag{9.16}$$

The operator $\mathcal{L} = \mathcal{D}_{TF}^*\mathcal{D}_{TF}$ is a second-order, elliptic, self-adjoint operator, and there is a Weitzenbock formula, which implies

$$\|\mathcal{D}_{TF}X\|_{L^2}^2 = \frac{1}{2}\|\nabla X\|_{L^2}^2 + \Big(\frac{1}{2} - \frac{1}{n}\Big)\|\text{div } X\|_{L^2}^2 - \frac{1}{2}\big(\text{Ric}_S(X), X\big)_{L^2}. \tag{9.17}$$

Compare with formulas (4.26)–(4.31) of Chap. 10.

The kernel of $\mathcal{L}$, which is equal to ker $\mathcal{D}_{TF}$, consists of conformal Killing fields, as noted in (3.39) of Chap. 2. It is a finite-dimensional subspace of $C^\infty(S,T)$. Let E be the inverse of $\mathcal{L}$ on the orthogonal complement of ker $\mathcal{L}$, and zero on ker $\mathcal{L}$. It follows that

$$E : H^s(S,T) \longrightarrow H^{s+2}(S,T), \tag{9.18}$$

for all $s \geq -1$, by the elliptic regularity results in Chap. 5, § 1, and more generally for all $s \in \mathbb{R}$, by the construction of $E \in OPS^{-2}(S)$ in Chap. 7. Now set

$$P_1 = \mathcal{D}_{TF}\ E\ \mathcal{D}_{TF}^*, \quad P_1 : H^s(S, S_0^2T^*) \to H^s(S, S_0^2T^*). \tag{9.19}$$

The operator P_1 is the orthogonal projection onto the range of $\mathcal{D}_{TF}$ in L^2, that is, the image of $\mathcal{D}_{TF}$ acting on $H^1(S,T)$, and $P_0 = I - P_1$ is the orthogonal projection onto the kernel of $\mathcal{D}_{TF}^*$ in $L^2(S, S_0^2 T^*)$. From (9.19) it follows that

$$P_0 : C^\infty(S, S_0^2 T^*) \longrightarrow C^\infty(S, S_0^2 T^*). \tag{9.20}$$

The set of smooth, divergence-free, trace-free, second-order, symmetric tensor fields Q_{jk} is precisely the image of P_0 in (9.20).

Now, if we have a Riemannian metric h_{jk} on S and a solution Q_{jk} to (9.13), with H constant, the scalar curvature may not satisfy (9.12). In [Yo1] and [OY1], following Lichnerowicz, who treated the case $H = 0$, it was shown that the triple (h_{jk}, H, Q_{jk}) leads to a *new* triple $(\overline{h}_{jk}, H, \overline{Q}_{jk})$, where $\overline{h}_{jk}$ is a conformal multiple of h_{jk}:

$$\overline{h}_{jk} = \varphi^4 h_{jk}, \tag{9.21}$$

H is unchanged, and $\overline{Q}_{jk}$ is a smooth multiple of Q_{jk}, involving a different power of φ, as we will see below. Then (9.12)–(9.13) hold for this new triple, provided φ satisfies a certain elliptic PDE, which we proceed to derive.

For these calculations, denote covariant differentiation associated to h and $\overline{h}$ by ∇ and $\overline{\nabla}$, respectively. Then

$$\overline{\nabla}_k \overline{Q}^{jk} = \nabla_k \overline{Q}^{jk} + (\overline{\Gamma}^j{}_{ik} - \Gamma^j{}_{ik})\overline{Q}^{ik} + (\overline{\Gamma}^k{}_{ik} - \Gamma^k{}_{ik})\overline{Q}^{ji}, \tag{9.22}$$

where the connection coefficients are related by

$$\overline{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} + \frac{2}{\varphi}\Big(\delta^i{}_j\, \partial_k\varphi + \delta^i{}_k\, \partial_j\varphi - h_{jk}\, \partial^i\varphi\Big). \tag{9.23}$$

Consequently, for any symmetric, second-order tensor field $\overline{Q}^{jk}$,

$$\overline{\nabla}_k \overline{Q}^{jk} = \nabla_k \overline{Q}^{jk} + \frac{10}{\varphi}\overline{Q}^{jk}\, \partial_k\varphi - \frac{2}{\varphi}\overline{Q}_k{}^k\, \partial^j\varphi.$$

The last term vanishes if $\overline{Q}_k{}^k = 0$. If, furthermore, $\overline{Q}^{jk} = \psi Q^{jk}$, then

$$\overline{\nabla}_k \overline{Q}^{jk} = \psi \nabla_k Q^{jk} + \Big(\partial_k\psi + \frac{10\psi}{\varphi}\partial_k\varphi\Big) Q^{jk}. \tag{9.24}$$

The last term here vanishes if $\psi = \varphi^{-10}$, so we have

$$\overline{Q}^{jk} = \varphi^{-10} Q^{jk} \Longrightarrow \overline{\nabla}_k \overline{Q}^{jk} = \varphi^{-10} \nabla_k Q^{jk} \tag{9.25}$$

when Q_{jk} is symmetric and has trace zero. Note that (9.25) implies $\overline{Q}_{jk} = \varphi^{-2} Q_{jk}$.

Thus, if Q_{jk} is constructed as above, as an element in the range of P_0, so it has trace zero and solves $\nabla_k Q_j{}^k = 0$, we also have $\overline{\nabla}_k \overline{Q}_j{}^k = 0$ whenever $\overline{h}_{jk}$ is related to h_{jk} by (9.21). The scalar curvatures of (S, h) and $(S, \overline{h})$ are related by

$$\overline{S}_S = \varphi^{-4} S_S - 8\varphi^{-5} \Delta\varphi, \tag{9.26}$$

where Δ is the Laplace operator on (S, h). Now we want to satisfy the analogue of (9.12), namely,

$$\overline{S}_S = \overline{Q}_{jk} \overline{Q}^{jk} - 6H^2 = \varphi^{-12} Q_{jk} Q^{jk} - 6H^2 = \varphi^{-12} f - 6H^2. \tag{9.27}$$

By (9.26), we want φ to be a smooth, positive solution to the PDE

$$\Delta\varphi = \frac{1}{8} S_S \varphi - \frac{1}{8} f \varphi^{-7} + \frac{3}{4} H^2 \varphi^5 = F(x, \varphi). \tag{9.28}$$

Equations of this form are discussed in § 1 of Chap. 14.

If $f > 0$ on S and $H \neq 0$, then Theorem 1.10 of Chap. 14 implies the solvability of (9.28), which is a special case of (1.50) of Chap. 14. We thus have

Proposition 9.2. *Let S be a compact, connected 3-manifold, with metric tensor h_{jk}, Q_{jk} a smooth, divergence-free, trace-free section of $S^2 T^*(S)$, and let H be a nonzero constant. Then there exists a positive $\varphi \in C^\infty(S)$ such that if*

$$\overline{h}_{jk} = \varphi^4 h_{jk}, \quad \overline{Q}_{jk} = \varphi^{-2} Q_{jk}, \tag{9.29}$$

then there is a Ricci-flat Lorentz manifold M, containing S as a spacelike hypersurface, with induced metric $\overline{h}_{jk}$ and second fundamental form

$$\overline{K}_{jk} = \overline{Q}_{jk} + H \overline{h}_{jk}, \tag{9.30}$$

provided $Q_{jk} Q^{jk} = f$ is not identically zero on S.

Proof. The argument above gives the result provided $f(x) > 0$ on S. It remains to weaken this condition on f. First, if the scalar curvature S_S of (S, h) is negative on $\{x \in S : f(x) = 0\} = \Sigma$, then Theorem 1.10 of Chap. 14 still implies the solvability of (9.28). On the other hand, we can make a preliminary conformal deformation of the metric tensor of S to make its scalar curvature negative on any proper closed subset of S, such as Σ, as long as Σ is not all of S, so Proposition 9.2 is proved.

If Q_{jk} is taken to be zero, then (9.28) becomes

$$\Delta\varphi = \frac{1}{8} S_S \varphi + \frac{3}{4} H^2 \varphi^5. \tag{9.31}$$

Integrating both sides, we see that if there is a positive solution φ, then

$$\int_S S_S(x)\varphi(x)\, dV(x) < 0.$$

In particular, (9.31) has no positive solution if $S_S \geq 0$ on S. Here is a positive result:

Proposition 9.3. *Let S be a compact, connected 3-manifold with metric tensor h_{jk}. Assume the scalar curvature of (S, h) satisfies $S_S(x) < 0$ on S. Let H be a nonzero constant. Then there is a positive $\varphi \in C^\infty(S)$ such that if $\overline{h}_{jk} = \varphi^4 h_{jk}$, then there is a Ricci-flat Lorentz manifold M, containing S as a spacelike hypersurface, with induced metric $\overline{h}_{jk}$ and second fundamental form*

$$\overline{K}_{jk} = H\overline{h}_{jk}. \tag{9.32}$$

Proof. The equation (9.31) has the same form as (1.49) in Chap. 14, as the equation for the conformal factor needed to alter (S, h), with scalar curvature S_S, to $(S, \overline{h})$, with scalar curvature $\overline{S}_S = -3H^2$. Thus solvability follows from Proposition 1.11 of Chap. 14.

If $H = 0$, then (9.28) becomes

$$\Delta\varphi = \frac{1}{8} S_S \varphi - \frac{1}{8} f \varphi^{-7}, \tag{9.33}$$

where we recall that $f = Q_{jk} Q^{jk}$. The solvability of (9.33) implies the identity $\int_S S_S \varphi\, dV = \int_S f\varphi^{-7}\, dV$, so there is no positive solution if $S_S \leq 0$ on S. On the other hand, Theorem 1.10 of Chap. 14 applies if $S_S(x) > 0$ on S and $f > 0$ on S, so we have the following:

Proposition 9.4. *Let S be a compact, connected 3-manifold with metric tensor h_{jk}. Assume the scalar curvature of (S, h) satisfies $S_S(x) > 0$ on S. Let Q_{jk} be a smooth, divergence-free, trace-free section of $S^2T^*(S)$. Then there is a positive $\varphi \in C^\infty(S)$ such that if $\overline{h}_{jk}$ and $\overline{Q}_{jk}$ are given by (9.29), then there is a*

Ricci-flat Lorentz manifold M, containing S as a spacelike hypersurface, with induced metric $\overline{h}_{jk}$ and second fundamental form

$$\overline{K}_{jk} = \overline{Q}_{jk}, \tag{9.34}$$

provided Q_{jk} is nowhere vanishing. In such a case, the mean curvature of $S \subset M$ vanishes.

Next, following [CBIM], we extend Proposition 9.3 to allow H to be non-constant, provided it does not vary too much. As before, we have a compact, 3-manifold S, with Riemannian metric tensor h, scalar curvature S_S. Let H be a smooth, real-valued function on S. We want to construct a positive $\varphi \in C^\infty(S)$ and a second-order, symmetric, trace-free tensor field Q_{jk} on S (i.e., $Q \in C^\infty(S, S_0^2 T^*)$) such that if we change the metric tensor to $\overline{h}_{jk} = \varphi^4 h_{jk}$, and set $\overline{Q}_{jk} = \varphi^{-2} Q_{jk}$, as in (9.29), then S is a spacelike hypersurface of a Ricci-flat Lorentz 4-manifold, with induced metric $\overline{h}_{jk}$ and with second fundamental form $\overline{K}_{jk} = \overline{Q}_{jk} + H\overline{h}_{jk}$, as in (9.30).

In order to achieve this, we need to satisfy the Gauss–Codazzi system

$$\begin{aligned} \overline{S}_S - \overline{Q}_{jk}\overline{Q}^{jk} + 6H^2 &= 0, \\ \overline{\nabla}_k \overline{Q}_j{}^k - 2\overline{\nabla}_j H &= 0, \end{aligned} \tag{9.35}$$

which one converts to

$$\begin{aligned} \Delta\varphi &= \frac{1}{8} S_S \varphi - \frac{1}{8}|Q|^2 \varphi^{-7} + \frac{3}{4} H^2 \varphi^5, \\ \nabla_k Q^{jk} - 2\varphi^6 h^{jk} H_{;k} &= 0, \end{aligned} \tag{9.36}$$

where $|Q|^2 = Q_{jk}Q^{jk}$. As before (and as noted in [Yo3] and [CBY]), we get from (9.35) to (9.36) via the identities (9.25) and (9.26). The first equation in (9.36) is the same as (9.28).

The second equation in (9.36) is

$$\mathcal{D}_{TF}^* Q + 2\varphi^6 \operatorname{grad} H = 0. \tag{9.37}$$

We look for Q in the form

$$Q = \mathcal{D}_{TF} X + Q^b, \quad \mathcal{D}_{TF}^* Q^b = 0, \tag{9.38}$$

where X is a vector field on S. As in (9.14), $\mathcal{D}_{TF} X = \operatorname{Def} X - (1/3)(\operatorname{div} X)h$. Pick $Q^b \in \mathcal{R}(P_0)$. Then (9.37) becomes

$$\mathcal{L}X = -2\varphi^6 \operatorname{grad} H, \tag{9.39}$$

where

$$\mathcal{L}X = \mathcal{D}_{TF}^* \mathcal{D}_{TF} X = -\text{div Def } X + \frac{1}{3} \text{ grad div } X. \tag{9.40}$$

Assume that $\mathcal{L}$ is invertible, that is, (S, h) has no conformal Killing fields. Thus, given φ, we solve (9.39) for X, $X = -2\mathcal{L}^{-1}(\varphi^6 \text{ grad } H)$. Then it remains to solve for φ:

$$\Delta\varphi = \frac{1}{8} S_S \varphi - \frac{1}{8} |E(\varphi^6) + Q^b|^2 \varphi^{-7} + \frac{3}{2} H^2 \varphi^5, \tag{9.41}$$

where

$$Eu = -2\mathcal{D}_{TF} \circ \mathcal{L}^{-1}(u \text{ grad } H), \quad E \in OPS^{-1}. \tag{9.42}$$

This has a slightly more complicated form than (9.28), though of course it reduces to (9.28) when H is constant.

We will establish the following:

Proposition 9.5. *Let S be a compact 3-manifold with metric tensor h. Assume the scalar curvature $S_S < 0$ on S. Let $H \in C^\infty(S)$ be a given positive function. Then we can find a positive $\varphi \in C^\infty(S)$, solving (9.41), with $Q^b = 0$, provided*

$$\|\mathcal{D}_{TF} \mathcal{L}^{-1} u\|_{L^\infty} \le A_0 \|u\|_{L^\infty}, \tag{9.43}$$

with

$$A_0 \|\nabla H\|_{L^\infty} < \sqrt{3} H_{min}. \tag{9.44}$$

As a preparation to proving this, we first obtain some a priori estimates for a positive solution $\varphi \in C^\infty(S)$ to (9.41), with $Q^b = 0$, that is, to

$$\Delta\varphi = f\big(x, E(\varphi^6), \varphi\big), \tag{9.45}$$

where

$$f\big(x, E(\varphi^6), \varphi\big) = \frac{1}{8} S_S \varphi - \frac{1}{8} |E(\varphi^6)|^2 \varphi^{-7} + \frac{3}{2} H^2 \varphi^5. \tag{9.46}$$

Note that if $\varphi(x_0) = \varphi_{\min} > 0$, then $\Delta\varphi(x_0) \ge 0$, so if (9.45) holds, then $f(x_0, E(\varphi^6), \varphi(x_0)) \ge 0$. Hence

$$\frac{3}{2} H(x_0)^2 \varphi(x_0)^5 \ge \frac{1}{8}\big(-S_S(x_0)\big)\varphi(x_0) \ge \frac{1}{8} \sigma \varphi(x_0) \tag{9.47}$$

if $S_S \le -\sigma < 0$ on S. This implies

$$\varphi \ge a_0 \ \text{ on } \ S; \quad a_0^4 = \frac{1}{12}\sigma H_{\max}^{-2}. \tag{9.48}$$

We next derive an upper bound on a solution to (9.48), using the hypothesis (9.43)–(9.44). Suppose $\varphi(x_1) = \varphi_{\max}$. Then $\Delta\varphi(x_1) \le 0$, so if (9.45) holds, then $f(x_1, E(\varphi^6), \varphi(x_1)) \le 0$. Now

$$\begin{aligned} &f\bigl(x_1, E(\varphi^6), \varphi(x_1)\bigr) \\ &\quad = \varphi_{\max}^{-7}\Bigl[\frac{3}{2}H(x_1)^2\varphi_{\max}^{12} - \frac{1}{8}|E(\varphi^6)|^2 + \frac{1}{8}S_S(x_1)\varphi_{\max}^8\Bigr]. \end{aligned} \tag{9.49}$$

The hypothesis (9.43)–(9.44) implies

$$\|E(\varphi^6)\|_{L^\infty} \le 2A_0\|\nabla H\|_{L^\infty}\|\varphi^6\|_{L^\infty} < 2\sqrt{3}H_{\min}\varphi_{\max}^6, \tag{9.50}$$

so

$$\begin{aligned} &\frac{3}{2}H(x_1)^2\varphi_{\max}^{12} - \frac{1}{8}|E(\varphi^6)|^2 \ge \eta\varphi_{\max}^{12}, \\ &\eta = \frac{1}{2}\bigl(3H_{\min}^2 - A_0^2\|\nabla H\|_{L^\infty}^2\bigr) > 0, \end{aligned} \tag{9.51}$$

and hence

$$f\bigl(x_1, E(\varphi^6), \varphi(x_1)\bigr) \ge \eta\varphi_{\max}^5 + \frac{1}{8}S_S(x_1)\varphi_{\max}. \tag{9.52}$$

If the left side of (9.52) is ≤ 0, this requires $\eta\varphi_{\max}^5 + (1/8)S_S(x_1)\varphi_{\max} \le 0$. Thus, if $|S_S| \le \gamma$, a solution to (9.45) must satisfy

$$\varphi \le a_1 \ \text{ on } \ S, \quad a_1^4 = \frac{\gamma}{8\eta} = \frac{\gamma/4}{3H_{\min}^2 - A_0^2\|\nabla H\|_{L^\infty}^2}. \tag{9.53}$$

For the rest of the discussion, we modify the formula (9.46), replacing it by

$$f\bigl(x, E(\varphi^6), \varphi\bigr) = \frac{1}{8}S_S\mu(\varphi) - \frac{1}{8}|E(\varphi^6)|^2\alpha(\varphi) + \frac{3}{2}H^2\varphi^5, \tag{9.54}$$

where

$$\begin{aligned} \mu(\varphi) &= \varphi, &\quad \varphi &\ge a_0, \\ \alpha(\varphi) &= \varphi^{-7}, &\quad \varphi &\ge a_0. \end{aligned} \tag{9.55}$$

In addition, we require the functions μ and α to be smooth and monotonic on $\mathbb{R}$, for α to be linear on $\varphi \leq 0$, for $\mu(\varphi) \geq \varphi$ on $\mathbb{R}$, and for $\mu(\varphi)$ to be some positive constant (say μ_0) for $\varphi \leq a_0/2$.

To prove Proposition 9.5, we will use the Leray–Schauder fixed-point theorem, in the following form (cf. Theorem B.5 in Chap. 14):

Theorem. *Let V be a Banach space, $F : [0,1] \times V \to V$ a continuous, compact map such that $F(0, v) = v_0$, independent of v. Suppose there exists $M < \infty$ such that, for all $(\tau, x) \in [0,1] \times V$,*

$$F(\tau, x) = x \Longrightarrow \|x\| < M. \tag{9.56}$$

Then $F_1 : V \to V$, given by $F_1(x) = F(1, x)$, has a fixed point.

We will apply this to $V = C(S)$ and

$$F(\tau, \varphi) = (\Delta - 1)^{-1}\big(\Psi_\tau(\varphi) - \varphi\big), \tag{9.57}$$

where, picking $b = (a_0 + a_1)/2$, we set

$$\Psi_\tau(\varphi) = (1 - \tau)(\varphi - b) + \tau f\big(x, E(\varphi^6), \varphi\big), \tag{9.58}$$

with f as in (9.54). Note that

$$F(0, \varphi) = -(\Delta - 1)^{-1} b = b. \tag{9.59}$$

Also,

$$F(\tau, \varphi) = \varphi \Longleftrightarrow \Delta\varphi = \tau f\big(x, E(\varphi^6), \varphi\big) + (1 - \tau)(\varphi - b). \tag{9.60}$$

To check (9.56), we need to estimate $\varphi_{\max}$ and $\varphi_{\min}$ whenever $\tau \in [0,1]$ and φ satisfies (9.60). The case $\tau = 0$ is clear, so we may assume $\tau > 0$. If $\varphi(x_0) = \varphi_{\min}$, then $\Psi(\varphi) \geq 0$ at x_0, so

$$\tau f\big(x_0, E(\varphi^6), \varphi(x_0)\big) + (1 - \tau)\big(\varphi(x_0) - b\big) \geq 0. \tag{9.61}$$

If $\varphi(x_0) < b$ and $\tau \in (0, 1]$, this requires $f(x_0, E(\varphi^6), \varphi(x_0)) \geq 0$, or (in place of (9.47))

$$\frac{3}{2} H(x_0)^2 \varphi_{\min}^5 \geq \frac{1}{8}\big(-S_S(x_0)\big)\mu(\varphi_{\min}). \tag{9.62}$$

This forces $\varphi_{\min}^5 \geq (\sigma/12) H_{\max}^{-2} \mu(\varphi_{\min})$, which in turn forces $\varphi_{\min} > 0$. Since $\mu(\varphi) \geq \varphi$ and $-S_S > 0$, this implies (9.47). Therefore, again we get the estimate (9.48).

If $\varphi(x_1) = \varphi_{\max}$, then $\Psi_\tau(\varphi) \le 0$ at x_1, so

$$\tau f\big(x_1, E(\varphi^6), \varphi(x_1)\big) + (1-\tau)\big(\varphi(x_1) - b\big) \le 0. \tag{9.63}$$

If $\varphi(x_1) > b$ and $\tau \in (0,1]$, this requires $f(x_1, E(\varphi^6), \varphi(x_1)) \le 0$, which is equal to (9.46) if $\varphi_{\max} > b$, so as before we have the estimate (9.53).

Thus the fixed-point theorem applies to our situation, so Proposition 9.5 is proved. Thus, in rough parallel with Proposition 9.3, we have the following:

Proposition 9.6. *Let S be a compact 3-manifold with metric tensor h_{jk}. Assume the scalar curvature $S_S < 0$ on S. Let $H \in C^\infty(S)$ be a positive function satisfying the hypothesis (9.43)–(9.44). Then there is a positive $\varphi \in C^\infty(S)$ such that if $\overline{h}_{jk} = \varphi^4 h_{jk}$, there is a Ricci-flat Lorentz 4-manifold M, containing S as a spacelike hypersurface, with induced metric $\overline{h}_{jk}$ and second fundamental form*

$$\overline{K}_{jk} = \overline{Q}_{jk} + H\overline{h}_{jk}, \tag{9.64}$$

where

$$\overline{Q} = -2\varphi^{-2} \cdot \mathcal{D}_{TF}\mathcal{L}^{-1}(\varphi^6 \nabla H). \tag{9.65}$$

In particular, the mean curvature of $S \subset M$ is H.

See [CBY] for a discussion of some cases where S is diffeomorphic to $\mathbb{R}^3$, and asymptotically flat, and [CO] for a further analysis, when also $H = 0$. In this case, one can make a preliminary conformal change of metric to achieve $S_S = 0$, so that (9.33) becomes

$$\Delta\varphi = -\frac{1}{8} f \varphi^{-7}.$$

Exercises

1. Show that the identities (9.3) and (9.6) for $G^0{}_j\big|_S$ imply Lemma 8.2.
2. Put together Proposition 9.4 and Birkhoff's theorem, and make some deductions, regarding (Q_{jk}) and solutions to (9.33), and symmetry properties that they *cannot* have.
3. Suppose that one wants to solve (1.1), namely,

$$G^{jk} = 8\pi T^{jk}. \tag{9.66}$$

Show that the equation (9.7)–(9.8) on S get replaced by

$$\begin{aligned} S_S - K_{jk}K^{jk} + K^j{}_j K^k{}_k &= 2\rho, \\ K_j{}^k{}_{;k} - 3H_{;j} &= J_j, \end{aligned} \tag{9.67}$$

where

$$\rho = 8\pi T_{00}\big|_S, \quad J_j = -8\pi T_{0j}\big|_S. \tag{9.68}$$

Study this system, particularly in the case $H = \text{const}$.

4. Extend Proposition 9.5 as follows. Replace (9.43)–(9.44) by the hypothesis that, for some $p > 3$,

$$\|\mathcal{D}_{TF}\mathcal{L}^{-1}u\|_{L^\infty} \le A_p\|u\|_{L^p}, \tag{9.69}$$

with

$$A_p\|\nabla H\|_{L^p} < \sqrt{3}\, H_{\min}. \tag{9.70}$$

5. Note that solving the constraint equations (9.7)–(9.8) with $K_{jk} = 0$ is reduced to solving (9.33) with $f = 0$, namely,

$$\Delta\varphi = \frac{1}{8}S_S\varphi.$$

Consider solutions to this, both on compact and on noncompact S. Relate the solution

$$\varphi(x) = 1 + \frac{M}{|x|}$$

on flat $\mathbb{R}^3$ (outside the origin) to the Schwarzschild metric.

10. Time slices and their evolution

Suppose M is a Lorentz manifold whose metric satisfies the empty-space Einstein equation (8.1), and on M we have a smooth function t such that $\tau = \text{grad}\, t$ is timelike. Thus M is foliated by the surfaces S_c, on which $t = c$, called *time slices*. One can choose local coordinates (t, x_1, x_2, x_3) on an open set $\mathcal{O} \subset M$ with respect to which the metric is

$$ds^2 = -\lambda(t,x)^2\, dt^2 + \sum_{j,k=1}^{3} g_{jk}(t,x)\, dx_j\, dx_k. \tag{10.1}$$

This can be done by picking local coordinates (x_1, x_2, x_3) arbitrarily on *one* slice S_c and then taking x_j to be constant on each integral curve of τ through such a coordinate patch. The function

$$\lambda(t,x) = \big[-\langle\tau,\tau\rangle\big]^{-1/2} \tag{10.2}$$

is called the *lapse function* of this foliation. Note that $\big(g_{jk}(c,x)\big)$ defines the Riemannian metric induced on S_c and that $N = -\lambda\tau = \lambda^{-1}\,\partial/\partial t$ is a unit timelike normal to S_c.

Each S_c has a second fundamental form $K_{jk}(c,x)$, and, by Proposition 9.1, for each c, $K_{jk} = K_{jk}(c)$ must satisfy the constraint equations (9.7)–(9.8), where the covariant derivatives are given by the Riemannian metric on S_c. Note that

$$\frac{\partial g_{jk}}{\partial t} = -2\lambda K_{jk}. \tag{10.3}$$

The following identity complements the constraint equation:

Proposition 10.1. *If the Einstein equation (8.1) holds, then*

$$\frac{\partial K_{jk}}{\partial t} = -\lambda_{;j;k} + \lambda(Ric^S_{jk} + 3HK_{jk} - 2K_{j\ell}K^\ell{}_k). \tag{10.4}$$

Proof. Calculating the components R^M_{j0k0} of the Riemann tensor of M, in coordinates $(x_0,\dots,x_3)$, with $x_0 = t$, one obtains, for $1 \le j,k \le 3$,

$$\lambda^{-2} R^M_{j0k0} = \lambda^{-1}\lambda_{;j;k} + \lambda^{-1}\,\partial_t K_{jk} + K_{j\ell}K^\ell{}_k. \tag{10.5}$$

Now (9.1) implies

$$\lambda^{-2} R^M_{j0k0} = \mathrm{Ric}^S_{jk} - \mathrm{Ric}^M_{jk} + 3HK_{jk} - K_{j\ell}K^\ell{}_k, \tag{10.6}$$

so, for any metric of the form (10.1), we have

$$\frac{\partial K_{jk}}{\partial t} = -\lambda_{;j;k} + \lambda(\mathrm{Ric}^S_{jk} + 3HK_{jk} - 2K_{j\ell}K^\ell{}_k) - \lambda\mathrm{Ric}^M_{jk}. \tag{10.7}$$

This proves (10.4).

Note that $\partial_t(K_j{}^k) = g^{k\ell}(\partial_t K_{j\ell}) + (\partial_t g^{k\ell})K_{j\ell}$. Using (10.3)–(10.4), we have

$$\frac{\partial K_j{}^k}{\partial t} = -\lambda_{;j}{}^{;k} + \lambda(\mathrm{Ric}^{Sk}_j + 3HK_j{}^k). \tag{10.8}$$

Taking the trace yields

$$3\frac{\partial H}{\partial t} = -\Delta\lambda + \lambda(S_S + 9H^2). \tag{10.9}$$

The importance of the evolution equations (10.3)–(10.4) is highlighted by the following result:

Proposition 10.2. *If the evolution equations (10.3)–(10.4) hold for* $0 \le t \le T$ *and the constraint equations (9.7)–(9.8) hold at* $t = 0$*, then the Einstein equation (8.1) holds for* $t \in [0,T]$ *(and hence so do the constraint equations).*

Proof. To begin, from (10.7) we see that if (10.4) holds, then $\mathrm{Ric}^M_{jk} = 0$, for $t \in [0, T],\ 1 \le j, k \le 3$. Hence, in view of the form of the metric (10.1),

$$\mathrm{Ric}_j{}^k = 0, \ \text{ for } 0 \le t \le T,\ 1 \le j,\ k \le 3. \tag{10.10}$$

From now on we drop the M from Ric^M. It remains to show that $\mathrm{Ric}_0{}^k = 0$ for $0 \le t \le T,\ 0 \le k \le 3$.

We will obtain a first order 4×4 system for $\mathrm{Ric}_0{}^k$, making use of the Ricci identity

$$\mathrm{Ric}_j{}^k{}_{;k} = \frac{1}{2} S_{M;j}, \tag{10.11}$$

which gives

$$\mathrm{Ric}_j{}^0{}_{;0} = -\mathrm{Ric}_j{}^1{}_{;1} - \mathrm{Ric}_j{}^2{}_{;2} - \mathrm{Ric}_j{}^3{}_{;3} + \frac{1}{2} S_{M;j}. \tag{10.12}$$

By (10.10) we have

$$S_M = \mathrm{Ric}_0{}^0, \quad S_{M;j} = \partial_j \mathrm{Ric}_0{}^0. \tag{10.13}$$

Now, if $\Gamma^\ell{}_{jk}$ are the connection coefficients of M, we have

$$\mathrm{Ric}_j{}^k{}_{;\ell} = \partial_\ell \mathrm{Ric}_j{}^k + \Gamma^k{}_{m\ell} \mathrm{Ric}_j{}^m - \Gamma^m{}_{j\ell} \mathrm{Ric}_m{}^k. \tag{10.14}$$

Thus, again by (10.10),

$$1 \le j, k \le 3 \Longrightarrow \mathrm{Ric}_j{}^k{}_{;\ell} = \Gamma^k{}_{0\ell} \mathrm{Ric}_j{}^0 - \Gamma^0{}_{j\ell} \mathrm{Ric}_0{}^k. \tag{10.15}$$

Hence, for $1 \le j \le 3$,

$$\mathrm{Ric}_j{}^0{}_{;0} = \frac{1}{2} \partial_j \mathrm{Ric}_0{}^0 - \sum_{k=1}^{3} (\Gamma^k{}_{0k} \mathrm{Ric}_j{}^0 - \Gamma^0{}_{jk} \mathrm{Ric}_0{}^k), \tag{10.16}$$

and we can replace the left side of (10.16) by

$$\mathrm{Ric}_j{}^0{}_{;0} = \partial_t \mathrm{Ric}_0{}^0 + \Gamma^0{}_{m0} \mathrm{Ric}_j{}^m - \Gamma^m{}_{j0} \mathrm{Ric}_m{}^0. \tag{10.17}$$

It is convenient to rewrite the terms on the right side of (10.12) when $j = 0$, using

$$\mathrm{Ric}_j{}^\ell = g^{\ell m} \mathrm{Ric}_{jm} = g^{\ell m} \mathrm{Ric}_{mj} = g^{ji} g^{\ell m} \mathrm{Ric}_m{}^i. \tag{10.18}$$

We obtain $\mathrm{Ric}_j{}^\ell{}_{;k} = g^{ji} g^{\ell m} \mathrm{Ric}_m{}^i{}_{;k}$, and hence

$$\mathrm{Ric}_0{}^\ell{}_{;k} = -\lambda^{-2} g^{\ell m}\, \partial_k \mathrm{Ric}_m{}^0 - \lambda^{-2} g^{\ell m} \Gamma^0{}_{ik} \mathrm{Ric}_m{}^i + \lambda^{-2} g^{\ell m} \Gamma^i{}_{mk} \mathrm{Ric}_i{}^0. \tag{10.19}$$

Consequently, we obtain a 4×4 system of the form

$$\begin{aligned} \partial_t \mathrm{Ric}_0{}^0 &= 2\lambda^{-2} \sum_{m,k=1}^{3} g^{km}\, \partial_k \mathrm{Ric}_m{}^0 + \mathcal{A}(\mathrm{Ric}), \\ \partial_t \mathrm{Ric}_j{}^0 &= \frac{1}{2} \partial_j \mathrm{Ric}_0{}^0 + \mathcal{B}_j(\mathrm{Ric}), \qquad 1 \le j \le 3, \end{aligned} \tag{10.20}$$

where $\mathcal{A}(\mathrm{Ric})$ and $\mathcal{B}_j(\mathrm{Ric})$ are linear in Ric. The system (10.20) is readily seen to be a linear, symmetrizable hyperbolic system; compare with the treatment of (3.3) in Chap. 16. Thus the quantities $\mathrm{Ric}_j{}^0$ vanish identically provided they vanish at $t = 0$. But the hypothesis that the constraint equations hold at $t = 0$ is equivalent to $G_j{}^0 = 0$ at $t = 0,\ 0 \le j \le 3$, by (9.3)–(9.6), and together with (10.10) this implies $S_M = 0$ at $t = 0$, and hence that

$$\mathrm{Ric}_j{}^0 = 0 \quad \text{at} \ \ t = 0,\ 0 \le j \le 3. \tag{10.21}$$

This finishes the proof of Proposition 10.2.

Note that if we regard the lapse function λ as an unknown, as well as g_{jk} and K_{jk}, which are 3×3 symmetric matrices, then (10.3)–(10.4) is a system of 12 equations in 13 unknowns. This underdetermined property is a consequence of one's ability to perform an arbitrary change of t-variable ($t' = t'(t)$) without affecting the foliation of M. We might insist that $\lambda = 1$, thus producing a determined 12×12 system, but that would lead to breakdown in finite time, since then (10.9) and the constraint equation (9.7) would imply

$$3 \frac{\partial H}{\partial t} = K_{jk} K^{jk} \ge 3H^2.$$

This breakdown might well be due to a bad choice of coordinates rather than to an actual breakdown for Einstein's equations.

Instead of trying to specify $\lambda(t,x)$ a priori, one might impose an extra equation involving λ. In [CBR] the following approach is taken. Let $\big(e_{jk}(x)\big)$ be an arbitrary Riemannian metric on S, and set

$$\lambda(t,x) = e(t,x)^{-1/2} g(t,x)^{1/2}, \tag{10.22}$$

where $e = \det(e_{jk})$, and $g = \det(g_{jk})$. If we set

$$k_{ij} = \lambda K_{ij}, \quad k = k^j{}_j = 3\lambda H, \tag{10.23}$$

a computation yields, for a metric of the form (10.1), the identity

$$
\begin{aligned}
\square k_{ij} &+ 3k_{\ell(i}\,\mathrm{Ric}^S{}_{j)}{}^{\ell} - 2R^S{}_i{}^{\ell}{}_j{}^m k_{\ell m} - 2k\,\mathrm{Ric}^S{}_{ij} - 2a_{(i}\nabla_{j)}k \\
&- 4k\nabla_i a_j - k_{\ell(i}\nabla_{j)}a^{\ell} - a^{\ell}\nabla_{\ell}k_{ij} - 3ka_i a_j - 4\lambda^{-2}k_{i\ell}k_{jm}k^{\ell m} \\
&= k_{\ell(i}\,\mathrm{Ric}^M{}_{j)}{}^{\ell} + 2k\,\mathrm{Ric}^M{}_{ij} - \nabla_{(i}\{\lambda^2 G^{M0}{}_{j)}\} - \partial_t \mathrm{Ric}^M{}_{ij},
\end{aligned}
\tag{10.24}
$$

when λ satisfies (10.22), where

$$
a_j = \lambda^{-1}\,\partial_j\lambda,
\tag{10.25}
$$

and ∇_j denotes the Levi–Civita connection on S_t, associated with the metric tensor $\big(g_{ij}(t)\big)$, and $\square k_{ij}$ is given by

$$
\square k_{ij} = \lambda^{-2}\,\partial_t^2 k_{ij} - g^{\ell m}\nabla^2_{\ell,m}k_{ij},
\tag{10.26}
$$

and we use the notation

$$
f_{(ij)} = f_{ij} + f_{ji}.
\tag{10.27}
$$

Now, when the right side of (10.24) is required to vanish, we can couple the resulting equation to (10.3), obtaining the system

$$
\partial_t g_{ij} = -2k_{ij},
\tag{10.28}
$$

$$
\begin{aligned}
\square k_{ij} = &-3k_{\ell(i}\,\mathrm{Ric}^S{}_{j)}{}^{\ell} + 2R^S{}_i{}^{\ell}{}_j{}^m k_{\ell m} + 2k\,\mathrm{Ric}^S{}_{ij} \\
&+ 2a_{(i}\nabla_{j)}k + 4k\nabla_i a_j + k_{\ell(i}\nabla_{j)}a^{\ell} \\
&+ a^{\ell}\nabla_{\ell}k_{ij} + 3ka_i a_j + 4\lambda^{-2}k_{i\ell}k_{jm}k^{\ell m}.
\end{aligned}
\tag{10.29}
$$

As before, λ is given by (10.22). This system has a hyperbolic character (compare with the discussion of the system (8.37)). The initial-value problem is well posed, and one has finite propagation speed. Furthermore, as shown in [CBR], if such a system is satisfied, then if we use the Lorentz metric (10.1), the Einstein tensor $G_{jk} = G^M_{jk}$ for this metric satisfies a homogeneous linear hyperbolic system of the following form (here, $1 \le j, k, \ell \le 3$):

$$
\begin{aligned}
\partial_t G^{jk} + \lambda^2(\nabla^j G^{k0} + \nabla^k G^{j0} - g^{jk}\nabla_{\ell}G^{\ell 0}) &= 0, \\
\square G^{j0} + f^j(G^{\mu\nu}, \nabla_{\ell}G^{k0}) &= 0, \\
\partial_t G^{00} + \nabla_j G^{j0} &= 0.
\end{aligned}
\tag{10.30}
$$

Here, f^j is linear in its arguments. In the middle equation, $0 \le \mu, \nu \le 3$, while $1 \le j, k, \ell \le 3$. Note that the last equation in (10.30) is just part of (1.55), a consequence of the Bianchi identity.

Suppose now that the system (10.28)–(10.29) is satisfied and also that, at $t = 0$, the constraint equations (9.7)–(9.8) and the (10.4) hold. The constraint equations give directly that $G^{0j} = 0$ at $t = 0$, for $0 \le j \le 3$. In view of (10.7), the equation (10.4) implies $\mathrm{Ric}^M_{jk} = 0$ at $t = 0$, for $1 \le j, k \le 3$. Now

$$S_M = -\lambda^{-2}\mathrm{Ric}^M_{00} + \sum_{1\le j,k\le 3} g^{jk}\, \mathrm{Ric}^M_{jk},$$

and $\mathrm{Ric}^M_{00} = G_{00} - (1/2)\lambda^2 S_M$, so we deduce that $S_M = 0$ at $t = 0$, and hence

$$G^{jk} = 0, \quad \text{at } t = 0, \quad 0 \le j,\ k \le 3. \tag{10.31}$$

Now the identity (1.55) (a consequence of Bianchi's identity) implies

$$\partial_t G^{j0} + \sum_{k=1}^{3} \nabla_k G^{jk} = 0 \ \text{ on } \ M, \quad 0 \le j \le 3. \tag{10.32}$$

In concert with (10.31), this implies

$$\partial_t G^{j0} = 0, \quad \text{at } t = 0, \tag{10.33}$$

for $0 \le j \le 3$. Thus all the Cauchy data for (10.30) vanish at $t = 0$. Thus, under our current hypotheses, we have a solution to Einstein's equations $G^{jk} = 0$ on M.

Another approach makes use of "maximal slicing," namely, requiring $H = 0$. In view of (10.9), this requires $\Delta\lambda = S_S\lambda$. Now, using (9.7), we see that when $H = 0$, $S_S = K_{jk}K^{jk} = |K|^2$, so we hence have the lapse equation

$$\Delta\lambda = |K|^2\lambda. \tag{10.34}$$

This has no nontrivial solution if S is compact, but it does if S is unbounded and "asymptotically flat." The use of the evolution equations with maximal slicing plays an important role in [CK].

The use of the evolution equations, with various approaches to the lapse function, and also some variants, involving a "shift vector," has played an important role in numerical work. A number of papers on this can be found in [EFH]. We also mention the recent work [CBY2], which has implications for both the theoretical and the numerical study of Einstein's equations.

Exercises

1. Derive the curvature identity (10.5).
2. Show that if S is a three-dimensional Riemannian manifold, with metric tensor g_{jk}, then

$$R^S_{ijk\ell} = g_{ik}\mathrm{Ric}^S_{j\ell} + g_{j\ell}\mathrm{Ric}^S_{ik} - g_{jk}\mathrm{Ric}^S_{i\ell} - g_{i\ell}\mathrm{Ric}^S_{jk} - \frac{1}{2}S_S(g_{ik}g_{j\ell} - g_{jk}g_{i\ell}).$$

3. Rewrite the proof of Proposition 10.2 in a coordinate-invariant manner. Show that $\mathrm{Ric}_0{}^0$ defines a t-dependent family of 0-forms r on S_t, that $\mathrm{Ric}_j{}^0$, $1 \le j \le 3$, defines a family of 1-forms ρ on S_t, and that (10.20) can be written in the form

$$\partial_t r = 2\lambda^{-2}\delta\rho + \mathcal{A}(r, \rho),$$
$$\partial_t \rho = \frac{1}{2} dr + \mathcal{B}(r, \rho),$$

where $\delta : \Lambda^1(S_t) \to \Lambda^0(S_t)$ is determined by the Riemannian metric on each slice S_t. Here, each S_t is identified with a single slice S_c, via the vector field τ.

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